Chapter 2: First order ordinary differential equations

Definition 2.1

We call a first-order differential equation, each equation of the form

$$F(x, y, y') = 0 \dots \dots \dots (*).$$

where $y: I \to \mathbb{R}$ is a function of the variable x defined in the interval I and y' is the first derivative of the function y and $F: (x, y, z) \to F(x, y, z)$ is a function of the variables x, y, z.

Definition 2.2

We call the solution of the differential equation $(*)$, in the interval I, each function ϕ is differentiable in the interval I and satisfies the following.

$$F\big(x, \phi(x), \phi'(x)\big) = 0.$$

Definition 2.3

The general solution of the differential equation in the interval I it represents the set of all solutions, i.e., the set of all functions which satisfy the equation in the interval I.

Example

$y' - \frac{2}{x}y - \frac{1}{2}x\sqrt{x} = 0$........(1)

(1) is a first-order differential equation.

The function $\phi:]0, +\infty[\to \mathbb{R}$ where $\phi(x) = x^2\sqrt{x}$ is a solution to equation (1) because ϕ is differentiable over the interval $]0, +\infty[$ and achieves $\phi'(x) - \frac{2}{x}\phi(x) - \frac{1}{2}x\sqrt{x} = 0$.

Indeed

$$\phi(x) = x^2\sqrt{x}\,; \;\; \phi'(x) = \frac{5}{2}x\sqrt{x}\,.$$

So

$$\phi'^{(x)} - \frac{2}{x}\phi(x) - \frac{1}{2}x\sqrt{x} = \frac{5}{2}x\sqrt{x} - \frac{2}{x}x^2\sqrt{x} - \frac{1}{2}x\sqrt{x} = \frac{5}{2}x\sqrt{x} - \frac{2}{x}x\sqrt{x} - \frac{1}{2}x\sqrt{x} = 0.$$

There are different types of first-order differential equations, but we will only mention four of them.

2.1 Separable Equations.

Definition 2.4

The first order **ODE** $y' = f(x,y)$ is said to be separable if $f(x,y)$ can be expressed as a product of a function of x times a function of y. That is, the equation is separable if the function f has the form: $f(x,y) = g(x)h(y)$, where g and h are continuous functions on some interval I.
The solution method is based on writing the equation in the form

$$\frac{1}{h(y)}dy = g(x)dx.$$

Of course, in dividing the equation by $h(y)$ we have to assume that $h(y) \neq 0$.

Example 1

The differential equation $y' = -\frac{x}{y}$ is separable since

$$f(x,y) = -\frac{x}{y} = (-x)\left(\frac{1}{y}\right).$$

Writing the equation in the form

$$ydy = xdx$$

and integrating

$$\int y\,dy = \int x\ dx$$

we get

$$x^2 + y^2 = C.$$

Example 2.

Show that the differential equation

$$y' = \frac{yx - y}{y+1}.$$

is separable. Then

1. Find the general solution and any singular solutions.
2. Find a solution which satisfies the initial condition $y(2) = 1$.

We have $f(x,y) = (x-1)\frac{y}{y+1}$.

For $y \neq 0$, writing the equation in the form:

$$\left(\frac{1}{y} + 1\right)dy = (x-1)dx$$

Integrating with respect to *x*, we get

$$\int \left(\frac{1}{y} + 1\right)dy = \int (x-1)\,dx$$

and

$$\ln|y| + y = \frac{1}{2}x^2 - x + C.$$

is the general solution. Again we have y defined implicitly as a function of x. Note that $y = 0$ is a solution of the differential equation (verify this), but this function is not included in the general solution

To find a solution that satisfies the initial condition, set $x = 2, y = 1$ in the general solution:

$$\ln|1| + 1 = \frac{1}{2}(2)^2 - 2 + C \text{ which implies } C = 1.$$

A particular solution that satisfies the initial condition is: $\ln|y| + y = \frac{1}{2}x^2 - x + 1$.

2.2 Homogeneous Equations

An ordinary differential equation $y' = f(x,y)$, is said to be a homogeneous differential equation if the following condition is satisfied $f(\lambda x, \lambda y) = f(x,y)$, for any $\lambda \in \mathbb{R}$.

Set $y = vx$; thus the general form of first order **ODE** becomes

$$y' = \frac{d(vx)}{dx} = v + xv' = f(x, vx)$$

On obtain

$$v' = \frac{f(x, vx) - v}{x} \ldots\ldots\ldots\ldots (*)$$

We can use variable separation to solve the equation $(*)$.

Example 1

Find the solution of the following equation

$$y' = \frac{y^2 + 2xy}{x^2}.$$

Set

$$f(x,y) = \frac{y^2 + 2xy}{x^2}$$

Clearly,

$$f(\lambda x, \lambda y) = \frac{(\lambda y)^2 + 2\lambda x \lambda y}{(\lambda x)^2} = f(x,y).$$

Therefore, this equation is homogenuos.

Now to find the solution, we set $y = vx$; and the equation can be written as follows

$$v' = \frac{f(x, vx) - v}{x} = \frac{\frac{v^2x^2 + 2xvx}{x^2} - v}{x} = \frac{v^2 + v}{x}.$$

By inspection, $v = -1$ and $v = 0$ are solutions (i.e. $y = -x$ and $y = 0$

For $v \neq -1$ and $v \neq 0$ we have

$$\frac{dx}{x} = \frac{dv}{v^2 + v}$$

if you integrate the two sides, we get

$$\ln|x| + c = \int \frac{1}{v^2 + v} dv = \int \frac{1}{v} - \frac{1}{v+1} dv = \ln|v| - \ln|v+1|$$

so

$$\ln|x| + c = \ln\left|\frac{v}{v+1}\right|$$

we get

$$\frac{v}{v+1} = kx$$

so

$$v = \frac{kx}{1 - kx}$$

Therefore,

$$y = \frac{kx^2}{1 - kx}$$

So the general solution of the original differential equation, is

$$y = \frac{kx^2}{1 - kx} \quad \text{or} \quad y = -x.$$

Example 2

Find the solution of the following equation

$$y\,' = \frac{1}{x^2} y^2 + \frac{1}{x} y + 1.$$

Set

$$f(x, y) = \frac{1}{x^2} y^2 + \frac{1}{x} y + 1.$$

Clearly,

$$f(x, y) = \frac{1}{(\lambda x)^2} (\lambda y)^2 + \frac{1}{(\lambda x)} (\lambda y) + 1 = f(x, y).$$

Now to find the solution, we set $y = vx$; and the equation can be written as follows

$$v\,' = \frac{f(x, vx) - v}{x} = \frac{\frac{1}{x^2}(xv)^2 + \frac{1}{x}(xv) + 1 - v}{x} = \frac{v^2 + 1}{x}.$$

So

$$\frac{dv}{v^2 + 1} = \frac{dx}{x}$$

if you integrate the two sides, we get

$$\int \frac{1}{v^2 + 1} dz = \int \frac{1}{x} dx.$$

So

$$\arctan v = lnx + C.$$

Or

$$v = tan(lnx + C).$$

So the general solution of the original differential equation, is

$$y = x\, tan(lnx + C).$$

First-Order Linear differential Equations

A first-order linear differential equation is one that can be written in the form

$$y' - a(x)y = b(x) \dots\dots\dots\dots (\ 1\ )$$

where a and b are continuous functions of x. Equation (1) is the linear equation's **standard form**.

Solving Linear Equations

We solve the equation

$$y' + a(x)y = b(x).$$

By multiplying both sides by a *positive* function $v(x)$ that transforms the left-hand side into the derivative of the product $v(x)y$. We will show how to find the function v later, but first we want to show how, once found, it provides the solution we seek.

Multiplying by $v(x)$ gives:

$$v(x)y' + a(x)v(x)y = v(x)b(x) \ (\ v(x) \text{ is chosen to make } vy' + avy = (vy)'\ ).$$

So

$$(v(x)y)' = \frac{d}{dx}(v(x)y) = v(x)b(x)$$

$$v(x)y = \int v(x)b(x)\,dx$$

so the solution would be

$$y = \frac{1}{v(x)}\int v(x)b(x)\,dx \dots\dots\dots\dots (\ 2\ )$$

Equation (2) expresses the solution of Equation (1) in terms of the function $v(x)$ and $b(x)$. We call $v(x)$ an **integrating factor** for Equation (1).

To find such an v, we have:

$$\frac{d}{dx}(vy) = v\frac{dy}{dx} + avy$$

$$v\frac{dy}{dx} + y\frac{dv}{dx} = v\frac{dy}{dx} + avy$$

$$\frac{dv}{dx} = av.$$

This is a separable differential equation for v, which we solve as follows:

$$\int \frac{dv}{v} = \int a\,dx$$

$$\ln v = \int a\,dx$$

$$v(x) = e^{\int a(x)dx}.$$

To solve the linear differential equation $y' + a(x)y = b(x)$, multiply both sides by the **integrating factor** $v(x) = e^{\int a(x)dx}$ and integrate both sides.

When you integrate the left-hand side product in this procedure, you always obtain the product $v(x)y$ of the integrating factor and solution function y this is after defining v.

Example 1

Solve the differential equation $\frac{dy}{dx} + 3x^2y = 6x^2$.

Solution

The given equation is linear since it has the form of Equation (1) with $a(x) = 3x^2$ and $b(x) = 6x^2$. An integrating factor is $v(x) = e^{\int a(x)dx} = e^{\int 3x^2 dx} = e^{x^3}$ Multiplying both sides of the differential equation by e^{x^3}, we get

$$e^{x^3}\frac{dy}{dx} + 3x^2 e^{x^3} y = 6x^2 e^{x^3}$$

or

$$\frac{d}{dx}\left(e^{x^3} y\right) = 6x^2 e^{x^3}$$

Integrating both sides, we have

$$e^{x^3} y = \int 6x^2 e^{x^3}\, dx = 2e^{x^3} + C$$

$$y = 2 + Ce^{-x^3}.$$

Eample 2

Find the solution of the initial-value problem $x^2 y' + xy = 1\,;\ x > 0;\ \ y(1) = 2$

Solution

We must first divide both sides by the coefficient of y' to put the differential equation into standard form: $y' + \frac{1}{x} y = \frac{1}{x^2};\ \ x > 0$ ……… (4)

The integrating factor is

$$v(x) = e^{\int a(x)dx} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x$$

Multiplication of Equation (4) by x gives

$$xy' + y = \frac{1}{x} \ \text{ or } \ (xy)' = \frac{1}{x}.$$

Then

$$xy = \int \frac{1}{x} dx = \ln x + C$$

and so

$$y = \frac{\ln x + C}{x}.$$

Since $y(1) = 2$ we have

$$2 = \frac{\ln 1 + C}{1} = C$$

Therefore the solution to the initial-value problem is

$$y = \frac{\ln x + 2}{x}.$$

Bernoulli Equations:

The differential equation

$$y' + a(x)y = b(x)y^n,\ \ n \neq 0,\ \ n \neq 1 \text{ ……… (5)}$$

where a and b are continuous functions on some interval I, is called a Bernoulli equation. To solve (5), multiply the equation by y^{-n} to obtain

$$y^{-n} y' + a(x) y^{1-n} = b(x) y^n.$$

The substitution $z = y^{1-n}$, $z' = (1-n)y^{-n}y'$ transforms (5) into the following linear equation in z and x:

$$\frac{1}{1-n} z' + a(x)z = b(x)$$

or

$$z' + (1-n)a(x)z = (1-n)b(x).$$

Example 1

Solve the Bernoulli equation $xy' + y = y^2 \ln x,\ \text{For } x > 0.$

In this example $a(x) = \frac{1}{x}$, $b(x) = \frac{1}{x}\ln x$, and $n = 2$. Therefore, we put $z = y^{1-n} = \frac{1}{y}$.

Then, $z' = -\frac{1}{y^2}y'$.

Inserting $z = \frac{1}{y}$ and $z' = -\frac{1}{y}y'$ into the differential equation, we get

$$z' - \frac{1}{x}z = -\frac{1}{x}\ln x$$

Thus, the resulting equation is a linear first order differential equation.

It can be solved using the integrating factor,

$$v(x) = e^{\int a(x)dx} = e^{-\int \frac{1}{x}dx} = \frac{1}{x}$$

Multiplying the differential equation by the integrating factor, we have

$$\frac{1}{x}z' - \frac{1}{x^2}z = -\frac{1}{x^2}\ln x$$

or

$$\left(\frac{z}{x}\right)' = -\frac{1}{x^2}\ln x$$

Integrating, we obtain

$$\begin{aligned}\frac{z}{x} &= -\int \frac{1}{x^2}\ln x\, dx \quad \text{(By integrating by parts)}\\ &= \frac{1}{x}\ln x + \int \frac{dx}{x^2}\\ &= \frac{1}{x}\ln x + \int \frac{dx}{x^2}\\ &= \frac{1}{x}\ln x - \frac{1}{x} + C.\end{aligned}$$

Multiplying by x, we have $z = \ln x - 1 + Cx$. Since $z = \frac{1}{y}$, the general solution to the problem is

$$y = \frac{1}{\ln x - 1 + Cx}.$$

Example 2

Fined every solution of the equation $xy' = 3y + x^5 y^{\frac{1}{3}}$.

Solution

Rewrite the differential equation as

$$y' = \frac{3}{x}y + x^4 y^{\frac{1}{3}}.$$

This is a Bernoulli equation for $n = \frac{1}{3}$. Divide the equation by $y^{\frac{1}{3}}$

$$y^{-\frac{1}{3}}y' = \frac{3}{x}y^{\frac{2}{3}} + x^4.$$

Define the new unknown function $z = y^{\frac{2}{3}}$, calculate its derivative, $z' = \frac{2}{3}y^{-\frac{1}{3}}y'$, and introduce them in the differential equation,

$$\frac{3}{2}z' = \frac{3}{x}z + x^4 \ \text{ or } \ z' - \frac{2}{x}z = \frac{2}{3}x^4$$

This is a linear equation for z. Integrate this equation using the integrating factor method. Then, the integrating factor is

$$v(x) = e^{\int a(x)dx} = e^{-\int \frac{2}{x}dx} = \frac{1}{x^2}$$

Therefore, the equation for z can be written as,

$$\frac{1}{x^2}z' - \frac{2}{x^3}z = \frac{2}{3}x^2$$

or

$$\frac{d}{dx}\left(\frac{1}{x^2}z\right) = \frac{2}{3}x^2.$$

Integrating, we obtain

$$\frac{1}{x^2}z = \frac{2}{3}\int x^2\,dx = \frac{2}{9}x^3 + C$$

and the solution of the differential equation is

$$z = \frac{2}{9}x^5 + Cx^2$$

Once z is known we compute the original unknown $y = \pm z^{\frac{3}{2}}$, where the double sign is related to taking the square root. We finally obtain

$$y = \pm\left(\frac{2}{9}x^5 + Cx^2\right)^{\frac{3}{2}}.$$