Chapter 1

Riemann integral and primitives

1.1 Definition of Riemann integral

1.1.1 partition

Definition.1.1

A partition P of the interval $[a, b]$ is a finite set of numbers $\{x_0, x_1, x_2, \dots\dots\dots\dots, x_{n-1}, x_n\}$ such that $a = x_0 < x_1 < x_2 < \dots\dots\dots\dots < x_{n-1} < x_n = b$

We write $\Delta x_i = x_i - x_{i-1}$.

We define the norm of partition P is the positive number $\|P\| = \max\limits_{1 \le i \le n}(x_i - x_{i-1})$.

Remark 1.1

When the n subintervals have equal length $\Delta x_i = \frac{b-a}{n}$

The i^{th} term of the partition is $x_i = a + i\frac{b-a}{n}$ (This makes $x_n = b$.)

1.1.2 Darboux sums

Definition 1.2

Suppose $f\colon [a, b] \to \mathbb{R}$ is bounded and P is a partition of $[a, b]$. Defin

$$m_i = \inf_{x_{i-1}<x<x_i} f(x) \quad ; \quad M_i = \sup_{x_{i-1}<x<x_i} f(x),$$

and

$$s(P, f) = \sum_{i=1}^{n} m_i\, \Delta x_i \quad ; \quad S(P, f) = \sum_{i=1}^{n} M_i\, \Delta x_i.$$

We call $s(P, f)$ the lower Darboux sum, and $S(P, f)$ the upper Darboux sum.

Lemma.1.1

Let P and Q be two partitions of $[a, b]$ such that $P \subset Q$ Then

$$s(P, f) \le s(Q, f),$$
$$S(P, f) \ge S(Q, f).$$

(The partition Q is called a refinement of P.).

Proof

First let us consider a particular case. Let P' be a partition formed from P by adding one extra point, say $c \in [x_{i-1}, x_i]$. Let $m_i' = \sup\limits_{x_{i-1} \le x \le c} f(x)$, $m_i'' = \sup\limits_{c \le x \le x_i} f(x)$.

Then $m_i' \ge m_i$, $m_i'' \ge m_i$, and we have

$$s(P', f) = \sum_{i=1}^{i-1} m_i\, \Delta x_{i-1} + m_i'(\mathrm{c} - x_{i-1}) + m_i''(x_i - \mathrm{c}) + \sum_{i=i+1}^{n} m_i\, \Delta x_{i+1}$$
$$\ge \sum_{i=1}^{i-1} m_i\, \Delta x_{i-1} + m_i(x_i - x_{i-1}) + \sum_{i=i+1}^{n} m_i\, \Delta x_{i+1} = s(P, f)$$

Similarly one obtains that

$$S(P', f) \le S(P, f).$$

Now to prove the assertion one has to P consequetly a finite number of points in order to form Q.

Lemma.1.2
Let P and Q be arbitrary partitions of $[a, b]$. Then
$$s(P, f) \leq S(Q, f).$$
Proof
Consider the partition $P \cup Q$. By Lemma 1 1 we have
$s(P, f) \leq s(P \cup Q, f) \leq S(P \cup Q, f) \leq S(Q, f)$.
Proposition 1.1
Let $f: [a, b] \to \mathbb{R}$ be a bounded function. Let $m, M \in \mathbb{R}$ be such that for all $x \in [a, b]$, we have $m < x < M$. Then for every partition P of $[a, b]$,
$$m(b - a) \leq s(P, f) \leq S(P, f) \leq M(b - a)$$
Proof
Let Pbe a partition of $[a, b]$. Note that $m \leq m_i \leq M_i \leq M$ for all i and $\sum_{i=1}^{n} \Delta x_i = (b - a)$.
Therefore,
$$m(b - a) = \sum_{i=1}^{n} m\, \Delta x_i \leq \sum_{i=1}^{n} m_i\, \Delta x_i \leq \sum_{i=1}^{n} M_i\, \Delta x_i \leq \sum_{i=1}^{n} M\, \Delta x_i = M(b - a)$$
Definition 1.3
As the sets of lower and upper Darboux sums are bounded, we define
Lower Darboux integral $\underline{\int_a^b} f = \sup s(P, f)$: P a partition of [a, b].
Upper Darboux integral $\overline{\int_a^b} f = \inf S(P, f)$: P a partition of [a, b].

Lemma.1.3
$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$
Proof
Fix a partition Q. Then by Lemma 1 2
$$\forall P: s(P, f) \leq S(Q, f).$$
Therefore
$$\underline{\int_a^b} f = \sup_P s(P, f) \leq S(Q, f).$$
And from the above
$$\forall Q: \underline{\int_a^b} f \leq S(Q, f).$$
Hence
$$\underline{\int_a^b} f \leq \inf_Q S(Q, f) = \overline{\int_a^b} f.$$
Proposition 1.2
Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Let $m, M \in \mathbb{R}$ be such that for all $x \in [a, b]$, we have $m \leq f(x) \leq M$. Then
$$m(b - a) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq M(b - a)$$
Proof.
By Proposition 1.1 , for every partition P,
$$m(b - a) \leq s(P, f) \leq S(P, f) \leq M(b - a)$$

The inequality $m(b-a) \leq s(P,f)$ implies $m(b\ -\ a)\ \leq\ \underline{\int_a^b} f$.The inequality $S(P,f) \leq M(b-a)$ implies $\overline{\int_a^b} f \leq\ M(b\ -\ a)$.

1.2.3 Integrables functions-Riemann integral

Definition 1.4

A function $f\colon [a,b] \to \mathbb{R}$ is called Riemann integrable if

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

The common value is called integral of f over $[a,b]$ and is denoted by $\int_a^b f\ (x)\ dx$.

Proposition 1.3

Let $f\ :\ [a,b]\ \to\ \mathbb{R}$ be a Riemann integrable function. Let $m, M\ \in\ \mathbb{R}$ be such that $m\ \leq\ f\ (x)\ \leq\ M$ for all $x\ \in\ [a,b]$. Then

$$m(b\ -\ a)\ \leq\ \int_a^b f\ (x)\ dx \leq\ M(b\ -\ a).$$

Proof

Is a direct consequence of Proposition **1.2.**

Example 1.1

We integrate constant functions. If $f(x) = c$ for some constant c, then we *take* $m\ =\ M\ =\ c$. In Proposition 1.3. Thus f is integrable on $[a,b]$ and

$$\int_a^b f\ (x)\ dx = c(b\ -\ a).$$

Theorem 1.1

A function $f\colon [a,b] \to \mathbb{R}$ is Riemann integrable if and only if for any $\varepsilon\ >\ 0$ there exists a partition P of $[a,b]$ such that $S(P,f) - s(P,f) < \varepsilon$.

Proof

Necessity

Let $\underline{\int_a^b} f = \overline{\int_a^b} f$, i.e. let us assume that f is integrable.

$\exists P_1, P_2$: $s(P_1,f) > \underline{\int_a^b} f - \frac{\varepsilon}{2}$ and $S(P_2,f) < \overline{\int_a^b} f + \frac{\varepsilon}{2}$.

Let $Q = P_1 \cup\ P_2$. Then

$$\underline{\int_a^b} f - \frac{\varepsilon}{2} < s(P_1,f) \leq S(P_1 \cup\ P_2,f) \leq S(P_1 \cup\ P_2,f) \leq S(P_2,f) < \overline{\int_a^b} f + \frac{\varepsilon}{2}.$$

Therefore (since $\underline{\int_a^b} f = \overline{\int_a^b} f$)

$$S(Q,f) - s(Q,f) < \varepsilon.$$

sufficiency

Fix $\varepsilon > 0$. Let P be a partition such that $S(P,f) - s(P,f) < \varepsilon$.

Note that

$$\overline{\int_a^b} f - \underline{\int_a^b} f = S(P,f) - s(P,f) < \varepsilon.$$

Therefore it follows that

$$\forall\varepsilon > 0\colon \overline{\int_a^b} f - \underline{\int_a^b} f < \varepsilon.$$

This implies that

$$\overline{\int_a^b} f = \underline{\int_a^b} f.$$

Example 1.2

Let us show $f(x) = x^2$ is integrable on $[a, b]$ for all $b > a > 0$. We will see later that continuous functions are integrable, but let us demonstrate how we do it directly.

Let ε be given. Take $n \in \mathbb{N}$ and let $x_i = a + i\frac{b-a}{n}$ form the partition $P = \{x_0, x_1, x_2, \dots\dots\dots\dots, x_{n-1}, x_n\}$ of $[a, b]$. Then $\Delta x_i = \frac{b-a}{n}$ for all i. As f is increasing, for every subinterval $[x_{i-1}, x_i]$,

$$m_i = \inf_{x_{i-1}<x<x_i} f(x) = \left(a + (i-1)\frac{b-a}{n}\right)^2$$
$$M_i = \sup_{x_{i-1}<x<x_i} f(x) = \left(a + i\frac{b-a}{n}\right)^2.$$

Then

$$\begin{aligned}
S(P, f) - s(P, f) &= \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i \\
&= \frac{b-a}{n} \sum_{i=1}^{n} \left(\left(a + i\frac{b-a}{n}\right)^2 - \left(a + (i-1)\frac{b-a}{n}\right)^2\right) \\
&= \frac{b-a}{n} \sum_{i=1}^{n} \left(\left(a + n\frac{b-a}{n}\right)^2 - \left(a + 0\frac{b-a}{n}\right)^2\right) = \frac{\mathrm{b}^3}{n} \\
&= \frac{b-a}{n}(b^2 - a^2).
\end{aligned}$$

Picking n to be such that, $\frac{b-a}{n}(b^2 - a^2) < \varepsilon$ the proposition is satisfied, and the function is integrable.

On the other hand, as we know from algebra (or can be proven by induction):

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}$$

So

$$\begin{aligned}
S(P, f) &= \frac{b-a}{n} \sum_{i=1}^{n} \left(a + i\frac{b-a}{n}\right)^2 \\
&= \frac{b-a}{n}\left[\sum_{i=1}^{n} a^2 + 2a\frac{b-a}{n} i + \left(\frac{b-a}{n}\right)^2 i^2\right] \\
&= \frac{b-a}{n}\left[a^2 n + 2a\frac{b-a}{n}\sum_{i=1}^{n} i + \left(\frac{b-a}{n}\right)^2 \sum_{i=1}^{n} i^2\right] \\
&= \frac{b-a}{n}\left[a^2 n + 2a\frac{b-a}{n}\frac{n(n+1)}{2} + \left(\frac{b-a}{n}\right)^2 \frac{n(2n+1)(n+1)}{6}\right]
\end{aligned}$$

$$= (b-a)\left[a^2 + a(b-a)\frac{(n+1)}{n} + (b-a)^2\frac{(2n+1)(n+1)}{6n^2}\right].$$

Similarly one obtains that

$$s(P,f) = (b-a)\left[a^2 + a(b-a)\frac{(n-1)}{n} + (b-a)^2\frac{(2n-1)(n-1)}{6n^2}\right].$$

So

$$\lim_{n\to\infty} S(P,f) = \lim_{n\to\infty} s(P,f) = \frac{(b-a)}{3}(b^2+ab+a^2) = \frac{1}{3}(b^2-a^2).$$

Finally we obtain

$$\int_a^b x^2 dx = \frac{1}{3}(b^2-a^2).$$

1.1.4 Riemann sums

Definition 1.5

Let $f: [a,b] \to \mathbb{R}$ be defined on the interval $[a,b]$ and let $P = \{x_0, x_1, x_2, \dots\dots\dots\dots, x_{n-1}, x_n\}$ be a partition of $[a,b]$.
Let $C = \{c_1, c_2, \dots\dots c_{i-1}, c_i, \dots\dots \quad c_{n-1}, c_n\}$ where c_i denote any value in the i^{th} subinterval ($c_i \in [x_{i-1}, x_i]$). The Riemann sum of a function f on $[a,b]$ that corresponds to P and the point system C is

$$R(P,f,\mathrm{C}) = \sum_{i=1}^{n} f(c_i)\,\Delta x_i.$$

Theorem 1.2
A function f is Riemann integrable on $[a,b]$ if there is a number L such that for each $\varepsilon > 0$ there is $\delta > 0$ such that if P is any partition of $[a,b]$ with $\|P\| < \delta$ then $|R(P,f,\mathrm{C}) - L| < \varepsilon$. (In other words $\lim_{\|P\|\to 0} R(P,f,\mathrm{C}) = L$).And we have $L = \int_a^b f(x)dx$.
The set of all Riemann integrable functions in $[a,b]$ is denoted by $\mathcal{R}([a,b])$.
Proof
Necessity
Using $|R(P,f,\mathrm{C}) - L| \le S(P,f) - s(P,f)$.
sufficiency
To do this, we will first show that

$$S(P,f) = \sup_{\mathrm{C}} R(P,f,\mathrm{C}), \quad s(P,f)\inf_{\mathrm{C}} R(P,f,\mathrm{C}).$$

Remark
If the function f is Riemann integrable on $[a,b]$ then the number $\int_a^b f(x)dx$ is the common limit of the two sequences

$$u_n = \frac{b-a}{n}\sum_{i=0}^{n-1} f(a + \frac{b-a}{n} i\,) \text{ and } v_n = \frac{b-a}{n}\sum_{i=1}^{n} f(a + \frac{b-a}{n} i\,).$$

Example 1.3

Calculate the limit of the sum $v_n = \sum_{i=0}^{n} \frac{n}{(n+i)^2}$.

We have

$$v_n = \sum_{i=0}^{n-1} \frac{n}{(n+i)^2} + \frac{1}{4n} = \frac{1}{n}\sum_{i=0}^{n-1} \frac{1}{\left(1+\frac{i}{n}\right)^2} + \frac{1}{4n}$$

by putting $[a,b] = [1,2]$; $f(x) = \frac{1}{x^2}$ then

$$v_n = \frac{b-a}{n}\sum_{i=0}^{n-1} f(a + \frac{b-a}{n} i) + \frac{1}{4n}.$$

So

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n} f(1+\frac{1}{n} i)\right) + 0 = \int_a^b f(x)dx = \int_1^2 \frac{1}{x^2}dx = \frac{1}{2}.$$

1.1.5 Examples of integrable functions

Theorem 1.3

Let f: [a, b] → ℝ be monotone. Then f is Riemann integrable.

Proof

Suppose that f is increasing so that $f(a) \le f(b)$.

If $f(a) = f(b)$ then f is constant, so f is Riemann integrable and $\int_a^b f(x)dx = f(a)(b-a)$.

If $f(a) < f(b)$ let $\varepsilon > 0$ and P a partition of $[a,b]$ such that $\|P\| < \delta = \frac{\varepsilon}{f(b)-f(a)}$.

For this partition we obtain

$$S(P,f) - s(P,f) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\,\Delta x_i$$

$$< \delta \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = \delta\big(f(b) - f(a)\big) = \varepsilon.$$

Theorem 1.4

Let f: $[a,b]$ → ℝ be continuous. Then f is Riemann integrable.

Proof

Let $P = \{x_0, x_1, x_2, \dots\dots\dots\dots, x_{n-1}, x_n\}$ be a partition of $[a,b]$ and
$m_i = \inf_{x_{i-1}\le x\le x_i} f(x)$; $M_i = \sup_{x_{i-1}\le x\le x_i} f(x)$.
Let $\varepsilon > 0$. A continuous function on closed interval $[x_{i-1}, x_i]$ is uniformly continuous and reaches its upper and lower bounds at least once, so there exists $\delta > 0$ such that

$\forall x_{i-1}, x_i \in [a,b]: |x_i - x_{i-1}| < \delta \Longrightarrow |f(x_i) - f(x_{i-1})| < \frac{\varepsilon}{b-a}$ and there is at least x_i' ; x_i'' are from the subinterval $[x_{i-1}, x_i]$ where $m_i = f(x_i')$; $M_i = f(x_i'')$.
Choose a partition P such that $\|P\| < \delta$, so

$$S(P,f) - s(P,f) = \sum_{i=1}^{n} (M_i - m_i)\, \Delta x_i = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\, \Delta x_i$$

$$< \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \Delta x_i = \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

Theorem 1.5

If $f, g: [a,b] \to \mathbb{R}$ are integrable, then $fg: [a,b] \to \mathbb{R}$ is integrable. If, in addition, $g \neq 0$ and $\frac{1}{g}$ is bounded, then $\frac{f}{g}: [a,b] \to \mathbb{R}$ is integrable.

1.2. Properties of the Riemann integral

Let $f, g: [a,b] \to \mathbb{R}$ Riemann integrable functions on $[a,b]$. The integral has the following three basic properties.

1) Linearity:

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b f(x)dx \quad , \quad \int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx$$

2) Monotonicity:

$$\text{If } \forall x \in [a,b]: f(x) \leq g(x), \text{then } \int_a^b f(x)dx \leq \int_a^b g(x)dx.$$

3) Additivity:If $\alpha, \beta, \gamma \in [a,b]$, then

a) $\displaystyle\int_\alpha^\beta f(x)dx = \int_\alpha^\gamma f(x)dx + \int_\gamma^\beta f(x)dx.$

b) $\displaystyle\int_\alpha^\alpha f(x)dx = 0.$

c) $\displaystyle\int_\alpha^\beta f(x)dx = -\int_\beta^\alpha f(x)dx.$

4) If f is continuous on $[a,b]$ and $\forall x \in [a,b]: f(x) \geq 0$ then

$$\left(\int_a^b f(x)dx = 0\right) \Longrightarrow (\forall x \in [a,b]: f(x) = 0).$$

5) $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$.

6) If f is continuous on $[a,b]$, then there exists $c \in [a,b]$, where

$$\int_a^b f(x)dx = f(c)(b-a).$$

1.3 Integrals and primitives

1.3.1 primitives

Definition 1.6

Let $f: [a,b] \to \mathbb{R}$ function, we say that function F is a primitive function of f over $[a,b]$ if and only if F is differentiable over $[a,b]$ and $\forall x \in [a,b]$: $F'(x) = f(x)$.

Proposition 1.4

If F_1 and F_2 are primitive functions of f on $[a,b]$ then$\forall x \in [a,b]$: $F_1(x) - F_2(x) = C$ where C is a real constant.

Example 1.4

The function $F(x) = \frac{1}{3}x^3$ is a primitive of the function $f(x) = x^2$ over $\mathbb{R}$ because

$$\forall x \in \mathbb{R}: F'(x) = \left(\frac{1}{3}x^3\right)' = x^2 = f(x).$$

1.3.2 Fundamental theorems of calculus

Theorem 1.6 (First theorem of calculus)

if $F: [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable in $]a,b[$ with $F' = f$ where $f: [a,b] \to \mathbb{R}$ is Riemann integrable, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof

Let $P = \{x_0, x_1, x_2, \dots\dots\dots\dots, x_{n-1}, x_n\}$ be a partition of $[a,b]$.

The function F is continuous on the closed interval $[x_{i-1}, x_i]$ and differentiable in the open interval $]x_{i-1}, x_i[$ with $F' = f$. By the mean value theorem, there exists

$c_i \in]x_{i-1}, x_i[$ such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1})$$
$$= f(c_i)(x_i - x_{i-1})$$

Since f is Riemann integrable, it is bounded and it follows that

$$m_i(x_i - x_{i-1}) \leq f(c_i)(x_i - x_{i-1}) \leq M_i(x_i - x_{i-1})$$

or

$$m_i(x_i - x_{i-1}) \leq F(x_i) - F(x_{i-1}) \leq M_i(x_i - x_{i-1})$$

where

$$M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x) \quad \text{and} \quad m_i = \inf_{x_{i-1} \leq x \leq x_i} f(x).$$

So

$$\sum_{i=1}^{n} m_i \, \Delta x_i \leq \sum_{i=1}^{n} \big(F(x_i) - F(x_{i-1})\big) \, \Delta x_i \leq \sum_{i=1}^{n} M_i \, \Delta x_i$$

Hence $s(P, f) \leq F(b) - F(a) \leq S(P, f)$ of every partition of $[a, b]$ wich implies that $\underline{\int_a^b f} \leq F(b) - F(a) \leq \overline{\int_a^b f}$. Since f is integrable i.e. $\underline{\int_a^b f} = \overline{\int_a^b f}$ we obtain

$$F(b) - F(a) = \int_a^b f(x) \, dx.$$

Theorem 1.7 **(Second theorem of calculus)**

Suppose that $f\colon [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and $F\colon [a, b] \to \mathbb{R}$ is defined by $\forall x \in [a, b]\colon F(x) = \int_a^x f(t) \, dt$.Then F is differentiable over $[a, b]$ and $\forall x \in [a, b]\colon F'(x) = f(x)$ (that is, F is a primitive function of f over $[a, b]$).

Proof

Let $x, h \in [a, b]$ and $h > 0$. Then

$$\frac{F(x + h) - F(x)}{h} = \frac{\int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt}{h} = \frac{1}{h} \int_x^{x+h} f(t) \, dt.$$

Let $\varepsilon > 0$. Since f is continuous at x there exists $\delta > 0$ such that

$$|f(t) - f(x)| < \varepsilon \quad \text{for} \quad |t - x| < \delta.$$

It follows that if $0 < h < \delta$ then

$$\left| \frac{F(x + h) - F(x)}{h} - f(x) \right| = \left| \frac{1}{h} \int_x^{x+h} f(t) \, dt - f(x) \right|$$

$$= \left| \frac{1}{h} \int_x^{x+h} \big(f(t) - f(x)\big) dt \right|$$

$$\leq \frac{1}{h} \int_x^{x+h} |f(t) - f(x)|\, dt$$

$$\leq \frac{1}{h} \sup_{x<t<x+h} |f(t) - f(x)| \left| \int_x^{x+h} 1 \right| dt$$

$$\leq \frac{1}{h} \varepsilon h = \varepsilon.$$

So

$$\lim_{\substack{h \to 0 \\ h>0}} \frac{F(x+h) - F(x)}{h} = f(x).$$

In the same way, we obtain

$$\lim_{\substack{h \to 0 \\ h<0}} \frac{F(x+h) - F(x)}{h} = f(x).$$

Which proves the result.

Corollary 1.1

Let $f: [a,b] \to \mathbb{R}$ be continuous in $[a,b]$ and F is a primitive function of f over $[a,b]$. Then

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

Proof

Proof is a direct consequence of Theorem 1.7.

Example 1.5

Since $F(x) = \frac{1}{3}x^3$ is primitive function of $f(x) = x^2$ over $\mathbb{R}$. Then

$$\forall a, b \in \mathbb{R}: \int_a^b f(x)\, dx = F(b) - F(a) = \frac{1}{3}b^3 - \frac{1}{3}a^3.$$

1.3.3 Change of variable

Theorem 1.8

Let $\varphi : [a, b] \rightarrow \mathbb{R}$ be a continuously differentiable function, let f be continuous over $\varphi([a, b])$, Then $\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$, where $b = \varphi(\beta)$; $a = \varphi(\alpha)$ $and x = \varphi(t)$; $dx = \varphi'(t)dt$.

Proof

The function $f(\varphi)\varphi'$ is continuous and therefore integrable. Let F be a primitive of f and then $F(\varphi)$ is a primitive of $f(\varphi(t))\varphi'(t)$. So according to the Corollary 1.1,

$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt = F\big(\varphi(\beta)\big) - F\big(\varphi(\alpha)\big) = F(b) - F(a) = \int_a^b f(x)dx.$$

Example 1.6

Calculate the integral $J = \int_0^1 \sqrt{1 - x^2}\, dx$. (Put $x = \varphi(t) = \sin t$).

$$x = \varphi(t) = \sin t \Longrightarrow dx = \cos t\, dt$$

$\sin\alpha = 0 \iff \alpha = 0, \pi, -\pi, 2\pi, -2\pi, \dots$ (The value of α can be chosen from among the values $0, \pi, -\pi, 2\pi, -2\pi \dots$).

$\sin\beta = 1 \iff \beta = \frac{\pi}{2}, \frac{-3\pi}{2}, \frac{5\pi}{2} \dots$ (The value of β can be chosen from among the values $\frac{\pi}{2}, \frac{-3\pi}{2}, \frac{5\pi}{2} \dots$).

So

$$J = \int_0^{\frac{\pi}{2}} \sqrt{1 - (\sin t)^2}\cos t\, dt = \int_0^{\frac{\pi}{2}} \sqrt{(\cos t)^2}\cos t\, dt.$$

Since $\forall x \in \left[0, \frac{\pi}{2}\right] : \cos t \geq 0$. Then

$$J = \int_0^{\frac{\pi}{2}} (\cos t)^2\, dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2t)\, dt$$

$$= \frac{1}{2}\left[t + \frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

Example 1.7

Calculate the integral $K = \int_0^4 \frac{\sqrt{x}}{\sqrt{x}+1} dx$. (Put $x = \varphi(t) = t^2$).

$$x = \varphi(t) = t^2 \Longrightarrow dx = 2tdt.$$

$\varphi(\alpha) = a \iff \alpha^2 = 0 \iff \alpha = 0$

$\varphi(\beta) = b \iff \beta^2 = 4 \iff \beta = -2, \beta = 2$ (The value of β can be chosen from among the values -2 , 2). So

$$K = \int_{0}^{-2} \frac{\sqrt{t^2}}{\sqrt{t^2}+1} 2tdt$$

Since $\forall t \in [-2,0]: t \leq 0$. Then

$$K = 2\int_{-2}^{0} \frac{t^2}{-t+1} dt = 2\int_{-2}^{0} (-t-1-\frac{1}{t-1})\, dt$$

$$= 2\left[-\frac{1}{2}t^2 - t - ln|t-1|\right]_{-2}^{0} = 2\ln 3.$$

1.3.4 Integration by parts

Theorem 1.9

Suppose that $u, v : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable in (a,b), and u', v' are integrable on $[a,b]$. Then

$$\int_a^b uv'\, dx = [uv]_a^b - \int_a^b u'v\, dx.$$

Proof

The function u v is continuous on $[a,b]$ and, by the product rule, differentiable

in (a,b) with derivative $(uv)' = u'v + uv'$. Since u, v, u' and v' are integrable on $[a,b]$. Theorem 1.4 implies that $u'v$, uv' and $(uv)'$, are integrable. From Theorem 1.5, we get that $\int_a^b (uv' + u'v)\, dx = \int_a^b uv'\, dx + \int_a^b u'v\, dx = [uv]_a^b$, which proves the result.

Example 1.8

calculate the integral $I = \int_0^1 Arc\tan x\, dx$.

$$\begin{cases} v' = 1 \\ u = Arc\tan x \end{cases} \Longrightarrow \begin{cases} v = x \\ u' = \dfrac{1}{x^2+1} \end{cases}$$

$$I = [uv]_0^1 - \int_0^1 u'v\, dx = [xArc\tan x]_0^1 - \int_0^1 \frac{1}{x^2+1} x\, dx$$

$$I = \left[xArc\tan x - \frac{1}{2} ln(x^2+1)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2} ln2.$$

Example 1.9

Calculate the integral $I = \int_1^2 x\ln\frac{x}{x+1} dx$.

$$\begin{cases} v' = x \\ u = \ln\dfrac{x}{x+1} \end{cases} \Rightarrow \begin{cases} v = \dfrac{1}{2}x^2 \\ u' = \dfrac{1}{x(x+1)} \end{cases}$$

$$I = [uv]_1^2 - \int_1^2 u'v\,dx$$

$$I = \left[\frac{1}{2}x^2\ln\frac{x}{x+1}\right]_1^2 - \int_1^2 \frac{1}{2}x^2\frac{1}{x(x+1)}dx$$

$$I = \left[\frac{1}{2}x^2\ln\frac{x}{x+1}\right]_1^2 - \int_1^2 \frac{1}{2}x^2\frac{1}{x(x+1)}dx$$

$$I = \frac{5}{2}\ln2 - 2\ln3 - \int_1^2 \frac{1}{2}\frac{x}{(x+1)}dx$$

$$I = \frac{5}{2}\ln2 - 2\ln3 - \frac{1}{2}\int_1^2 1 - \frac{1}{(x+1)}dx$$

$$I = \frac{5}{2}\ln2 - 2\ln3 - \frac{1}{2}[x - \ln(x+1)]_1^2$$

$$I = \frac{5}{2}\ln2 - 2\ln3 - \frac{1}{2}[1 + \ln2 - \ln3] = 2\ln2 - \frac{3}{2}\ln3 - \frac{1}{2}.$$

1.4 Primitives and integration techniques

1.4.1 The Indefinite Integral

Definition 1.7
The set of all primitive functions of f is the **indefinite integral** of f with respect to x and denoted by $\int f(x)\,dx$ where
$\int f(x)\,dx$ is read " the integral of f w.r.t x ".
Note: The above definition says that if a function F is an primitive of f, then

$$\int f(x)\,dx = F(x) + C \quad \text{where C is a real constant.}$$

Example 1.10

$$\int x^3\,dx = \frac{1}{4}x^4 + C.$$

1.4.2 primitives of usual functions

Table of usual primitives

$\int f(x)\,dx$	f
$\dfrac{1}{\alpha+1}x^{\alpha+1} + C$	$(\alpha \in \mathbb{R}^* - \{-1\}$حيث$)x^{\alpha}$

$\ln\lvert x\rvert + C$	$\dfrac{1}{x}$
$e^x + C$	e^x
$-\cos x + C$	$\sin x$
$\sin x + C$	$\cos x$
$-\ln\lvert\cos x\rvert + C$	$\tan x$
$\tan x + C$	$\dfrac{1}{\cos^2 x}$
$-\text{cotan}\, x + C$	$\dfrac{1}{\sin^2 x}$
$\cosh x + C$	$\sinh x$
$\sinh x + C$	$\cosh x$
$\dfrac{1}{a}\text{Arctan}\, \dfrac{x}{a} + C$	$\dfrac{1}{x^2 + a^2}$
$\text{Arcsin}\, \dfrac{x}{a} + C$	$\dfrac{1}{\sqrt{a^2 - x^2}}$
$\dfrac{1}{2a}\ln\left\lvert\dfrac{x+a}{x-a}\right\rvert + C$	$\dfrac{1}{x^2 - a^2}$
$\dfrac{1}{\alpha + 1}\big(u(x)\big)^{\alpha+1} + C$	$\big(u(x)\big)^{\alpha} u'(x)$ $\big($حيث $\alpha \in \mathbb{R}^* - \{-1\}$ و $u \in C^1(I)\big)$
$\ln\lvert u(x)\rvert + C$	$\dfrac{u'(x)}{u(x)}$ $\big($حيث $\forall x \in I: u(x) \neq 0$ و $u \in C^1(I)\big)$
$e^{u(x)} + C$	$u'(x)e^{u(x)}$ $\big($حيث $u \in C^1(I)\big)$
$G\big(u(x)\big)u'(x) + C$ G is a primitive of g over I	$g\big(u(x)\big)u'(x)$ Where $u \in C^1(I)$ and g *is continuous over* $u(I)$

1.4.3 change the variable in indefinite integrals

Theorem 1.10

Let $h: I \longrightarrow J$ C^1-diffeomorphism. We put $x = h(t)$ and $dx = h'(t)dt$ then

$$\int f(x)\, dx = \int f\big(h(t)\big)\, h'(t)dt \quad \text{and } t = h^{-1}(x).$$

Note

A function $h: I \longrightarrow J$ is called C^1-diffeomorphism if

a) h is a bejiction of I on J;

b) h and h^{-1} admit derivatives of order 1, continuous, respectively on I and J.

Example 1.11

Calculate $I = \int \sqrt{1-x^2}\, dx$.

We put $x = h(t) = \sin t$ where h: $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\longrightarrow]-1,1[$ (h is C^1-diffeomorphism), and $dx = \cos t\, dt$.

So

$$I = \int \sqrt{1 - sin^2 t} \cos t\, dt = \int \sqrt{\cos^2 t} \cos t\, dt.$$

Since $\forall t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[: \cos t \geq 0$ we get

$$I = \int cos^2 t\, dt = \frac{1}{2}\int (1 + \cos 2t)\, dt$$

$$= \frac{1}{2}\left(t + \frac{1}{2}\sin 2t\right) + C = \frac{1}{2}t + \frac{1}{2}\cos t \sin t + C$$

$$= \frac{1}{2}t + \frac{1}{2}\sqrt{1 - sin^2 t}\sin t + C.$$

Substituting $t = h^{-1}(x) = Arcsinx$ we get the following result:

$$I = \frac{1}{2}Arcsinx + \frac{1}{2}x\sqrt{1-x^2} + C.$$

Example 1.12

Calculate $J = \int \frac{x}{\sqrt{x}+1} dx$.

We put $x = h(t) = t^2$ where h: $]0, +\infty[\longrightarrow]0, +\infty[$ (h is C^1-diffeomorphism), and $dx = 2tdt$.

So

$$J = \int \frac{t^2}{\sqrt{t^2} + 1} 2tdt$$

Since $\forall t \in]0, +\infty[: t > 0$ we get

$$J = \int \frac{t^2}{\sqrt{t^2} + 1} 2tdt$$

$$= 2\int \frac{t^3}{\mathrm{t} + 1} dt$$

$$= 2\int \left(t^2 - \frac{1}{\mathrm{t}+1} - \mathrm{t} + 1\right) dt$$

$$= 2\left(\frac{1}{3}t^3 - \ln(\mathrm{t}+1) - \frac{1}{2}t^2 + \mathrm{t}\right)$$

$$J = \frac{2}{3}t^3 - 2\ln(\mathrm{t}+1) - t^2 + 2\mathrm{t} + C.$$

Substituting $t = h^{-1}(x) = \sqrt{x}$ we get the following result:

$$J = \frac{2}{3}x\sqrt{x} - x + 2\sqrt{x} - 2\ln(\sqrt{x}+1) + C.$$

1.4.4 Integration by parts in indefinite integrals

Theorem 1.11

Let I be a interval for $\mathbb{R}$ and v, u are functions of class C^1 on the interval I then

$$\int uv'\,dx = uv - \int u'v\,dx.$$

Example 1.13

Calculate $I = \int xe^{2x}\,dx$. By putting:

$$\begin{cases} v' = e^{2x} \\ u = x \end{cases} \Rightarrow \begin{cases} v = \frac{1}{2}e^{2x} \\ u' = 1 \end{cases}$$

$$I = \int xe^{2x}\,dx = uv - \int u'v\,dx$$

$$= x\frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}\,dx$$

$$= \left(\frac{1}{2}x - \frac{1}{4}\right)e^{2x} + C.$$

Example 1.14

Calculate $J = \int e^x \sin x\,dx$. By putting:

$$\begin{cases} v' = \sin x \\ u = e^x \end{cases} \Rightarrow \begin{cases} v = -\cos x \\ u' = e^x \end{cases}.$$

We get

$$J = -e^x \cos x + \int e^x \cos x\,dx.$$

Again we put

$$\begin{cases} v' = \cos x \\ u = e^x \end{cases} \Rightarrow \begin{cases} v = \sin x \\ u' = e^x \end{cases},$$

so

$$J = -e^x \cos x + e^x \sin x - \int e^x \sin x\,dx$$

$$J = -e^x \cos x + e^x \sin x - J$$

$$2J = -e^x \cos x + e^x \sin x$$

we obtain

$$= -e^x \cos x + e^x \sin x + C.$$

Example 1.15*

Calculate $J = \int x\sqrt{x}\, dx$. By putting:

$$\begin{cases} v' = x \\ u = \sqrt{x} \end{cases} \Longrightarrow \begin{cases} v = \frac{1}{2}x^2 \\ u' = \frac{1}{2\sqrt{x}} \end{cases}.$$

We get

$$J = \frac{1}{2}x^2\sqrt{x} - \int \frac{1}{2\sqrt{x}}\frac{1}{2}x^2\, dx$$

$$= \frac{1}{2}x^2\sqrt{x} - \frac{1}{4}\int x\sqrt{x}\ dx$$

$$= \frac{1}{2}x^2\sqrt{x} - \frac{1}{4}J$$

$$J + \frac{1}{4}J = \frac{1}{2}x^2\sqrt{x}$$

we obtain

$$J = \frac{2}{5}x^2\sqrt{x} + C.$$

1.5 Special integration methods

1.5.1 Integration of a rational function

Decomposition of a rational function into simple elements

Definition 1.8

Let P, Q be two real polynomials, $Q(x) \neq 0$. Function $x \to \frac{P(x)}{Q(x)}$ is called rational function or rational fraction

Definition 1.9

The functions $x \to \frac{A}{(x-a)^k}$, $x \to \frac{Mx+N}{(x^2+px+q)^k}$ where $k \in \mathbb{N}^*, a, A, M, N, p, q \in \mathbb{R}, p^2 - 4q < 0$, are called simple elements, of the first and second species respectively.

Theorem 1.12

Any rational fraction $\frac{P}{Q}$ is represented in the unique form $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$

the polynomials S, R being respectively the quotient and the remainder of the division of P by Q.

Theorem 1.13

Let $\deg P < \deg Q$,

$Q(x) = (x - a_1)^{m_1}(x - a_2)^{m_2} \dots \dots (x - a_k)^{m_k}.(x^2 + p_1 x + q_1)^{n_1}$

$(x^2 + p_2 x + q_2)^{n_2} \dots \dots (x^2 + p_s x + q_s)^{n_s}, p_j^2 - 4q_j < 0, \forall\, 1 \le j \le s.$

Then the fraction $\frac{P}{Q}$ is represented in the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{(x - a_1)^2} + \cdots + \frac{A_{m_1}}{(x - a_1)^{m_1}}$$
$$+ \frac{B_1}{x - a_2} + \frac{B_2}{(x - a_2)^2} + \cdots + \frac{B_{m_2}}{(x - a_2)^{m_2}}$$
$$+ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$
$$+ \frac{C_1}{x - a_k} + \frac{C_2}{(x - a_k)^2} + \cdots + \frac{C_{m_k}}{(x - a_k)^{m_k}}$$
$$+ \frac{M_1^1 x + N_1^1}{x^2 + p_1 x + q_1} + \frac{M_2^1 x + N_2^1}{(x^2 + p_1 x + q_1)^2} + \cdots + \frac{M_{n_1}^1 x + N_{n_1}^1}{(x^2 + p_1 x + q_1)^{n_1}}$$
$$+ \frac{M_1^2 x + N_1^2}{x^2 + p_2 x + q_2} + \frac{M_2^2 x + N_2^2}{(x^2 + p_2 x + q_2)^2} + \cdots + \frac{M_{n_2}^2 x + N_{n_2}^2}{(x^2 + p_2 x + q_2)^{n_2}}$$
$$+ \cdots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$
$$+ \frac{M_1^s x + N_1^s}{x^2 + p_s x + q_s} + \frac{M_2^s x + N_2^s}{(x^2 + p_s x + q_s)^2} + \cdots + \frac{M_{n_s}^s x + N_{n_s}^s}{(x^2 + p_s x + q_s)^{n_2}}.$$

where $A, B, C, M, N, p, q, a,$ are real canstants.

Examples 1.16

1) $\frac{P(x)}{Q(x)} = \frac{4x^4 - 4x^3 - 3x^2 - 12x + 13}{(2x-1)^2(x-2)} = x + 2 + \frac{1}{x-2} + \frac{4}{2x-1} + \frac{-4}{(2x-1)^2}.$

$\frac{P(x)}{Q(x)} == x + 2 + \frac{12x^2 - 28x + 17}{(2x-1)^2(x-2)} = S(x) + \frac{R(x)}{Q(x)}$. Where $\deg R < \deg Q$.So

$\frac{R(x)}{Q(x)} = \frac{1}{x-2} + \frac{4}{2x-1} + \frac{-4}{(2x-1)^2}$.

2) $\frac{P(x)}{Q(x)} = \frac{5x^7-x^6+6x^5+11x^4+29x^3+66x^2+29x+27}{(x-1)^3(x+2)^2(x^2+x+1)^2}$ where $\deg P < \deg Q$. So

$\frac{P(x)}{Q(x)} = \frac{1}{x-1} + \frac{-1}{(x-1)^2} + \frac{2}{(x-1)^3} + \frac{-2}{x+2} + \frac{3}{(x+2)^2} + \frac{x-1}{x^2+x+1} + \frac{2x+1}{(x^2+x+1)^2}$.

Integration of a rational fraction

To calculate the integral of a fraction $\frac{P(x)}{Q(x)}$, we first write this fraction as the sum of a polynomial and a finite number of rational fractions in the form $\frac{A}{(x-a)^k}$ or $\frac{Mx+N}{((x-\alpha)^2+\beta^2)^k}$ where k is a non zero natural number and $\beta, \alpha, N, M, A, a$ are real numbers, so the integral rational fractions returns to calculate integrals of the type $\int \frac{A}{(x-a)^k} dx$ and $\int \frac{Mx+N}{((x-\alpha)^2+\beta^2)^k} dx$.

Calculate the integral $\int \frac{A}{(x-a)^k} dx$

$$\int \frac{1}{x-a} dx = ln|x-a| + C$$

$$\forall\, k > 1\text{:}\ \int \frac{1}{(x-a)^k} dx = \frac{-1}{(k-1)(x-a)^{k-1}} + C.$$

Calculate the integral $\int \frac{Mx+N}{((x-\alpha)^2+\beta^2)^k} dx$.

Calculating this integral after changing the variable $x = \alpha + \beta t$ leads to calculating integrals of two types: $I_k = \int \frac{t}{(1+t^2)^k} dt$ And $J_k = \int \frac{1}{(1+t^2)^k} dt$, where we have:

$$I_1 = \frac{1}{2} ln(1+t^2) + C \quad \text{and} \quad \forall\, k > 1\text{:}\quad I_k = \frac{-1}{2(k-1)(1+t^2)^{k-1}} + C.$$

As for the integration $J_k = \int \frac{1}{(1+t^2)^k} dt$, we use integration by parts and obtain the following recurrence relation:

$$J_1 = Arctanx + C \quad \text{and} \quad \forall\, k \geq 1\text{:}\ 2kJ_{k+1} = (2k-1)J_k + \frac{t}{(1+t^2)^k} \dots\dots (*)$$

Example 1.17

Calculate the integral $I = \int \frac{2x^4-x^3+2x^2-1}{(x^2+1)(x-1)} dx$.

By Euclidean division we get:

$$\frac{2x^4 - x^3 + 2x^2 - 1}{x^3 - x^2 + x - 1} = 2x + 1 + \frac{x^2 + x}{(x^2 + 1)(x - 1)}.$$

We put $\frac{x^2+x}{(x^2+1)(x-1)} = \frac{Mx+N}{x^2+1} + \frac{A}{x-1}$ we get $M = 0, N = 1, A = 1.$

So

$$I = \int \left(2x + 1 + \frac{1}{x^2 + 1} + \frac{1}{x - 1}\right) dx$$

$$= x^2 + x + Arctanx + ln|x - 1| + C.$$

Example 1.18

Calculate the integral $J = \int \frac{x^2-6x+11}{(x+1)(x-2)^2} dx.$

We put $\frac{x^2-6x+11}{(x+1)(x-2)^2} = \frac{a}{x+1} + \frac{b}{x-2} + \frac{c}{(x-2)^2}$ we get $a = 2, b = -1, c = 1.$

So

$$J = \int \left(\frac{2}{x + 1} - \frac{1}{x - 2} + \frac{1}{(x - 2)^2}\right) dx$$

$$= 2\ln(x + 1) - \ln(x - 2) - \frac{1}{x - 2} + C.$$

Example 1.19*

Calculate the integral $J = \int \frac{8x^6-8x^5+2x^4+23x^3-15x^2+7x+2}{(x+1)^2\left(2x^2-2x+1\right)^2} dx$

By Euclidean division we get:

$$\frac{8x^6 - 8x^5 + 2x^4 + 23x^3 - 15x^2 + 7x + 2}{(x + 1)^2(2x^2 - 2x + 1)^2} = 2 + \frac{-8x^5 + 10x^4 + 15x^3 - 17x^2 + 11x}{(x + 1)^2(2x^2 - 2x + 1)^2}.$$

We put

$$\frac{-8x^5 + 10x^4 + 15x^3 - 17x^2 + 11x}{(x + 1)^2(2x^2 - 2x + 1)^2} = \frac{a}{x + 1} + \frac{b}{(x + 1)^2} + \frac{cx + d}{2x^2 - 2x + 1}$$

$$+ \frac{cx + d}{2x^2 - 2x + 1}.$$

We get:

$$a = -2, b = -1, c = 0, d = 3, e = 1, f = 0.$$

So

$$I=\int\left(2+\frac{-2}{x+1}+\frac{-1}{(x+1)^2}+\frac{3}{2x^2-2x+1}+\frac{x}{(2x^2-2x+1)^2}\right)dx$$

$$I=2x-2ln|x+1|+\frac{1}{x+1}+\int\left(\frac{3}{2x^2-2x+1}+\frac{x}{(2x^2-2x+1)^2}\right)dx.$$

Since $2x^2-2x+1=2\left(\left(x-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2\right)$, to calculate the integral on the second side we put $x=\frac{1}{2}+\frac{1}{2}t$.

So

$$\int\left(\frac{3}{2x^2-2x+1}+\frac{x}{(2x^2-2x+1)^2}\right)dx=3\int\frac{1}{t^2+1}dx+\int\frac{t+1}{(t^2+1)^2}dx$$
$$=3Arc\ tan\,x+\int\frac{t}{(t^2+1)^2}dx+\int\frac{1}{(t^2+1)^2}dx$$

$$\int\left(\frac{3}{2x^2-2x+1}+\frac{x}{(2x^2-2x+1)^2}\right)dx=3\mathrm{Arc}\tan t-\frac{1}{2}\frac{1}{t^2+1}+\underbrace{\int\frac{1}{(t^2+1)^2}dx}_{J_2}$$

Substituting $k=1$ in the regressive relationship $(*)$ we get: $2J_2=J_1+\frac{t}{(1+t^2)^1}=\mathrm{Arc}\tan t+\frac{t}{1+t^2}$ and from there $J_2=\frac{1}{2}\mathrm{Arc}\tan t+\frac{t}{2(t^2+1)}$.

So

$$\int\left(\frac{3}{2x^2-2x+1}+\frac{x}{(2x^2-2x+1)^2}\right)dx=\frac{7}{2}\mathrm{Arc}\tan t+\frac{t-1}{2(t^2+1)}.$$

Substituting $t=2x-1$ we get

$$\int\left(\frac{3}{2x^2-2x+1}+\frac{x}{(2x^2-2x+1)^2}\right)dx=\frac{7}{2}\mathrm{Arc}\tan(2x-1)+\frac{x-1}{2(2x^2-2x+1)}.$$

So

$$I=2x-2ln|x+1|+\frac{1}{x+1}+\frac{7}{2}\mathrm{Arc}\tan(2x-1)+\frac{x-1}{2(2x^2-2x+1)}+C.$$

1.5.2 Integration of the type $\int \boldsymbol{R}(\sin x,\cos x)\,\boldsymbol{dx}$:

Where $\boldsymbol{R}(\boldsymbol{x},\boldsymbol{y})$ is a rational fraction in the variables x and y.
This integral can be converted to a rational fractional integral using the change in the variable $t=\tan\frac{x}{2}$, where:

$$\cos x=\frac{1-t^2}{1+t^2}\ \ ;\ \ \sin x=\frac{2t}{1+t^2}\ \ ;\ \ dx=\frac{2}{1+t^2}dt.$$

Example 1.20

Calculate the integral $J = \int \frac{cos^2 x}{5 - 4sin\, x} dx$

by putting $t = tan\frac{x}{2}$ we get

$$J = \int \frac{\left(\frac{1-t^2}{1+t^2}\right)^2}{5 - 4\left(\frac{2t}{1+t^2}\right)} \left(\frac{2}{1+t^2}\right) dt = \int \frac{2(t^2-1)^2}{(5t^2-8t+5)(t^2+1)^2} dt.$$

We put

$$\frac{2(t^2-1)^2}{(5t^2-8t+5)(t^2+1)^2} = \frac{at+b}{t^2+1} + \frac{ct+d}{(t^2+1)^2} + \frac{et+f}{5t^2-8t+5}.$$

on obtain $a = 0\,,\ b = \frac{5}{8}\,,\ c = 1\,,\ d = 0\,,\ e = 0\,,\ f = -\frac{9}{8}.$

So

$$J = \int \frac{\frac{5}{8}}{t^2+1} + \frac{t}{(t^2+1)^2} + \frac{-\frac{9}{8}}{5t^2-8t+5} dt = \frac{5}{8}\operatorname{Arc}\tan t - \frac{1}{2(t^2+1)} + \underbrace{\int \frac{-\frac{9}{8}}{5t^2-8t+5} dt}_{I}.$$

Calculate the integral I.

Since $5t^2 - 8t + 5 = 5\left((\mathrm{t} - (\frac{4}{5}))^2 + (\frac{3}{5})^2\right)$, we put $t = \frac{4}{5} + \frac{3}{5}y$ so

$$I = -\frac{3}{8}\int \frac{1}{y^2+1} dy = -\frac{3}{8}\operatorname{Arc}\tan y = -\frac{3}{8}\operatorname{Arc}\tan\left(\frac{5}{3}t - \frac{4}{3}\right).$$

And

$$J = \frac{5}{8}\operatorname{Arc}\tan t - \frac{1}{2(t^2+1)} - \frac{3}{8}\operatorname{Arc}\tan\left(\frac{5}{3}t - \frac{4}{3}\right).$$

Substituting $t = tan\frac{x}{2}$ we get

$$J = \frac{5}{16}x - \frac{1}{2}\cos^2\frac{x}{2} - \frac{3}{8}\operatorname{Arc}\tan\left(\frac{5}{3}\tan\frac{x}{2} - \frac{4}{3}\right) + C.$$

1.5.3 Integration of the type $\int \boldsymbol{R}\left(\boldsymbol{x}, \left(\frac{\boldsymbol{ax+b}}{\boldsymbol{cx+d}}\right)^{\frac{\boldsymbol{m}}{\boldsymbol{n}}}, \left(\frac{\boldsymbol{ax+b}}{\boldsymbol{cx+d}}\right)^{\frac{\boldsymbol{p}}{\boldsymbol{q}}}, \dots\dots, \left(\frac{\boldsymbol{ax+b}}{\boldsymbol{cx+d}}\right)^{\frac{\boldsymbol{r}}{\boldsymbol{s}}}\right) \boldsymbol{dx}$

Where $\boldsymbol{R}(\boldsymbol{x}, y, \dots\dots, z)$ is a rational fraction in the variables $x\,, y, \dots\dots, z$ and $\frac{m}{n}, \frac{p}{q}, \dots\dots, \frac{r}{s}$ are rational numbers.To calculate this type of integration, we use a change in the variable $t = \left(\frac{ax+b}{cx+d}\right)^{\frac{1}{k}}$, where k is the Least Common Multiple (LCM) of the numbers $n, q, \dots\dots\dots, s$.

Example 1.21

Calculate the integral $I = \int \frac{1+\sqrt{x+1}}{\sqrt[3]{x+1}} dx$.

We put $t = (x+1)^{\frac{1}{6}}$ and from it $x = t^6 - 1$ and $dx = 6t^5 dt$ so

$$I = \int \frac{1+t^3}{t^2} 6t^5 dt = 6\int t^6 + t^3 dt = \frac{6}{7}t^7 + \frac{3}{2}t^4 + C.$$

So

$$I = \frac{6}{7}(x+1)^{\frac{7}{6}} + \frac{3}{2}(x+1)^{\frac{4}{6}} + C.$$

Example 1.22

Calculate the integral $J = \int x\sqrt{\frac{x-1}{x+1}} dx$.

We put $t = \sqrt{\frac{x-1}{x+1}}$ and from it $x = \frac{-t^2-1}{t^2-1}$ and $dx = \frac{4t}{(t^2-1)^2} dt$.

So

$$J = \frac{-t^2-1}{t^2-1} t \frac{4t}{(t^2-1)^2} dt = \int \frac{-4(t^4+t^2)}{(t^2-1)^3} dx.$$

We put

$$\frac{-4(t^4+t^2)}{(t^2-1)^3} = \frac{a}{t-1} + \frac{b}{(t-1)^2} + \frac{c}{(t-1)^3} + \frac{d}{t+1} + \frac{e}{(t+1)^2} + \frac{f}{(t+1)^3}$$

we get

$$a = -\frac{1}{2}\ ,b = -\frac{3}{2}\ ,c = -1\ ,\ d = \frac{1}{2}\ ,\quad e = -\frac{3}{2}\ ,\quad f = 1.$$

So

$$J = \int \frac{-\frac{1}{2}}{t-1} + \frac{-\frac{3}{2}}{(t-1)^2} + \frac{-1}{(t-1)^3} + \frac{\frac{1}{2}}{t+1} + \frac{-\frac{3}{2}}{(t+1)^2} + \frac{1}{(t+1)^3} dt$$

$$= -\frac{1}{2}\ln|t-1| + \frac{3}{2(t-1)} + \frac{1}{2(t-1)^2} + \frac{1}{2}\ln|t+1| + \frac{3}{2(t+1)} - \frac{1}{2(t+1)^2} + C.$$

Substituting $t = \sqrt{\frac{x-1}{x+1}}$ we get

$$J = \frac{1}{2}\ln\left|\frac{\sqrt{\frac{x-1}{x+1}}+1}{\sqrt{\frac{x-1}{x+1}}-1}\right| + \left(\frac{1}{2}x^2 - \frac{1}{2}x\ -1\right)\sqrt{\frac{x-1}{x+1}} + C.$$

1.5.4 Integration of the type $\int \boldsymbol{R}(\boldsymbol{x}, \sqrt{\boldsymbol{ax^2+bx+c}})\,\boldsymbol{dx}$.

After writing the trinomial $\boldsymbol{ax^2+bx+c}$ in canonical form, this integral takes one of the following forms:

$$\int \boldsymbol{R}(\boldsymbol{x}, \sqrt{(x-\alpha)^2+\beta^2})\,dx, \int \boldsymbol{R}(\boldsymbol{x}, \sqrt{(x-\alpha)^2-\beta^2})\,dx \text{ and } \int \boldsymbol{R}(\boldsymbol{x}, \sqrt{\beta^2-(x-\alpha)^2})\,dx.$$

To calculate the integral $\int \boldsymbol{R}(\boldsymbol{x}, \sqrt{(x-\alpha)^2+\beta^2})\,dx$, we use a change in the variable $x - \alpha = \beta\sinh t$.

To calculate the integral $\int \boldsymbol{R}(\boldsymbol{x}, \sqrt{(x-\alpha)^2-\beta^2})\,dx$, we use a change in the variable $x - \alpha = \pm\beta\cosh t$.(According to the interval of integration).

To calculate the integral $\int \boldsymbol{R}(\boldsymbol{x}, \sqrt{\beta^2-(x-\alpha)^2})\,dx$, we use a change in the variable $x - \alpha = \beta\cos t$.(or $x-\alpha = \beta\sin t$).

Example 1.23

Calculate the integral $L = \int \sqrt{x^2+4x+3}\,dx$.

We have $x^2+4x+3 = (x+2)^2-1$ and from there

If $x+2 \leq -1$ (i.e. if $x \in]-\infty,-3]$) we put $x+2 = -\cosh t$ where $t \in [0,+\infty[$.

If $x+2 \geq 1$ (i.e. if $x \in [-1,+\infty[$) we put $x+2 = \cosh t$ where $t \in [0,+\infty[$.

For $x \in]-\infty,-3] \cup [-1,+\infty[$ then $x+2 = \mp\cosh t$ and $dx = -\sinh t\,dt$.

So

$$L = \int \sqrt{\cosh^2 t - 1}\,(\mp\sinh t)dt = \int \sqrt{\sinh^2 t}\,(\mp\sinh t)dt = \int \mp\sinh^2 t\,dt$$

$$= \frac{1}{2}\int \mp(-\cosh 2t+1)\,dt = \mp\left(-\frac{1}{4}\sinh 2t + \frac{1}{2}t\right) = \mp\left(-\frac{1}{2}\cosh t\sinh t + \frac{1}{2}t\right)$$

$$= \mp\frac{1}{2}\left[\mp(x+2)\sqrt{(x+2)^2-1}\right] \mp \frac{1}{2}\operatorname{Arg}\cosh[\mp(x+2)]$$

$$= \frac{1}{2}(x+2)\sqrt{x^2+4x+3} \pm \frac{1}{2}\ln\left|\mp(x+2)+\sqrt{x^2+4x+3}\right| + C.$$

$$= \frac{1}{2}(x+2)\sqrt{x^2+4x+3} - \frac{1}{2}\ln\left|x+2+\sqrt{x^2+4x+3}\right| + C.$$

(Note that $-\frac{1}{2}\ln\left|(x+2)+\sqrt{x^2+4x+3}\right| = \frac{1}{2}\ln\left|-(x+2)+\sqrt{x^2+4x+3}\right|$).

Example 1.24

Calculate the integral $L = \int x\sqrt{x^2+x+1}\,dx$.