Chapter 03

1 Optimization with inequality constraints

1.1 Introduction

Optimization with inequality constraints is a fundamental problem encountered in various disciplines, ranging from engineering and economics to operations research and machine learning.

The goal of optimization with inequality constraints is to navigate the solution space efficiently to identify the optimal solution that optimizes the objective function while respecting the imposed constraints. This task can be challenging due to the complexity of the solution space and the need to balance competing objectives.

Example

Project Management: In project management, optimizing resource allocation is crucial to ensure projects are completed efficiently within time and budget constraints. Constraints could include limits on manpower, budgetary restrictions, and deadlines. By formulating resource allocation as an optimization problem with inequality constraints, project managers can allocate resources optimally to different tasks while ensuring that no resource exceeds its capacity and that project deadlines are met.

Transportation : Optimization with inequality constraints is used extensively in transportation and logistics to plan the most efficient routes for vehicles while considering factors such as vehicle capacity, time windows for deliveries, and road network constraints. By formulating vehicle routing problems as optimization problems with inequality constraints, transportation companies can minimize costs associated with fuel consumption and vehicle wear and tear while meeting customer demands and adhering to regulatory restrictions.

1.2 General Problem

Constrained optimization problems are formulated as

minimize $f(x)$

subject to $g_j(x) \leq 0$, $j = 1, ..., p$,

$h_i(x) = 0$,$i = 1, ..., m$,

where g_j, h_i: $\mathbb{R}^n \to \mathbb{R}$ are inequality and equality constraint functions, respectively.

Note that $g : \mathbb{R}^n \to \mathbb{R}^p$ and $h : \mathbb{R}^n \to \mathbb{R}^m$

Remark 1 *The constraints can be written as* $g(x) \leq 0$ *and* $h(x) = 0$*, respectively*

The feasible set is $\Omega := \{x \in \mathbb{R}^n : g(x) \leq 0, h(x) = 0\}$.

Definition 2 *(Active constraints)*

$g_i(x) \leq 0$ is active in $x^* \in \Omega$ if $g_i(x^*) = 0$,
$g_i(x) \leq 0$ is inactive in $x^* \in \Omega$ if $g_i(x^*) < 0$.

Definition 3 *(Regular point)*
We call x a regular point in Ω if $\nabla h_i(x)$; $\nabla g_j(x)$; are linearly independent

Now we consider the first order necessary condition (FONC) for the optimization
problem with both equality and inequality constraints:

Theorem 4 *Suppose $f, g, h \in C^1$, x^* is a regular point and local minimizer of f, then $\exists \lambda \in \mathbb{R}^m$; $\mu \in \mathbb{R}^p$ such that*

$$\begin{aligned} \nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla h_i(x^*) + \sum_{j=1}^{p} \mu_j \nabla g_j(x^*) &= 0 \\ h(x^*) &= 0 \\ g(x^*) &\leq 0 \\ \forall j &\in \{1, 2, ..., p\}, \quad \mu_j \geq 0 \\ \forall j &\in \{1, 2, ..., p\}, \quad \mu_j g_j(x^*) = 0 \end{aligned}$$

Proof. See Edwin K. P. Chong, Stanislaw H. Zak, An Introduction to Optimization, 2nd Edition, A Wiley-lnterscience Publication ■

Theorem 5 *Suffcient optimality conditions*
Suppose the functions f ; h_i ; $i =; ...; p$; $g_j; j = 1, ..., m$ are twice di erentiable.
Let x be a feasible point of NLP. If there are Lagrange multipliers λ and $\mu \geq 0$ such that:
(i) the KKT conditions are satisfied at (x, λ, μ); and
(ii) and the hessian of the Lagrangian is positive definite$\big($i.e $d^T H_L d > 0\big)$ for d from the subspace

$$S = \left\{ d \in \mathbb{R}^n \diagup d^T \nabla h_i(x) = 0;\ d^T \nabla g_j(x) = 0,\ h_i; i =; ...; p; g_j; j = 1, ..., m,\ \mu_j > 0 \right\}$$

In the convex case, we have the following result:

Theorem 6 *We suppose that f, h_i for $i = 1, ..., p$ and g_j for $j = 1, ..., q$ are class C^1 and that f and g_j, $j = 1, ..., q$ are convex and h_i*
, $i = 1, ..., p$ are affine. We also assume that x^* is a regular point point. So
(x^* is minimizer of f)$\Leftrightarrow$ (KKT conditions are satisfied).

Proof. $(\Leftarrow)$ By the condition of convex function, we have the Lagrangian L is convex

$$\forall x \in \mathbb{R}^n : L(x^*, \lambda^*, \mu^*) \leq L(x, \lambda^*, \mu^*)$$

If x in Ω, $h_i(x) = 0$ and $g_j(x) \leq 0$, we get

$$\sum_{i=1}^{m} \lambda_i^* h_i(x^*) + \sum_{j=1}^{p} \mu_j^* g_j(x^*) = \sum_{j=1}^{p} \mu_j^* g_j(x^*) \leq 0$$

In the other hand,

$$f(x^*) = L(x^*, \lambda^*, \mu^*) \leq L(x, \lambda^*, \mu^*) = f(x)$$

Then x^* is minimizer of f ■

Example 7 *Solve the following optimization problem:*

$$\begin{aligned} \min f(x) &= x_1^2 - x_2^2 \\ & st \\ x_1 + 2x_2 + 1 &= 0 \\ x_1 - x_2 &\leq 3 \end{aligned}$$

Solution 8 *Lagrange function*

$$L(x, \lambda, \mu) = x_1^2 - x_2^2 + \lambda (x_1 + 2x_2 + 1) + \mu (x_1 - x_2 - 3)$$

Optimality condition:

$$\begin{cases} \frac{\partial L}{\partial x_1} = 2x_1 + \lambda + \mu = 0 \\ \frac{\partial L}{\partial x_1} = -2x_2 + 2\lambda - \mu = 0 \\ x_1 + 2x_2 + 1 = 0 \\ x_1 - x_2 \leq 3 \\ \mu (x_1 - x_2 - 3) = 0 \\ \mu \geq 0 \end{cases}$$

1) If $\mu = 0$

$$\begin{cases} 2x_1 + \lambda = 0 \\ -2x_2 + 2\lambda = 0 \\ x_1 + 2x_2 + 1 = 0 \end{cases}$$

Then

$$\begin{cases} \frac{-\lambda}{2} = x_1 \\ \lambda = x_2 \\ \lambda = \frac{-2}{3} \end{cases}$$

Hence $(x_1, x_2, \lambda, \mu) = \left(\frac{1}{3}, \frac{-2}{3}, \frac{-2}{3}, 0\right)$
2) If $(x_1 - x_2 = 3)$

$$\begin{cases} 2x_1 + \lambda + \mu = 0 \\ -2x_2 + 2\lambda - \mu = 0 \\ x_1 + 2x_2 + 1 = 0 \\ x_1 - x_2 = 3 \\ \mu \geq 0 \end{cases}$$

so

$$\begin{cases} x_1 = \frac{-\lambda-\mu}{2} \\ x_2 = \frac{2\lambda-\mu}{2} \\ \left(\frac{-\lambda-\mu}{2}\right) + 2\left(\frac{2\lambda-\mu}{2}\right) + 1 = 0 \\ \left(\frac{-\lambda-\mu}{2}\right) - \left(\frac{2\lambda-\mu}{2}\right) = 3 \end{cases}$$

And

$$\begin{cases} \left(\frac{-\lambda-\mu}{2}\right) + 2\left(\frac{2\lambda-\mu}{2}\right) + 1 = 0 \\ \left(\frac{-\lambda-\mu}{2}\right) - \left(\frac{2\lambda-\mu}{2}\right) = 3 \end{cases}$$

$$\begin{cases} \mu = \frac{-4}{3} \\ \lambda = -2 \end{cases}$$

Contradiction with $\mu \geq 0$

To confirme that $(x_1, x_2, \lambda, \mu) = \left(\frac{1}{3}, \frac{-2}{3}, \frac{-2}{3}, 0\right)$ *is optmal solution, we prove the hessian of the Lagrangian is positive definite* $\left(i.e\ d^T H_L d > 0\right)$ *for* d *from the subspace*

$$S = \left\{d \in \mathbb{R}^n \diagup d^T \nabla h_i(x) = 0; d^T \nabla g_j(x) = 0, \mu > 0\right\}$$

$$\begin{aligned} S &= \left\{d \in \mathbb{R}^2 \diagup \begin{pmatrix} d_1 & d_2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0\right\} \\ &= \left\{d \in \mathbb{R}^2 \diagup d_1 = -2d_2\right\}. \end{aligned}$$

Then

$$\begin{aligned} d^T H_L d &= \begin{pmatrix} -2d_2 & d_2 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} -2d_2 \\ d_2 \end{pmatrix} \\ &= d_2 > 0. \end{aligned}$$

Therefore, $(x_1, x_2, \lambda, \mu) = \left(\frac{1}{3}, \frac{-2}{3}, \frac{-2}{3}, 0\right)$ *is an optimal solution.*