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First year Licence **Introduction to probability and descriptive statistics**

Answers of the third series : Combinatorial analysis

Answer 01 : 1)Show that :

$$C_n^1 + C_n^3 + ... = C_n^0 + C_n^2 + ... \quad for\ any\ \ n$$

We have

$$(x+y)^n = \sum_{i=0}^{n} C_n^k \ x^{n-k} \ y^k$$

We pose $x = 1$ and $y = -1$, and the previous equation reduces to :

$$0 = \sum_{i=0}^{n} C_n^k \ (-1)^k$$

Wich can be written

$$C_n^0 + C_n^2 + ... = C_n^1 + C_n^3 + ...$$

2) Prove that

$$C_n^1 + 2C_n^2 + ... + nC_n^n = n\, 2^{n-1}$$

This time we begin with the expansion of $(1+x)^n$:

$$(1+x)^n = \sum_{i=0}^{n} C_n^k \ 1^{n-k} \ x^k$$

Differentiating both sides of the previous equation with respect to x gives

$$n\ (1+x)^{n-1} = \sum_{i=1}^{n} C_n^k \ k \ x^{k-1}$$

Now, let $x = 1$, we find :

$$n2^{n-1} = \sum_{i=1}^{n} C_n^k \ k \ x^{k-1} = C_n^1 + 2C_n^2 + ... + nC_n^n$$

Answer 03 : Roll 3 different dice simultaneously and by using the results obtained we construct a 3-digit number.

1. $6 \times 6 \times 6 = 6^3$
2. The number of numbers are less than 500 and greater than 200 is : 3×6^2.
3. 3×6^2.

4. $6 \times 5 \times 4 = P_6^3$

Answer 04 : Passwords have 3 different letters followed by 2 different symbols of the following set $\{@, \%, \$, \star\}$ then 2 numbers.

a) $26 \times 25 \times 24 \times 4 \times 3 \times 10 \times 10 = P_{26}^3 \times P_4^2 \times 10^2$.

b) $6 \times 25 \times 24 \times 4 \times 3 \times 10 \times 5 = 6 \times P_{25}^2 \times P_4^2 \times 10 \times 5$.

Answer 05 : 1)a) For the word "Maths"? the number of arrangement is : 5!

b) For the word $''proposition''$ we have $\{p, p, r, o, o, o, s, i, i, t, n\}$ so the number of arrangement is : $\dfrac{11!}{2! \times 3! \times 2!}$.

c) For the word "theorem " we have : $\dfrac{7!}{2!}$.

d) For the word "arrangement" we have : $\dfrac{11!}{2! \times 2! \times 2! \times 2!}$.

Answer 06 : Twenty books are to be arranged on a shelf ; eleven on travel, five on cooking, and four on gardening.

1. 20!.
2. We have three groups so 3!. For the books on travel we have 11!, for the books on cooking we have 5! and for the books on gardening we have 4!. So the number of arrangements is : $3! \times 11! \times 5! \times 4!$.
3. $11! \times (5 + 4 + 1)!$ or $11! \times (20 - 11 + 1)!$.

Answer 07 :

1) There are C_{70}^{12} choices for the first group. Having chosen 12 for the first group, there are C_{58}^{12} choices for the second group and so on. The total is :

$$C_{70}^{12} \times C_{58}^{12} \times C_{46}^{12} \times C_{34}^{12} \times C_{22}^{12} \times C_{10}^{10} = \frac{70!}{(12!)^5 \times 10!}$$

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answer of the fourth series : Probability space

Answer 01 : Express each of the following events by using the events A, B and C :

1. Exactly A occurs : $A \cap \overline{B} \cap \overline{C}$.
2. A and B occur : $A \cap B \cap \overline{C}$.
3. All three events occur : $A \cap B \cap C$.
4. Exactly one of the three events occurs : $\left(A \cap \overline{B} \cap \overline{C}\right) \cup \left(\overline{A} \cap B \cap \overline{C}\right) \cup \left(\overline{A} \cap \overline{B} \cap C\right)$.
5. Exactly two of the three events occurs : $\left(A \cap B \cap \overline{C}\right) \cup \left(A \cap \overline{B} \cap C\right) \cup \left(\overline{A} \cap B \cap C\right)$..
6. None of the three events occurs : $\left(\overline{A} \cap \overline{B} \cap \overline{C}\right)$.
7. At least one of the events occurs : $A \cup B \cup C$.
8. At most one of the events occurs : $4 \cup 6$.

Answer 03 :

1. let $(\Omega, \mathcal{F}, P)$ be a probability space.

   i) We want to prove that $\mathcal{G} = \{A \in \mathcal{F}, \quad P(A) = 0 \quad ou \quad P(A) = 1\}$ is a tribe on Ω.

   Cond.1. $\mathcal{G}$ is no vide because $\Omega \in \mathcal{F}$ and $P(\Omega) = 1$, so $\Omega \in \mathcal{G}$.

   Cond.2. Let $A \in \mathcal{G}$ be an event and we show that $\overline{A} \in \mathcal{G}$.

   We have

$$\begin{aligned}
A \in \mathcal{G} &\implies A \in \mathcal{F} \quad and \quad P(A) = 0 \quad or \quad P(A) = 1 \\
&\implies \overline{A} \in \mathcal{F} \quad and \quad P(\overline{A}) = 1 - P(A) = 1 - 0 = 1 \quad or \quad P(\overline{A}) = 0 \\
&\implies \overline{A} \in \mathcal{F} \quad and \quad P(\overline{A}) = 0 \quad or \quad P(\overline{A}) = 1 \\
&\implies \overline{A} \in \mathcal{G}.
\end{aligned}$$

   Cond.3. Let $(A_i)_{i \geq 0}$ be a sequence of events of $\mathcal{G}$. We want to prove that $\cup_i A_i \in \mathcal{G}$.

$$\begin{aligned}
For \;\; all \;\; i \geq 0, \;\; A_i \in \mathcal{G} &\implies \forall\, i \geq 0, \;\; A_i \in \mathcal{F} \quad and \quad P(A_i) = 0 \quad or \quad P(A_i) = 1 \\
&\implies \cup_i\, A_i \in \mathcal{F} \text{..........(1)} \; becaus \;\; \mathcal{F} \; is \;\; a \;\; tribe
\end{aligned}$$

   If $P(A_i) = 0$, for all $i \geq 0$, so $P(\cup_i\, A_i) = 0$..........(2)

   If **at least** $A_i, i \geq 0$, such that $P(A_i) = 1$, we find $P(\cup_i\, A_i) = 1$..........(3)

   So from (1), (2), and (3) we deduce that $\cup_i A_i \in \mathcal{G}$.

   ii) Let $A \in \mathcal{F}$ be an event. We want to show that

$$\mathcal{F}_A = \{A \cap B\ , \quad B \in \mathcal{F}\}$$

   is a tribe on Ω.

Cond.1. We have $A \in \mathcal{F}_A$ because $A \cap \Omega = A$ and $\Omega \in \mathcal{F}$. Or,
we have $\emptyset \in \mathcal{F}_A$ because $A \cap \emptyset$ and $\emptyset \in \mathcal{F}$.

Cond.2. Let $C \in \mathcal{F}_A$ be an avent and we want to prove that $\overline{C} \in \mathcal{F}_A$.
First, note that $\overline{C}$ is the complement of C with respect to A, so we can write

$$\overline{C} = A \setminus C = A \cap \overline{C}$$

So we have

$$\begin{aligned}
C \in \mathcal{F}_A &\Longrightarrow \exists\, B \in \mathcal{F}, C = A \cap B \\
&\Longrightarrow \exists\, B \in \mathcal{F},\quad \overline{C} = A \cap \left(\overline{A \cap B}\right) \\
&\Longrightarrow \exists\, B \in \mathcal{F},\quad \overline{C} = A \cap \left(\overline{A} \cup \overline{B}\right) \\
&\Longrightarrow \exists\, B \in \mathcal{F},\quad \overline{C} = \left(A \cap \overline{A}\right) \cup \left(A \cap \overline{B}\right) \\
&\Longrightarrow \exists\, B \in \mathcal{F},\quad \overline{C} = A \cap \overline{B} \quad et \quad \overline{B} \in \mathcal{F} \\
&\Longrightarrow \overline{C} = A \cap \overline{B} \in \mathcal{F}_A
\end{aligned}$$

Cond.3. Let $(A_i)_{i \geq 0}$ be a sequence of events of $\mathcal{F}_A$.
We want to prove that $\cup_i A_i \in \mathcal{F}_A$. We have :

$$\begin{aligned}
For\ \ all\ \ i \geq 0,\ \ A_i \in \mathcal{F}_A &\Longrightarrow \forall\, i \geq 0,\ \exists\, B_i \in \mathcal{F}, A_i = A \cap B_i \\
&\Longrightarrow \cup_i\, A_i = \cup_i \left(A \cap B_i\right) \\
&\Longrightarrow \cup_i\, A_i = A \cap \left(\cup_i\, B_i\right) \\
&\Longrightarrow \cup_i\, A_i \in \mathcal{F}_A,\ \ because\ \ \cup_i\, B_i \in \mathcal{F}
\end{aligned}$$

2. ★ Let $(\Omega, \mathcal{F}_i)$, $i \in I$, be a probable space. Prove that $\mathcal{F} = \cap_{i \in I} \mathcal{F}_i$ is a tribe on Ω.

Answer 04 : Let $(\Omega, \mathcal{F}, P)$ be a probability space.

1) Let B be an event. Show that P_B is a probability on Ω such as :

$$P_B(A) = \frac{P(A \cap B)}{P(B)} = P(A \mid B).$$

Cond.1. $0 \leq P_B(A) \leq 1$, because
$A \cap B \subseteq B \Rightarrow 0 \leq P(A \cap B) \leq P(B) \Rightarrow 0 \leq \dfrac{P(A \cap B)}{P(B)} \leq 1.$

Cond.2. $P_B(\Omega) = 1$, $because\ \ P_B(\Omega) = \dfrac{P(\Omega \cap B)}{P(B)} = \dfrac{P(B)}{P(B)} = 1$

Cond.3. Let $(A_i)_{i \geq 0}$ be a sequence of events of $\mathcal{F}$, such as these events are disjoints two by two.
We want to prove that $P_B(\cup_i A_i) = \sum_i\ P_B(A_i)$. We have :

$$\begin{aligned} P_B(\cup_i A_i) &= \frac{P\Big((\cup_i A_i)\ \cap B\Big)}{P(B)} \\ &= \frac{P\Big(\cup_i\ (A_i\ \cap B)\Big)}{P(B)} \end{aligned}$$

Note that the events A_i, $i = 1, ..., n$, are disjoints two by two, then the events $A_i \cap B$ are also disjoints two by two. So :

$$\begin{aligned} P_B(\cup_i A_i) &= \sum_i \frac{P(A_i\ \cap B)}{P(B)} \\ &= \sum_i P_B(A_i) \end{aligned}$$

2) ★ Let a be a reel and $\Omega = \mathbb{R}$. Show that P_a is a probability on Ω such as :

$$P_a(A) = \delta_a(A) = \begin{cases} 1 & if \quad a \in A \\ 0 & if \quad non \end{cases}$$

Cond.1. $0 \leq P_a(A) \leq 1$.

Cond.2. $P_a(\Omega) = 1$, *because* $a \in \mathbb{R} = \Omega$.

Cond.3. Let $(A_i)_{i\geq 0}$ be a sequence of events of $\mathcal{F}$ two by two disjoints.

We want to prove that $P_a(\cup_i A_i) = \sum_i\ P_a(A_i)$. We have :

$$P_a(\cup_i\ A_i) = \begin{cases} 1 & si \quad a \in \cup_i\ A_i \\ 0 & si \quad non \end{cases}$$

Case 1 : $P_a(\cup_i A_i) = 1$, so :

$$\begin{aligned} a \in\ \cup_i\ A_i &\Leftrightarrow \exists! i, a \in A_i,\ \ and\ \ for\ \ all\ \ j \neq i, a \notin A_j \\ &\qquad because\ A_i \cap A_j = \emptyset,\ \forall\ i \neq j \\ &\Leftrightarrow \exists! i, P_a(A_i) = 1,\ \ et\ \ \forall\ j \neq i, P_a(A_j) = 0 \end{aligned}$$

so

$$\begin{aligned} \sum_i\ P_a(A_i) &= P_a(A_0) + ... + P_a(A_{i-1}) + P_a(A_i) + P_a(A_{i+1)} + ... \\ &= 0 + ... + 0 + 1 + 0 + ... = 1 \end{aligned}$$

Case 2 : $P_a(\cup_i A_i) = 0$, so :

$$\begin{aligned} a \notin\ \cup_i\ A_i &\Leftrightarrow \forall i \geq 0, a \notin A_i \\ &\Leftrightarrow \forall i \geq 0, P_a(A_i) = 0 \end{aligned}$$

so

$$\begin{aligned} \sum_i\ P_a(A_i) &= P_a(A_0) + P_a(A_1) + P_a(A_2) + ... \\ &= 0 + 0 + 0 + ... = 0 \end{aligned}$$

Answer 05 : Let $(\Omega, \mathcal{F}, P)$ be a probability space and $A_1, ..., A_n$ n events of $\mathcal{F}$. Prove that :
1) By using " la démonstration par récurrence " , we prove the following proposition $P(n)$:

$$P(n) : P\Big(\cup_{i=1}^{n} A_i \Big) \leq \sum_{i=1}^{n} P(A_i)$$

Step 1 : We want to prove the proposition for $n = 2$ (i.e, we show that $P(n = 2)$ is true).
So, we have :

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2) \leq P(A_1) + P(A_2) = \sum_{i=1}^{2} P(A_i).$$

Step 2 : We assume that $P(n)$ is true, and we prove that the $P(n+1)$ is true.

$$P\Big(\cup_{i=1}^{n+1} A_i \Big) = P\Big((\cup_{i=1}^{n} A_i) \cup A_{n+1} \Big)$$

We pose $B = \cup_{i=1}^{n} A_i$, so :

$$\begin{aligned} P\Big(B \cup A_{n+1} \Big) &= P(B) + P(A_{n+1}) - P(B \cap A_{n+1}) \\ &\leq P(B) + P(A_{n+1}) = P(\cup_{i=1}^{n} A_i) + P(A_{n+1}) \quad \textit{by using the } P(n) \\ &\leq \sum_{i=1}^{n} P(A_i) + P(A_{n+1}) \\ &= \sum_{i=1}^{n+1} P(A_i). \end{aligned}$$

2) We have :

$$P\Big(\cap_{i=1}^{n} A_i \Big) = P\Big(\overline{\cup_{i=1}^{n} \overline{A_i}} \Big) = 1 - P\Big(\cup_{i=1}^{n} \overline{A_i} \Big)$$

From 1), we find :

$$P\Big(\cup_{i=1}^{n} \overline{A_i} \Big) \leq \sum_{i=1}^{n} P\Big(\overline{A_i} \Big)$$

$$\Rightarrow 1 - P\Big(\cup_{i=1}^{n} \overline{A_i} \Big) \geq 1 - \sum_{i=1}^{n} P(\overline{A_i})$$

$$\Rightarrow P\Big(\cap_{i=1}^{n} A_i \Big) \geq 1 - \sum_{i=1}^{n} P(\overline{A_i})$$

Answer 06 : Let
M the event " observe a marked ball " so we have $P(M) = 3/10$,
B the event " observe a black ball " so we have $P(B) = 7/10$,
R the event " observe a red ball " so we have $P(R) = 1 - P(B) = 3/10$, and $P(R \cap M) = 2/10$.

1. $P(R \cup M) = P(R) + P(M) - P(R \cap M) = \Big(1 - P(N)\Big) + P(M) - P(R \cap M) = 4/10.$
2. $P(R \cap \overline{M}) = P(R \setminus M) = P(R) - P(R \cap M) =$ ....

3. **The 1$^{\text{st}}$ method :**

$$P(B\cap\overline{M})=P(\overline{R}\cap\overline{M})=P(\overline{R\cup M})=1-P(R\cup M)=........$$

The 2$^{\text{nd}}$ method :

$$P(B\cap\overline{M})=P(B\setminus M)=P(B)-P(B\cap M)$$

Find $P(B\cap M)$: note that $\{B,R\}$ is the partition of Ω so :

$P(M)=P(M\cap B)+P(M\cap R)\rightarrow P(B\cap M)=P(M)-P(M\cap R)=(3/10)-(2/10)=1/10.$

Answer 08 :

1. Let W_n the event " draw two white balls ", so :

$$P(W_n)=\frac{Card(W_n)}{card(\Omega)}=\frac{C_n^2}{C_{n+8}^2}=\frac{n\times(n-1)}{(n+8)\times(n+7)}$$

2. Let R the event " draw two red balls " and G the event " draw two green balls ", so :

$$P(n)=P(W_n)+P(R)+P(G)=\frac{n\times(n-1)}{(n+8)\times(n+7)}+\frac{card(R)+card(G)}{card(\Omega)}$$

$$=\frac{n\times(n-1)}{(n+8)\times(n+7)}+\frac{C_5^2+C_3^2}{C_{n+8}^2}=\frac{n^2-n+26}{(n+8)\times(n+7)}$$

$\lim_{n\to\infty}P(n)=1$. When $n\to\infty$, we can say that the event "both balls are the same color" is almost certain.

Answer 09 : A queue is formed randomly by n people. Two friends A and B are in this queue. So, the number of arrangements for n people is : $n!$.

1. The probability that the two friends are located one behind the other is :

$$\frac{(n-1)\times 2!\times(n-2)!}{n!}=\frac{(n-1)!\times 2!}{n!}=\frac{2}{n}$$

The two friends A and B form a set, and the number of arrangements the $n-2$ people is $(n-2)!$ possible. Since A can be placed in line with B behind him or A behind B, so there are $(n-1)\times 2!$ possible cases.

Or,

$$\frac{2!\times(n-2+1)!}{n!}=\frac{2}{n}$$

2. The probability that the two friends are r places apart (separated by $r-1$ people) is :

$$\frac{(n-r)\times 2!\times(n-2)!}{n!}=\frac{2\times(n-r)}{n\times(n-1)}$$

The possible cases of finding person A then $r-1$ people then person B or the opposite are : $(n-r)\times 2!$.

Answer 10 :

1. Note that the probability of each face appearing is proportional to the number written on it, so :

$$p_1 = 1 \times p, \quad p_2 = 2p, \quad p_3 = 3p, \quad p_4 = 4p, \quad p_5 = 5p, \quad p_6 = 6p$$

   And we have : $\sum_{i=1}^{6} p_i = 1 \Rightarrow p = \frac{1}{21}$. We deduce the following table :

Face i	1	2	3	4	5	6
$p_i = P(i)$	$\frac{1}{21}$	$\frac{2}{21}$	$\frac{3}{21}$	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{6}{21}$

2. Let A the event "obtain an even number", $A = \{2, 4, 6\}$, so

$$P(A) = P(\{2\} \cup \{4\} \cup \{6\}) = P(\{2\}) + P(\{4\}) + P(\{6\}) = p_2 + p_4 + p_6 = \frac{12}{21}$$

   .

Answer 11 : There are n students in a class. Let A_n the event " at least 2 students have the same birthday from n students ", so :

$$\begin{aligned} P(A_n) &= 1 - P(\overline{A_n}) \\ &= 1 - \frac{card(\overline{A_n})}{card(\Omega)} \\ &= 1 - \frac{P_{365}^{n}}{365^n} \end{aligned}$$

If $n = 2$: $P(A_{n=2}) = 1 - \frac{365 \times 364}{365^2} = 0.0027$

Question : Find n, such as $P(A_n) = 0.5$.

Exercise 12 : We have 20 problems. The student can solve 11 of them. The instructor selects 5 questions randomly.

1. Let A the event " the student can solve all five problems on the exam", so :

$$P(A) = \frac{card(A)}{Card(\Omega)} = \frac{C_{11}^5}{C_{20}^5} = ...$$

2. Let B the event " the student can solve exactly one problem", so :

$$P(B) = \frac{card(A)}{Card(\Omega)} = \frac{C_{11}^1 \times C_9^4}{C_{20}^5} = ...$$

3. Let C the event " the student can solve at least two problems", so :

$$P(C) = 1 - P(\overline{C}) = 1 - \frac{card(\overline{C})}{Card(\Omega)} = 1 - \frac{C_9^5 + C_{11}^1 \times C_9^4}{C_{20}^5} = ...$$

4. Let D the event " the student can solve at most one problem", so :

$$P(D) = P(\overline{C}) = ...$$