SOLUTIONS TO EXERCISES SERIES NO 1

Exercise 1

Reminder 1

1.1 Partition P of the interval $[a,b]$ when the subintervals $[x_{i-1},x_i]$ are equal in length, then

$$\forall i \in \{1,2,3 \dots\dots\dots\dots n-1,n\}: \Delta x_i = \frac{b-a}{n} \;;\; x_i = a + \frac{b-a}{n} i.$$

1.2 Lower Darboux sum: $s_n = \sum_{i=1}^{n} m_i \, \Delta x_i$ and. Upper Darboux sum: $S_n = \sum_{i=1}^{n} M_i \, \Delta x_i$ with

$$M_i = \sup_{x_{i-1} \le x \le x_i} f(x) \text{ and } m_i = \inf_{x_{i-1} \le x \le x_i} f(x).$$

1.3 A function $f: [a,b] \to \mathbb{R}$ is Riemann integrable if and only if

$$\lim_{n\to\infty} (S_n - s_n) = 0.$$

a) $[a,b] = [1,2]$; $f(x) = \frac{1}{x}$.

$$x_i = 1 + \frac{1}{n} i \quad ; \quad \Delta x_i = \frac{1}{n}$$

Since f is decreasing in the domain $[1,2]$, then:

$$M_i = \sup_{x_{i-1} \le x \le x_i} f(x) = f(x_{i-1}) = \frac{n}{n+i-1} \quad \text{and} \quad m_i = \inf_{x_{i-1} \le x \le x_i} f(x) = f(x_i) = \frac{n}{n+i}.$$

So

$$S_n = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{n} \frac{1}{n+i-1} \quad ; \quad s_n = \sum_{i=1}^{n} m_i \Delta x_i = \sum_{i=1}^{n} \frac{1}{n+i}$$

so

$$S_n - s_n = \sum_{i=1}^{n} \frac{1}{n+i-1} - \sum_{i=1}^{n} \frac{1}{n+i} = \frac{1}{n} - \frac{1}{2n} = \frac{1}{2n}$$

and so

$$\lim_{n\to\infty} (S_n - s_n) = 0 \implies \text{It is integrable.}$$

Exercise 2

Reminder 2

2.1 If the function f is Riemann integrable on $[a,b]$ then the number $\int_a^b f(x)dx$ is the common limit of the two sequences:

$$u_n = \frac{b-a}{n} \sum_{k=0}^{n-1} f(a + \frac{b-a}{n} k \,) \text{ ‘} v_n = \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{b-a}{n} k\right).$$

a) $u_n = \sum_{k=0}^{n} \frac{n}{(n+k)^2}$

$$u_n=\sum_{k=0}^{n}\frac{n}{(n+k)^2}=\frac{1}{n}\sum_{k=0}^{n}\frac{1}{\left(1+\frac{k}{n}\right)^2}=\frac{b-a}{n}\sum_{k=0}^{n}f(a+\frac{b-a}{n}k).$$

Where $a=1$; $b=2$; $f(x)=\frac{1}{x^2}$ then

$$\lim_{n\to\infty}u_n=\int_1^2 f(x)dx=\int_1^2\frac{1}{x^2}dx=\left[-\frac{1}{x}\right]_1^2=\frac{1}{2}.$$

Exercise 3

Reminder 3

3.1 Change of variable

Let $\varphi:\ [a,b]\ \to\ \mathbb{R}$ be a continuously differentiable function, let f be continuous over $\varphi([a,b])$, Then $\int_a^b f(x)dx=\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$, where $b=\varphi(\beta)$; $a=\varphi(\alpha)$ and $x=\varphi(t)$; $dx=\varphi'(t)dt$.

3.2 Change the variable in indefinite integrals

Let $h: I\longrightarrow J$ C^1-diffeomorphism. We put $x=h(t)$ and $dx=h'(t)dt$ then

$$\int f(x)\,dx=\int f\big(h(t)\big)\,h'(t)dt \quad \text{and } t=h^{-1}(x).$$

3.3 Integration of the type $J_n=\int\frac{1}{(1+t^2)^n}dt$

$$\forall\ n\geq 1:\ 2nJ_{n+1}=(2n-1)J_n+\frac{t}{(1+t^2)^n}\ \ ;\ J_1=\text{Arctan}x+C.$$

3.4 Integration of the type $\int R(sinx,cosx)\,dx$

using the change in the variable $t=\tan\frac{x}{2}$, where:

$$\cos x=\frac{1-t^2}{1+t^2}\ \ ;\ \ \sin x=\frac{2t}{1+t^2}\ \ ;\ dx=\frac{2}{1+t^2}dt.$$

3.5 Integration of the type $\int R\left(x,\left(\frac{ax+b}{cx+d}\right)^{\frac{m}{n}},\left(\frac{ax+b}{cx+d}\right)^{\frac{p}{q}},\dots\dots,\left(\frac{ax+b}{cx+d}\right)^{\frac{r}{s}}\right)$

use a change in the variable $t=\left(\frac{ax+b}{cx+d}\right)^{\frac{1}{k}}$, where k is the Least Common Multiple (LCM) of the numbers $n,q,\dots\dots\dots,s$.

a) $I=\int\frac{1}{3\sqrt[3]{x+1}-x+1}dx.$

We put $t=(x+1)^{\frac{1}{3}}$ then $x+1=t^3\Longrightarrow dx=3t^2dt$ so

$$I=\int\frac{3t^2}{-t^3+3t+2}dt=\int\frac{1}{3t-t^3+2}3t^2dt.$$

We have

$$-t^3+3t+2=(2-t)(t+1)^2.$$

We put

$$\frac{3t^2}{3t-t^3+2}=\frac{3t^2}{(2-t)(t+1)^2}=\frac{a}{2-t}+\frac{b}{t+1}+\frac{c}{(t+1)^2}$$

And we get

$$a=\frac{4}{3}\quad,\quad b=-\frac{5}{3}\quad,\quad c=1.$$

So

$$I=-\frac{4}{3}\int\frac{1}{t-2}dx-\frac{5}{3}\int\frac{1}{t+1}dx+\int\frac{1}{(t+1)^2}dx$$

$$I=-\frac{4}{3}ln|t-2|-\frac{5}{3}ln|t+1|-\frac{1}{t+1}+C.$$

Substituting $t=(x+1)^{\frac{1}{3}}$ we get:

$$I=-\frac{4}{3}ln\left|\sqrt[3]{x+1}-2\right|-\frac{5}{3}ln\left|\sqrt[3]{x+1}+1\right|-\frac{1}{\sqrt[3]{x+1}+1}+C.$$

b) $J=\int_{-1}^{\frac{1}{2}}\sqrt{x^2+2x+5}\,dx.$

We have $x^2+2x+5=(x+1)^2+2^2$, we put $x+1=2\sinh t$ so $dx=2\,cosh\,t\,dt$.

And

$$x=\frac{1}{2}\Leftrightarrow(-1+2\sinh t=-1)\Leftrightarrow t=0$$

$$x=-1\Leftrightarrow(-1+2\sinh t=-1)\Leftrightarrow t=\ln 2.$$

So

$$J=\int_{-1}^{\frac{1}{2}}\sqrt{x^2+2x+5}\,dx=\int_0^{ln2}\sqrt{4\sinh^2 t+4}\,(2\cosh t\,dt)=4\int_0^{ln2}\sqrt{\cosh^2 t}\cosh t\,dt.$$

Since $\forall t\in\ [0,\ln 2]:\ \cosh t>0$ then

$$J=4\int_0^{\ln 2}\cosh^2 t\,\mathrm{dt}=4\int_0^{\ln 2}\frac{\cosh 2t+1}{2}dt=[\sinh 2t+2t]_0^{\ln 2}=\sinh(2\ln 2)+2\ln 2$$

$$J=2\ln 2+\frac{15}{8}.$$

c) $k=\int\frac{sin\,x}{1+sin\,x}dx.$

We put $t=\tan\frac{x}{2}$, where $\cos x=\frac{1-t^2}{1+t^2}$; $\sin x=\frac{2t}{1+t^2}$; $dx=\frac{2}{1+t^2}dt.$

Then

$$k = \int \frac{\sin x}{1+\sin x}dx = \int \frac{\frac{2t}{1+t^2}}{1+\frac{2t}{1+t^2}}\frac{2}{1+t^2}dt = \int \frac{4t}{(t^2+1)(t+1)^2}dt.$$

And we put

$$\frac{4t}{(t^2+1)(t+1)^2} = \frac{at+b}{t^2+1} + \frac{c}{t+1} + \frac{d}{(t+1)^2}$$

we get

$$a = 0 \quad , \quad b = 2 \quad , \quad c = 0 \quad , \quad d = -2.$$

So

$$k = 2\int \frac{1}{t^2+1}dx - 2\int \frac{1}{(t+1)^2}dx = 2\text{Arc}\tan t + \frac{2}{t+1} + C.$$

Substituting $t = \tan\frac{x}{2}$ we get:

$$k = x + \frac{2}{tan\frac{x}{2}+1} + C.$$

<u>Exercise 4</u>

Reminder 4

4.1 Integration by parts

Suppose that $u, v : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable in (a,b), and u', v' are integrable on $[a,b]$. Then

$$\int_a^b uv'\,dx = [uv]_a^b - \int_a^b u'v\,dx.$$

4.2 Integration by parts in indefinite integrals

Let I be a interval for $\mathbb{R}$ and v, u are functions of class C^1 on the interval I then

$$\int uv'\,dx = uv - \int u'v\,dx.$$

a) $I = \int x^2 \ln\frac{x-1}{x}dx.$

Suppose that:

$$\begin{cases} v' = x^2 \\ u = ln\frac{x-1}{x} \end{cases} \Rightarrow \begin{cases} v = \frac{1}{3}x^3 \\ u' = \frac{1}{x(x-1)} \end{cases}.$$

So

$$I = uv - \int u'v\,dx$$

$$= \frac{1}{3}x^3 \ln\frac{x-1}{x} - \int \frac{1}{x(x-1)}\frac{1}{3}x^3\,dx$$

$$= \frac{1}{3}x^3 \ln\frac{x-1}{x} - \int \frac{1}{x(x-1)}\frac{1}{3}x^3\,dx$$

$$= \frac{1}{3}x^3 \ln\frac{x-1}{x} - \frac{1}{3}\int \frac{x^2}{x-1}\,dx$$

By Euclidean division we get:

$$\frac{x^2}{x-1} = x + 1 + \frac{1}{x-1}.$$

So

$$I = \frac{1}{3}x^3 \ln\frac{x-1}{x} - \frac{1}{3}\int \left(x + 1 + \frac{1}{x-1}\right)dx$$

$$= \frac{1}{3}x^3 \ln\frac{x-1}{x} - \frac{1}{6}x^2 - \frac{1}{3}x - \frac{1}{3}\ln|x-1| + C.$$

c) $J = \int_0^1 x\text{Arc}\tan x\,dx$.

Assume that:

$$\begin{cases} v' = x \\ u = \text{Arc}\tan x \end{cases} \Rightarrow \begin{cases} v = \dfrac{1}{2}x^2 \\ u' = \dfrac{1}{x^2+1} \end{cases}.$$

So

$$J = \int_0^1 uv'\,dx = \left[\frac{1}{2}x^2 \text{Arc}\tan x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{1}{x^2+1}x^2\,dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\int_0^1 1 - \frac{1}{x^2+1}\,dx = \frac{\pi}{8} - \frac{1}{2}[x - \text{Arc}\tan x]_0^1$$

$$= \frac{\pi}{4} - \frac{1}{2}.$$

Exercise 5

a) $\int \frac{x^5+3x^4+3x}{(x^2+1)(x+2)^2}\,dx$.

By Euclidean division we get:

$$\frac{x^5+3x^4+3x}{(x^2+1)(x+2)^2}=x-1+\frac{-x^3+x^2+3x+4}{(x^2+1)(x+2)^2}.$$

By putting

$$\frac{-x^3+x^2+3x+4}{(x^2+1)(x+2)^2}=\frac{ax+b}{x^2+1}+\frac{c}{x+2}+\frac{d}{(x+2)^2}$$

we get

$$a=0\ \ ,\ b=1\ ,\ c=-1\ ,\qquad d=2.$$

So

$$I=\int\left(x-1+\frac{1}{x^2+1}+\frac{-1}{x+2}+\frac{2}{(x+2)^2}\right)dx$$

$$=\frac{1}{2}\,x^2-x+\operatorname{Arc}\tan x-\ \ln|x+2|-\frac{2}{x+2}+C.$$

b) $J=\int_1^3\sqrt{x}\ln\frac{x+1}{x}\,dx.$

Assume that

$$\begin{cases}v'=\sqrt{x}\\ u=\ln\dfrac{x+1}{x}\end{cases}\Longrightarrow\begin{cases}v=\dfrac{2}{3}x^{\frac{3}{2}}\\ u'=\dfrac{-1}{x^2+x}\end{cases}$$

We get

$$J=\left[\frac{2}{3}x^{\frac{3}{2}}\ln\frac{x+1}{x}\right]_1^3-\frac{2}{3}\int_1^3\frac{-1}{x^2+x}x^{\frac{3}{2}}\,dx$$

$$=\left[\frac{2}{3}x^{\frac{3}{2}}\ln\frac{x+1}{x}\right]_1^3-\frac{2}{3}\int_1^3\frac{-1}{x^2+x}x^{\frac{3}{2}}\,dx$$

$$=2\sqrt{3}\ln\frac{4}{3}-\frac{2}{3}ln2+\frac{2}{3}\underbrace{\int_1^3\frac{\sqrt{x}}{x+1}\,dx}_{A}.$$

Calculation of integral A.

We put $t=\sqrt{x}\ \ \Longrightarrow\ \ x=t^2$ and $dx=2tdt$.So

$$x=3\Leftrightarrow(t^2=3)\Leftrightarrow t=\sqrt{3}\,;\ t=-\sqrt{3}\ \ \text{(rejected)}$$

$$A=\int_1^{\sqrt{3}}\frac{t}{t^2+1}2tdt=2\int_1^{\sqrt{3}}1-\frac{1}{t^2+1}dt$$

$$=\ 2[x-Arc\tan x]_1^{\sqrt{3}}=2\left(\sqrt{3}-\frac{\pi}{3}\right)-2\left(1-\frac{\pi}{4}\right)=2\sqrt{3}-2-\frac{\pi}{6}$$

so

$$J=2\sqrt{3}ln\frac{4}{3}-\frac{2}{3}ln2+\frac{4}{3}\sqrt{3}-\frac{4}{3}-\frac{\pi}{9}.$$

c) $k=\int_0^{\frac{\pi}{2}}\frac{\sin 2x}{1+\cos x+\sin x}dx.$

$$k=\int_0^{\frac{\pi}{2}}\frac{\sin 2x}{1+\cos x+\sin x}dx=2\int_0^{\frac{\pi}{2}}\frac{\cos x\sin x}{1+\cos x+\sin x}dx.$$

By putting $t=\tan\frac{x}{2}$ then

$$x=\frac{1}{2}\Longrightarrow t=\tan\frac{\pi}{4}=1$$

$$x=0\Longrightarrow t=\tan 0=0.$$

So

$$k=2\int_0^1\frac{\left(\frac{1-t^2}{1+t^2}\right)\left(\frac{2t}{1+t^2}\right)}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}}\left(\frac{2}{1+t^2}dt\right)=\int_0^1\frac{-4t(t-1)}{(t^2+1)^2}dt$$

Assume that

$$\frac{-4t(t-1)}{(t^2+1)^2}=\frac{at+b}{t^2+1}+\frac{ct+d}{(t^2+1)^2}.$$

We get

$$a=0\ \ ,\ \ b=-4\ \ ,\ \ c=4\ \ ,\ \ d=4.$$

So

$$k=\int_0^1\frac{-4}{t^2+1}+\frac{4t+4}{(t^2+1)^2}dt=\int_0^1\frac{-4}{t^2+1}+\frac{4t}{(t^2+1)^2}+\frac{4}{(t^2+1)^2}dt$$

$$=\left[-4Arc\tan t+\frac{-2}{t^2+1}\right]_0^1+\int_0^1\frac{4}{(t^2+1)^2}dt=-\pi+1+4\underbrace{\int_0^1\frac{1}{(t^2+1)^2}dt}_{B}.$$

Calculation of integral B.

To calculate the integral B, we use integration by parts, or we can use the relationship

$$\forall\, n \geq 1:\ 2nJ_{n+1} = (2n-1)J_n + \frac{t}{(1+t^2)^n}\ \ ;\ J_1 = \text{Arctan}x + C\ \text{ where}$$

$$J_n = \int \frac{1}{(1+t^2)^n}\,dt.$$

Substituting $n = 1$ into the previous relationship we get:

$$2J_2 = J_1 + \frac{t}{(1+t^2)^1} = \text{Arc}\tan x + \frac{t}{1+t^2},$$

and from there

$$J_2 = \int \frac{1}{(t^2+1)^2}\,dt = \frac{1}{2}\text{Arc}\tan t + \frac{t}{2(t^2+1)}.$$

So

$$k = -\pi + 1 + 4\left[\frac{1}{2}Arc\,tan\,t + \frac{t}{2(t^2+1)}\right]_0^1 = 2 - \frac{\pi}{2}.$$

Exercise 6

1) We have f is continuous on $]-\infty, 0[$ and on $]0, +\infty[$. And

$$\lim_{\substack{x\to 0 \\ <}} f(x) = \lim_{\substack{x\to 0 \\ <}}\left(\sqrt{x^2 - 2x}\right) = 0 = f(0).$$

So f is continuous on $\mathbb{R}$.

2) $\int_{-1}^{1} f(x)\,dx = \underbrace{\int_{-1}^{0} f(x)\,dx}_{I} + \underbrace{\int_{0}^{1} f(x)\,dx}_{J}.$

Calculation of integral I.

$$I = \int_{-1}^{0} \sqrt{x^2 - 2x}\,dx = \int_{-1}^{0} \sqrt{(x-1)^2 - 1}\,dx.$$

We put $x - 1 = -\cosh t \Longrightarrow dx = -\sinh t\,dt$ and

$$x = 0 \Leftrightarrow t = 0$$

$$x = -1 \Leftrightarrow t = \ln\left(\sqrt{3} + 2\right) \text{أو } t = \ln\left(-\sqrt{3} + 2\right).$$

So

$$I = \int_{\ln(\sqrt{3}+2)}^{0} \sqrt{\cosh^2 t - 1}\,(-\sinh t\mathrm{dt}) = \int_{0}^{\ln(\sqrt{3}+2)} \sqrt{\cosh^2 t - 1}\,\mathrm{sinhdt}.$$

Since $\sinh t > 0$ in the interval $]0, +\infty[$. Then

$$I = \int_{0}^{\ln(\sqrt{3}+2)} \sinh^2 t\, dt$$

$$= \int_{0}^{\ln(\sqrt{3}+2)} \frac{\cosh 2\,t - 1}{2}\, dt$$

$$= \left[\frac{\sinh 2t + 2t}{4}\right]_{0}^{\ln(\sqrt{3}+2)}$$

$$= \sqrt{3} - \frac{1}{2}\ln\left(\sqrt{3} + 2\right).$$

Calculation of integral J (Using integration by parts).

Assume that

$$\begin{cases} v' = \sqrt{x} \\ u = \ln(x+1) \end{cases} \Rightarrow \begin{cases} v = \frac{2}{3}x^{\frac{3}{2}} \\ u' = \frac{1}{x+1} \end{cases}.$$

So

$$J = \left[\frac{2}{3}x^{\frac{3}{2}}\ln(x+1)\right]_0^1 - \frac{2}{3}\int_0^1 \frac{1}{x+1}x^{\frac{3}{2}}\,dx$$

$$= \frac{2}{3}\ln 2 - \frac{2}{3}\underbrace{\int_0^1 \frac{x\sqrt{x}}{x+1}\,dx}_{A}.$$

$$J = \left[\frac{2}{3}x^{\frac{3}{2}}\ln(x+1)\right]_0^1 - \frac{2}{3}\int_0^1 \frac{1}{x+1}x^{\frac{3}{2}}\,dx$$

Calculation of integral A.

We put $t = \sqrt{x} \Rightarrow x = t^2 \Rightarrow dx = 2tdt.$

So

$$A = \int_0^1 \frac{t^3}{t^2+1}2t\,dt$$

$$= 2\int_0^1 \frac{t^4}{t^2+1}dt$$

$$= 2\int_0^1 \frac{1}{t^2+1} + t^2 - 1dt$$

$$= 2\left[Arc\,tan\,t + \frac{1}{3}t^3 - t\right]_0^1$$

$$= \frac{1}{2}\pi - \frac{4}{3}.$$

So

$$J = \frac{2}{3}\ln 2 - \frac{1}{3}\pi + \frac{8}{9}.$$

And

$$\int_{-1}^{1} f(x)\,dx = \sqrt{3} - \frac{1}{2}\ln\left(\sqrt{3}+2\right) + \frac{2}{3}\ln 2 - \frac{1}{3}\pi + \frac{8}{9}.$$