University of Oum El Bouaghi

Faculty of Economic Sciences, Commercial Sciences and Management Sciences

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Mathematics 2

Series 01 (Ordinary Differential Equations).

Introduction

The subject of differential equations can be described as the study of equations involving derivatives. It can also be described as the study of anything that changes. The reason for this goes back to differential calculus, where one learns that the derivative of a function describes the rate of change of the function. Thus any quantity that varies can be described by an equation involving its derivative, whether the quantity is a position, velocity, temperature, population or volume.

Definition 01 : An ordinary differential equation (ODE) is an equation involving a independant variable, a unknown function of one variable (dependent variable) and some its derivatives, while a partial differntial equation (PDE) can be defined as is an equation involving an unkown function of two or more variables and certian of its partial derivatives.

Examples :

$$y^2\frac{dy}{dt} = e^t \qquad \frac{d^2y}{dx^2} + 3x^2y^2\frac{dy}{dx} = 0.$$

Remark 1 : in the ODEs we may refer for simplisity $\frac{dy}{dx} = y'$or $\frac{dy}{dx} = y_x$, therefore the equations in the previous example can be rewritten in this way

$$y^2y' = e^t \quad , \quad y'' + 3x^2y^2y' = 0.$$

Definition 2 : The order highest derivative which appears in any differntial equation is called The order of differential equation.

Examples :

1) $y'' + y' + y = sinx,\ is\ a\ second\ order\ ODE.$
2) $y' + y^2 = 0,\ is\ a\ first\ order\ ODE.$
3) $y''' + xy' = ln(x),\ is\ a\ third\ order\ ODE.$

Definition 3 : The function $y = y(t)$, is called is a solution to a ODE on the open interval I, if it satisfies the equation and defined on I.

Example :

Let $f(x) = \frac{1}{(c-x)}, c \in IR$. We have $D = IR - \{c\}$.

This function is a solution of the following EDO : $y' = y^2$.

First order linear equations
Separation of variables

If one can re-arrange an ordinary differential equation into the following standard form:

$$\frac{dy}{dx} = f(x)g(y),$$

then the solution may be found by the technique of **SEPARATION OF VARIABLES**:

$$\int \frac{dy}{g(y)} = \int f(x)\, dx\, .$$

This result is obtained by dividing the standard form by $g(y)$, and then integrating both sides with respect to x.

Examples :

$\frac{dy}{dx} = y$ IS separable,

$\frac{dy}{dx} = x^2 - y^2$ IS NOT.

Example :

Find the solution of $\frac{dy}{dx} = e^{2x+y}$

Write equation as: $\frac{dy}{dx} = e^{2x}e^{y}\ (\equiv f(x)g(y))$

Separate variables and integrate: $\int \frac{dy}{e^y} = \int e^{2x} dx$

i.e. $-e^{-y} = \frac{1}{2}e^{2x} + C$

i.e. $e^{-y} = -\frac{1}{2}e^{2x} - C$

i.e. $-y = \ln\left(-\frac{1}{2}e^{2x} - C\right)$

i.e. $y = -\ln\left(-\frac{1}{2}e^{2x} - C\right).$

Explicit and implicit solution

- **Explicit solution is a solution where the dependent variable $y(x)$ can be separated.**
- **Implicit is when the dependent variable $y(x)$ cannot be separated.**

Example :

Let : $y' = \frac{y}{x(y+1)}$.

We have :

$$\frac{y+1}{y} dy = \frac{1}{x} dx \rightarrow \int \frac{y+1}{y} dy = \int \frac{1}{x} dx \rightarrow y + ln(y) = ln(x) + c,$$

Is an implicit solution.

initial conditions

Initial conditions are values of the solution and/or its derivative(s) at specific points. As we will see eventually, solutions to “nice enough” differential equations are unique and hence only one solution will meet the given initial conditions.

Example : $2y' - 4xy = 0,\ y(0) = 1.$

We have $y' = 2xy \rightarrow \ln|y| = x^2 + c \rightarrow y = ke^{x^2}.$

Now apply the IC, this gives :$1 = k$,

And so the solution is : $y = e^{x^2}.$

General Solution

The general solution to a differential equation is the most general form that the solution can take and doesn’t take any initial conditions into account.

Example : $y(x) = \frac{3}{4} + \frac{c}{x^2}$ *is* the general solution to $2xy' + 4y = 3.$

Second order linear equations

Homogeneous equations

In this Tutorial, we will practise solving equations of the form:

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0.$$

i.e. second order (the highest derivative is of second order),
linear (y and/or its derivatives are to degree one) with
constant coefficients (a, b and c are constants that may be zero).

A trial solution of the form $y = Ae^{mx}$ yields an “auxiliary equation”:

$$am^2 + bm + c = 0,$$

that will have two roots (m_1 and m_2).

The general solution y of the o.d.e. is then constructed from the possible forms (y_1 and y_2) of the trial solution. The auxiliary equation may have:

i) real different roots,
m_1 and $m_2 \longrightarrow y = y_1 + y_2 = Ae^{m_1x} + Be^{m_2x}$

or ii) real equal roots,
$m_1 = m_2 \longrightarrow y = y_1 + xy_2 = (A + Bx)e^{m_1x}$

Example01 :

$$\boxed{2y'' + 3y' - 2y = 0}$$

This equation is a homogeneous second order equation, i.e. $2m^2 + 3m - 2 = 0$: AUXILIARY EQUATION (A.E.)

i.e. $(2m-1)(m+2) = 0$

i.e. $m_1 = \frac{1}{2}$ and $m_2 = -2$: TWO DIFFERENT REAL ROOTS

i.e. $y_1 = Ae^{\frac{1}{2}x}$ and $y_2 = Be^{-2x}$: TWO INDEPENDENT SOLUTIONS

Example #2. Solve the differential equation: $y'' - 2y' + y = 0$

Solution:

Characteristic equation:

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$r = 1, r = 1 \quad \text{(Repeated roots)}$$

$$\Rightarrow y_1 = C_1 e^x \text{ and } y_2 = C_2 x e^x$$

So the general solution is: $\boxed{y = C_1 e^x + C_2 x e^x}$

NonHomogeneous Linear Equations

The solution of a second order nonhomogeneous linear differential equation of the form

$$ay'' + by' + cy = G(x)$$

where a, b, c are constants, $a \neq 0$ and $G(x)$ is a continuous function of x on a given interval is of the form

$$y(x) = y_p(x) + y_c(x)$$

where $y_p(x)$ is a particular solution of $ay'' + by' + cy = G(x)$ and $y_c(x)$ is the general solution of the complementary equation/ corresponding homogeneous equation $ay'' + by' + cy = 0$.

METHODS FOR FINDING THE PARTICULAR SOLUTION (yp)

If $G(x)$ is a polynomial it is reasonable to guess that there is a particular solution, $y_p(x)$ which is a polynomial in x of the same degree as $G(x)$ (because if y is such a polynomial, then $ay'' + by' + c$ is also a polynomial of the same degree.)

Method to find a particular solution: Substitute $y_p(x) =$ a polynomial of the same degree as G into the differential equation and determine the coefficients.

Example Solve the differential equation: $y'' + 3y' + 2y = x^2$.

$$y_c(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} = c_1 e^{-x} + c_2 e^{-2x}$$

We now need a particular solution $y_p(x)$.

- We consider a trial solution of the form $y_p(x) = Ax^2 + Bx + C$.
- Then $y_p'(x) = 2Ax + B, \quad y_p''(x) = 2A$.
- We plug y_p'', y_p' and y_p into the equation to get
  $2A + 3(2Ax + B) + 2(Ax^2 + Bx + C) = x^2$
  $$\longrightarrow \quad 2Ax^2 + (6A + 2B)x + (2A + 3B + 2C) = x^2.$$
- Equating coefficients, we get
  $2A = 1 \longrightarrow \boxed{A = \frac{1}{2}}, \quad 6A + 2B = 0, \quad 2A + 3B + 2C = 0.$

$y(x) = y_p(x) + y_c(x)$