Chapter 1: Electrostatics الكهرباء الساكنة

1. General concepts مفاهيم عامة

1.1. Definition

Electrostatics is the branch of physics which studies electrical phenomena created by electrical charges in the rest state (static, immobile) and their interactions (forces).

1.2. Electrification phenomenon ظاهرة التكهرب

The phenomenon of electrification was discovered by the Greek mathematician Thales of Miletus (624-547 BCE) when he observed the attraction of straw twigs to yellow amber.

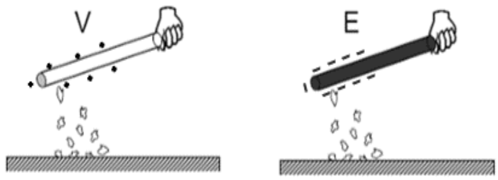
We have all noticed that when the hair is combed, the comb becomes able to attract the small pieces of paper, we therefore say that the comb is electrified.

1.2.1. Electrification modes: أنماط التكهرب

- Electrification by friction التكهرب بالاحتكاك
- Electrification by contact التكهرب بالتماس
- Electrification by influence التكهرب بالتأثير

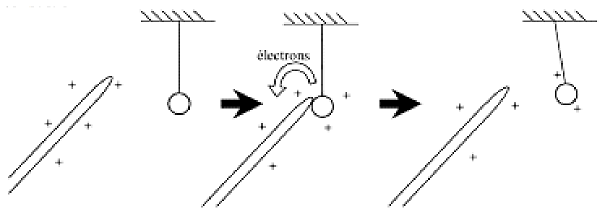
A) Electrification by friction:

We rub a rod of (glass (V), ebonite (E), plastic, amber, etc.) with (wool, silk, cloth, etc.). The rod becomes capable of attracting fine objects (paper, hair, etc.), which means that it is electrified and that it has a charge (+: glass) or (-: ebonite).



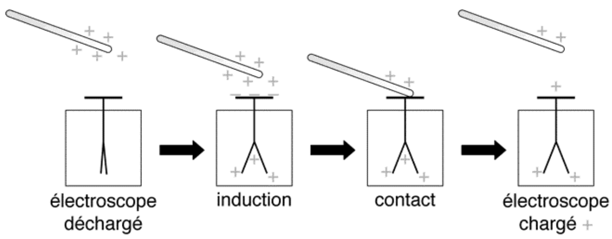
B) Electrification by contact:

The already electrified glass rod is brought into contact with a polystyrene ball of an electrically neutral electrostatic pendulum.



When we put the two objects in contact, there will be an attraction. Right after contact, the rod and the ball will repel each other because both are now positively electrified.

C) Electrification by influence:



We approach the already electrified glass rod to the metal disk of the electroscope, we notice that the aluminum sheets repel each other, that is to say they are charged with the same sign.

Conclusion

- We deduce that these materials have acquired a new property called "electrification"
- This property creates an attraction much more intense than the universal attraction produced by the masses.
- There are two types of electrification, positive and negative.

1.2.2. Explanation of the electrification phenomenon

The atoms of materials contain in their natural states an equivalent number of electrons and protons, these materials are therefore electrically neutral. But if this equilibrium is disrupted by an increase or decrease for any reason such as electrification, then matter becomes charged. The phenomenon of electrification is explained by a movement of electrons.

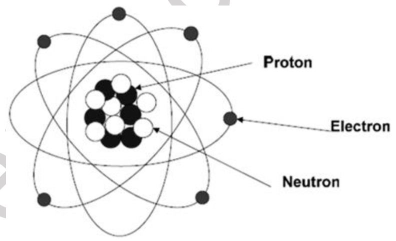
- ❖ The body that gains (receives) electrons becomes negatively charged (-).
- ❖ The body that loses (gives) electrons becomes positively charged (+).

2. The electric charge الشحنة كهربائية

2.1. The elementary electric charge الشحنة الكهربائية العنصرية

The electrical properties of matter find their principle at the level of the atom. Each atom is made up of a nucleus, around which a cloud of negatively charged electrons orbits, the nucleus in turn is made up of protons with positive charges (p^+) and neutrons with zero charges (n^0). The electrons (e^-) and the protons (p^+) carry the same electric charge in absolute value which we note by "e". This electric charge is called "the elementary electric charge $''e''$ or quanta of the electric charge; whose value is given by:

$$\boldsymbol{e = 1.602 \times 10^{-19} C}$$



2.2. The punctual electric charge الشحنة الكهربائية النقطية

A punctual electric charge denoted "q" can take any value, it presents an integer multiple of the elementary electric charge "e", we write:

$$\boldsymbol{q = \pm n\, e}$$

The electric charge of a body q is expressed in Coulomb "C".

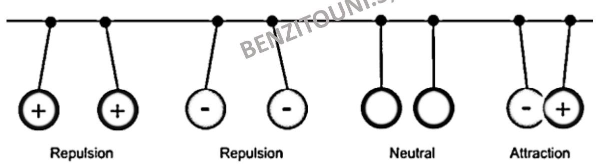
2.3. Principle of conservation of electric charge

Whatever the phenomena produced by the electric charges q_i, in the isolated system:

$$\sum_{i=1}^{n} qi = C^t$$

2.4. Attraction and repulsion between charges التجاذب و التنافر بين الشحنات

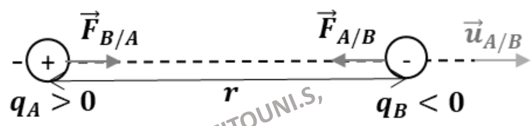
- ❖ Two bodies carrying a charge of the same sign repel each other يتنافران
- ❖ Two bodies carrying a charge of the opposite sign attract each other يتجاذبان



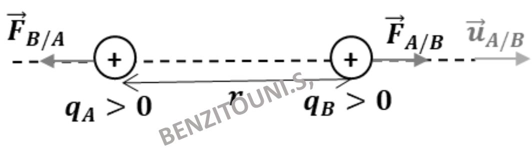
3. Coulomb's law قانون كولوم

Let be two particles carrying electric charges q_A and q_B, separated by a distance $''r''$ interact with each other. This interaction is modeled by a force called the electrostatic force $\vec{F_e}$. This force is attractive if the charges have the opposite sign and is repulsive if the charges have the same sign.

According to the principle of action/reaction (Newton's 3rd law):



- ➢ $\vec{F}_{A/B}$ = - $\vec{F}_{B/A}$ (The two forces have opposite directions).
- ➢ $\|\vec{F}_{A/B}\|$ = $\|\vec{F}_{B/A}\|$ (The two forces have the same intensity).
- ➢ $\vec{u}_{A/B}$ (The forces are radial, carried by the axis which joins the two charges).



According to several experiments, the intensity of the electrostatic force exerted between the two charges q_A and q_B is inversely proportional to the square of the distance separating the two particles and it is proportional to the charge of the two particles.

3.1. The electrostatic force $\vec{F_e}$ القوة الكهربائية/كهرو ستاتيكية

The electrostatic force $\vec{F_e}$ exerted between two charges is given by the mathematical expression, called Coulomb's law:

$$\overrightarrow{F_{A/B}} = -\overrightarrow{F_{B/A}} = k\,\frac{q_A\, q_B}{r^2}\,\overrightarrow{u_{AB}}$$

$$F_{A/B} = F_{B/A} = k\,\frac{|q_A|\,|q_B|}{r^2}$$

- $\overrightarrow{u_{AB}}$: the unit vector of $\overrightarrow{AB}$ which indicates the direction of the force.
- $\boldsymbol{k}$: Coulomb’s constant or the constant of proportionality. Defined by:

$$k = \frac{1}{4\,\pi\,\varepsilon_0}\ ; \quad \text{where :}\ \ k = 9 \times 10^9\ N.m^2.C^{-2}$$

- $\boldsymbol{\varepsilon_0}$, is the vacuum permittivity equal to $\varepsilon_0 = 8.85 \times 10^{-12}\ C^2/Nm^2$

Exercise-1

1. Calculate the electrostatic force exerted on the electron of the hydrogen atom ${}^1_1H$
2. Compare this force to the force of gravitational attraction.

We give: $e = 1.602\ \times 10^{-19}C;\quad me = 9\ \times 10^{-31}kg;\quad mp = 1850\ me;\quad r_{atom} = 5 \times 10^{-11}m$

$$G = 6.67\ \times 10^{-11}\ \ N.m^2.kg^{-2};\quad k = 9\ \times 10^9\ N.m^2.C^{-2}$$

Solution-1

1- **The electrostatic force F_e**

$$F_e = k\,\frac{|q_e|\,|q_p|}{r^2} = k\,\frac{e^2}{r^2} = 0.92\ \times 10^{-7}N$$

2- **Comparison**

$$F_G = G\,\frac{m_e m_p}{r^2} = G\,\frac{1850\,m_e^2}{r^2} = 4\ \times 10^{-47}N$$

By comparison: $\frac{F_e}{F_G} \sim 10^{40}\ times \Rightarrow\ F_e \gg F_G$

3.2. The components of the electrostatic force مركبات القوة الكهربائية

In the Cartesian system $(\vec{\imath}, \vec{\jmath}, \vec{k})$; the point M is identified by the position vector $\vec{r}$; where:

$$\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

Likewise, the distance r between two charges q_A and q_B is identified by:

$$\vec{r} = \overrightarrow{AB} = (x_B - x_A)\vec{\imath} + (y_B - y_A)\vec{\jmath} + (z_B - z_A)\vec{k}$$

and:

$$r = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

The electrostatic force is therefore written:

$$\overrightarrow{F_{A/B}} = k\,\frac{q_A\,q_B}{r^2}\overrightarrow{u_{AB}}\ ;\ \text{ where : } \overrightarrow{u_{AB}} = \frac{\overrightarrow{AB}}{AB} = \frac{\vec{r}}{r}$$

$$\Rightarrow \boldsymbol{\overrightarrow{F_{A/B}} = k\,\frac{q_A\,q_B}{r^3}\vec{r}}$$

Exercise-2

In the Cartesian system, two charges $q_A = 10^{-7}C$ and $q_B = -2\times 10^{-7}C$; are located respectively in: $A(2,-1,3)$ and $B(-1,2,0)$.

1- Determine the components of the electrostatic force $\overrightarrow{F_e}$.

Solution-2

We have: $\overrightarrow{F_e} = F_x\,\vec{\imath} + F_y\,\vec{\jmath} + F_z\,\vec{k}$

In addition: $\overrightarrow{F_{A/B}} = k\,\frac{q_A\,q_B}{r^3}\vec{r}$

with : $\begin{cases}\vec{r} = -3\,\vec{\imath} + 3\,\vec{\jmath} - 3\vec{k} \\ r = 3\sqrt{3}\end{cases}$; N.A : $\begin{cases}\overrightarrow{F_e} = \left(-\frac{2\sqrt{3}}{9}\,\vec{\imath} + \frac{2\ \sqrt{3}}{9} - \frac{2\sqrt{3}}{9}\vec{k}\right)\ \times 10^{-5} \\ F_e = \frac{2\sqrt{3}}{9}\times 10^{-5}\quad N\end{cases} \Rightarrow \begin{cases}F_x = \frac{-2\sqrt{3}\times 10^{-5}}{9} \\ F_y = \frac{2\sqrt{3}\times 10^{-5}}{9} \\ F_z = \frac{-2\sqrt{3}\times 10^{-5}}{9}\end{cases}$

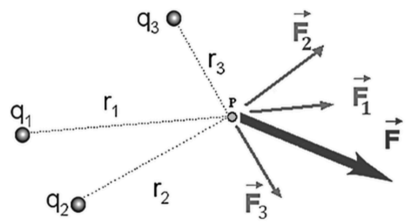
3.3. Principle of superposition مبدا التراكب

If we have "n" electric charges at rest, and in a vacuum, the principle of superposition allows us to make the vector sum of the electrostatic forces exerted at point M, have a charge q_M.

$$\overrightarrow{F_M} = \sum_{i=1}^{n}\overrightarrow{F_i}$$

$$\overrightarrow{F_M} = \overrightarrow{F_1} + \overrightarrow{F_2} + \cdots + \overrightarrow{F_n}$$

$$\overrightarrow{F_M} = k\frac{q_1 q_M}{r_1^2}\overrightarrow{u_1} + k\frac{q_2 q_M}{r_2^2}\overrightarrow{u_2} + \cdots \ldots \ldots \ldots \ldots k\frac{q_n q_M}{r_n^2}\overrightarrow{u_n}$$



4. Electrostatic field $\vec{E}$ مفهوم الحقل الكهربائي/كهرو ستاتيكي

In the presence of a charged particle the physical properties of the space surrounding this particle are changed, we then say that the electric charge "q" creates an electrostatic field $\vec{E}$.

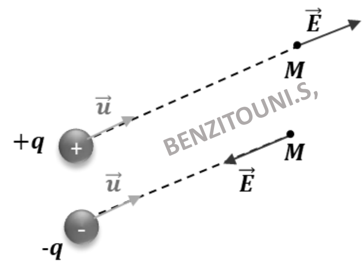
4.1. Electrostatic field $\vec{E}$ created by a punctual electric charge

Let be an electric charge "q" acts on another electric charge "Q" placed at a point "M" in space by an electrostatic force $\overrightarrow{F_e}$, given by:

$$\overrightarrow{F_e} = k\,\frac{q\,Q}{r^2}\vec{u}$$

We therefore write:

$$\overrightarrow{F_e} = Q\,\left[k\,\frac{q}{r^2}\vec{u}\right]$$



The term in parentheses presents the electrostatic field $\vec{E}$ created by the charge q at point M, and which takes the form:

$$\overrightarrow{E_M} = \frac{kq}{r^2}\vec{u}$$

The unit of electrostatic field is expressed in (V/m)

The relation between the force and the electrostatic field is therefore:

$$\overrightarrow{F_e} = Q\,\vec{E}$$

4.2. The electrostatic field lines خطوط الحقل الكهربائي

The field lines are oriented lines at which the field vector $\vec{E}$ is tangent at any point, where:

- **2-** The lines are continued.
- **3-** The lines start from the positive charge and end at the negative charge.
- **4-** The arrows indicate the direction of $\vec{E}$.

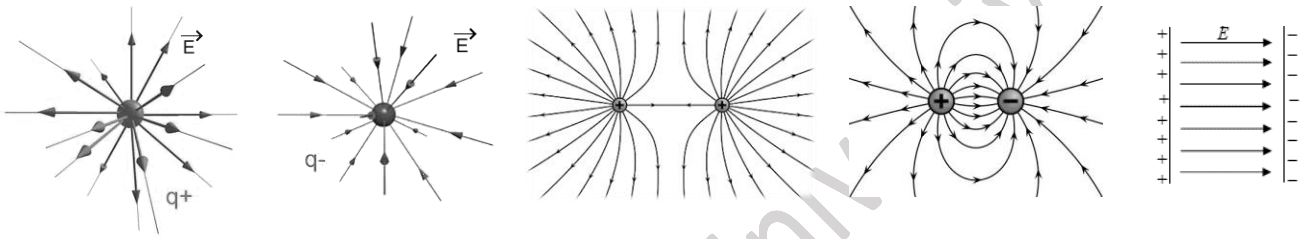


Figure. Electrostatic field lines

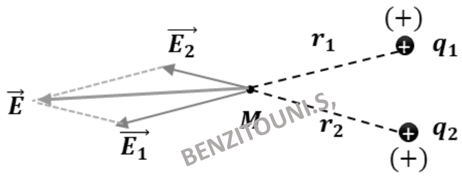
4.3. Electrostatic field $\vec{E}$ created by a set of charges

Likewise, if we have $''n''$ electric charges $''q_i''$, to determine the electrostatic field $\vec{E}$ produced by this set of charges at point $''M''$, we use the principle of superposition, which allows us to make the vector sum of the fields $\vec{E_i}$, we write:

$$\overrightarrow{E_M} = \sum_{i=1}^{n} \vec{E_i}$$

$$\overrightarrow{E_M} = \overrightarrow{E_1} + \overrightarrow{E_2} + \cdots + \overrightarrow{E_n}$$

$$\overrightarrow{E_M} = \frac{kq_1}{r_1^2}\overrightarrow{u_1} + \frac{kq_2}{r_2^2}\overrightarrow{u_2} + \cdots \ldots \frac{kq_n}{r_n^2}\overrightarrow{u_n}$$



4.4. Electrostatic field created by a continuous distribution of charges

Matter is formed by a large number of charged particles, these charges can be distributed uniformly along a straight line (line, wire), on a surface plane or in a volume.

Electrostatic equilibrium requires that the distribution of charges in matter is continuous, we therefore distinguish three types of charge distribution:

1- Linear distribution (l)
2- Surface distribution (S)
3- Volume distribution (V)

To simplify the study, we divide the distribution of charges into small parts charged with charge "dq", then we calculate "dE" produced by these charges "dq" and finally we make the integral according to the type of distribution.

The elementary electrostatic field $\overrightarrow{dE}$ created at point "M" is given by:

$$\overrightarrow{dE} = \frac{k\,dq}{r^2}\,\vec{u}$$

$$\overrightarrow{E_M} = \int \overrightarrow{dE}$$

$$\overrightarrow{E_M} = \int \frac{k\,dq}{r^2}\,\vec{u}$$

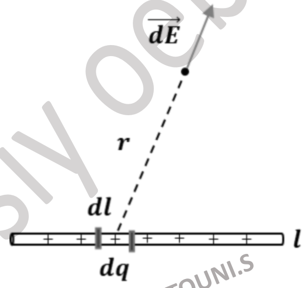
a) Linear distribution توزيع خطي

If the charges are continuously distributed on a wire of length "l", we introduce the notion of the linear charge density "λ", where: $dq = \lambda\,dl$.

The elementary electrostatic field $\overrightarrow{dE}$ created at point "M" is given by:

$$\overrightarrow{dE} = \frac{k\,dq}{r^2}\,\vec{u}$$

$$\overrightarrow{E_M} = \int \overrightarrow{dE} = \int \frac{k\,\lambda\,dl}{r^2}\,\vec{u}$$



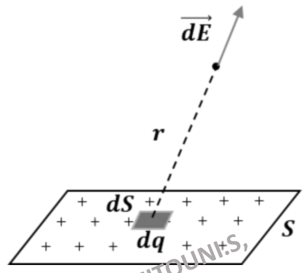
b) Surface distribution توزيع سطحي

If the charges are continuously distributed on a surface plane "S", we introduce the notion of the surface charge density "σ", where: $dq = \sigma\,dS$

The elementary electrostatic field $\overrightarrow{dE}$ created at point "M" is given by:

$$\overrightarrow{dE} = \frac{k\,\sigma\,dS}{r^2}\,\vec{u}$$

$$\overrightarrow{E_M} = \iint \frac{k\,\sigma\,dS}{r^2}\,\vec{u}$$



c) Volume distribution توزيع حجمي

If the charges are continuously distributed in a volume "V", we introduce the notion of the volume charge density "ρ", where: $dq = \rho\,dV$.

The elementary electrostatic field $\overrightarrow{dE}$ created at point "M" is given by:

$$\overrightarrow{dE} = \frac{k\,\rho\,dV}{r^2}\,\vec{u}$$

$$\overrightarrow{E_M} = \iiint \frac{k\rho\,dV}{r^2}\,\vec{u}$$

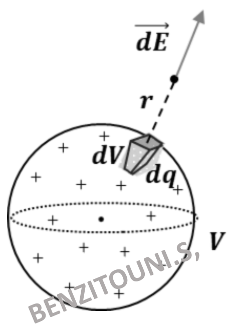


Table.1. Elements of length $\vec{dl}$, surface dS and volume dV in different coordinate systems

Cartesian coordinate system (x, y, z)	**Polar coordinate system (ρ, θ)**
$\vec{dl} = dx\,\vec{i} + dy\vec{j} + dz\,\vec{k}$ $dS = dxdy$ $dV = dx\,dy\,dz$	$\vec{dl} = dr\,\overrightarrow{u_r} + r\,d\theta\,\overrightarrow{u_\theta}$, $dS = dr\,r\,d\theta$
Cylindrical coordinate system (ρ, θ, z)	**Spherical coordinate system (r, θ, φ)**
$\vec{dl} = d\rho\,\overrightarrow{u_\rho} + \rho d\theta\,\overrightarrow{u_\theta} + dz\,\vec{k}$ $dS_\rho = \rho d\theta dz$; (lateral Surface element) $dS_z = d\rho \rho d\theta$.(Surface element of Base) $dV = d\rho\,\rho\,d\theta\,dz$	$\vec{dl} = dr\,\overrightarrow{u_r} + r\,d\theta\,\overrightarrow{u_\theta} + r\,sin\,\theta\,d\varphi\,\overrightarrow{u_\varphi}$ $dS = r^2 sin\theta d\theta d\varphi$ $dV = r^2\,dr\,sin\,\theta\,d\theta\,d\varphi$

5. Electrostatic potential الكمون الكهربائي

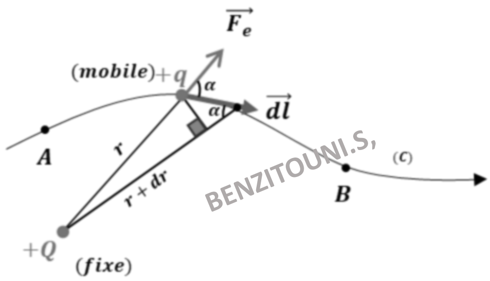
In the mechanics of material point, we know that if the force $\vec{F_c}$ is conservative, the work of this force does not depend on the path followed and we say that this force F_c derives from a potential energy E_p. The electrostatic force $\vec{F_e}$ is a conservative force, it therefore derives from a potential energy E_p, and we write:

$$\vec{F_e} = -\overrightarrow{grad}\,E_p$$

5.1. Electrostatic potential energy الطاقة الكامنة الكهربائية

Let be a point charge "q" moves in an electrostatic field "$\vec{E}$" created by another fixed charge "Q", it therefore subjected to an electrostatic force $\vec{F_e}$, where:

$$\vec{F_e} = q\,\vec{E}$$



The elementary work carried out by the force $\vec{F_e}$ to make an elementary displacement $\vec{dl}$, is written as follows:

$$dW = \vec{F_e}.\vec{dl}$$

$$\Rightarrow dW = F_e.dl.\cos\alpha$$

According to the diagram: $\begin{cases} \cos\alpha = \frac{dr}{dl} \\ Fe = \frac{k\,q\,Q}{r^2} \end{cases} \quad \Rightarrow \quad dW = \frac{k\,q\,Q}{r^2}\,dr$

$$\Rightarrow W(\vec{F_e}) = \int dW = \int \frac{k\,q\,Q}{r^2}\,dr$$

$$\Rightarrow W(\vec{F_e}) = -\frac{k\,q\,Q}{r} + c$$

$$if: r \to \infty, (W = 0) \Rightarrow (c = 0).$$

$$\Rightarrow \boldsymbol{W(\vec{F_e}) = -\frac{k\,q\,Q}{r}}$$

According to the principle of conservation of potential energy: $\Delta E_p = -\sum W(\vec{F_c})$

$$\Rightarrow \Delta E_p = -W(\vec{F_e}) = \frac{k\,q\,Q}{r}$$

Therefore, we write the expression of the potential energy of the electrostatic interaction between two charges q and Q as follows:

$$\boldsymbol{E_p = \frac{k\,q\,Q}{r}}$$

The unit of electrostatic potential energy is expressed in Joule (J).

5.2. Electrostatic potential of a punctual charge

The electrostatic potential "V" is a scalar field directly related to the potential energy E_p of a charge "q" located in an electrostatic field "$\vec{E}$" created by a fixed charge "Q".

We have:

$$E_p = q\left[\frac{kQ}{r}\right]$$

$$\Rightarrow E_p = q\,V$$

$$\boldsymbol{V = \frac{kQ}{r}}$$

The unit of electrostatic potential is expressed in volts (V).

We consider a charge $''q_A''$ located at point $A\ (x_A, y_A, z_A),$ the potential created by this charge at point $M\ (x, y, z)$ in the Cartesian system is given by:

$$V = \frac{kq_A}{\sqrt{(x - x_A)^2 + (y - y_A)^2 + (z - z_A)^2}}$$

5.3. Relation between field $\vec{E}$ and potential V

The relation between the electrostatic field and the electrostatic potential can be determined from the following relations:

$$\Rightarrow \begin{cases} W(\vec{F_e}) = -\Delta E_p \\ E_p = q\,V \end{cases} \quad \Rightarrow \begin{cases} dW = -dE_p = -(q\,dV) \\ dE_p = q\,dV \end{cases}$$

$$dW = -q\,dV$$

$$\Rightarrow\ \vec{F_e}.\vec{dl} = -q\,dV$$

$$\Rightarrow\ dV = -\frac{\vec{F_e}}{q}.\vec{dl}$$

$$\Rightarrow\ dV = -\vec{E}.\vec{dl}$$

In the cartesian coordinate system:

$$\begin{cases} \vec{E} = Ex\,\vec{i} + Ey\,\vec{j} + Ez\,\vec{k} \\ \vec{dl} = dx\,\vec{i} + dy\,\vec{j} + dz\,\vec{k} \\ dV = \frac{\partial V}{\partial x}dx\,\vec{i} + \frac{\partial V}{\partial y}dy\,\vec{j} + \frac{\partial V}{\partial z}dz\,\vec{k} \end{cases} \Rightarrow \frac{\partial V}{\partial x}dx\,\vec{i} + \frac{\partial V}{\partial y}dy\,\vec{j} + \frac{\partial V}{\partial z}dz\,\vec{k} = -\left(E_x\,dx + E_y\,dy + E_z\,dz\right)$$

$$\frac{\partial V}{\partial x} = -E_x; \qquad \frac{\partial V}{\partial y} = -E_y; \qquad \frac{\partial V}{\partial z} = -E_z$$

The relation between $\vec{E}$ and V can be established in the following form:

$$\vec{\boldsymbol{E}} = -\overrightarrow{\boldsymbol{grad}}\,\boldsymbol{V}$$

Table.2. $\vec{E} = -\overrightarrow{grad}\,V$ in different coordinate systems.

Cartesian coordinate system $(\boldsymbol{x},\boldsymbol{y},\boldsymbol{z})$	**Polar coordinate system** $(\boldsymbol{\rho},\boldsymbol{\theta})$	**Cylindrical coordinate system** $(\boldsymbol{\rho},\boldsymbol{\theta},\boldsymbol{z})$	**Spherical coordinate system** $(\boldsymbol{r},\boldsymbol{\theta},\boldsymbol{\varphi})$
$\begin{cases} E_x = -\dfrac{\partial V}{\partial x} \\ E_y = -\dfrac{\partial V}{\partial y} \\ E_z = -\dfrac{\partial V}{\partial z} \end{cases}$	$\begin{cases} E_r = -\dfrac{\partial V}{\partial r} \\ E_\theta = -\dfrac{1}{r}\dfrac{\partial V}{\partial \theta} \end{cases}$	$\begin{cases} E_\rho = -\dfrac{\partial V}{\partial \rho} \\ E_\theta = -\dfrac{1}{\rho}\dfrac{\partial V}{\partial y} \\ E_z = -\dfrac{\partial V}{\partial z} \end{cases}$	$\begin{cases} E_r = -\dfrac{\partial V}{\partial r} \\ E_\theta = -\dfrac{1}{r}\dfrac{\partial V}{\partial \theta} \\ E_\varphi = -\dfrac{1}{r\sin\theta}\dfrac{\partial V}{\partial \varphi} \end{cases}$

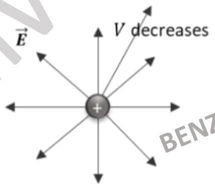
Remarks:

1- The potential V decreases along the field line.

2- The sign (-) indicates that the field $\vec{E}$ is directed towards low potentials.

3- The field lines are perpendicular to the equipotential lines:

$V = C^t \;\Rightarrow\; dV = 0 \;\Rightarrow\; \vec{E}.\vec{dl} = 0 \;\Longrightarrow\; \vec{E} \perp \vec{dl}$



5.4. Electrostatic potential created by a set of charges

According to the principle of superposition, the potential V_M generated by "n" charges at point "M" is:

$V_M = V_1 + V_2 + V_3 + \cdots \ldots \ldots V_n$

$V_M = \sum_{i=1}^{n} V_i = \sum_{i=1}^{n} \dfrac{kq_i}{r_i}$

5.5. Electrostatic potential created by a continuous distribution of charges

$$V_M = \int dV$$

$$V_M = \int \frac{kdq}{r}$$

5.6. Rotational of the electrostatic field $\vec{\boldsymbol{E}}$

$$\overrightarrow{rot}\,\vec{E} = \vec{\nabla} \wedge \vec{E} = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix}$$

If the field $\vec{E}$ derives from a potential V: $\quad \vec{E} = -\overrightarrow{grad}\,V$

$$\Rightarrow \overrightarrow{rot}\,\vec{E} = -\overrightarrow{rot}\left(\overrightarrow{grad}\,V\right) = \vec{0}$$

Therefore; if $\overrightarrow{rot}\,\vec{E} = \vec{0}$, the field is irrotational

6. Electric dipole ثنائي قطب كهربائي

6.1. Definition

The electric dipole is a system of two identical charges with different signs. We call $\overrightarrow{P}$ the dipole moment which is always directed from the negative charge $(-)$ to the positive charge $(+)$. (See **table.3**)

We write:

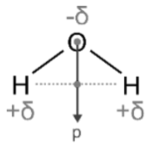
$$\vec{P} = q\,\vec{a}$$

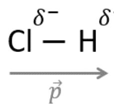
Table.3. Experimental dipole moment of some molecules.

Molecules	H_2O	HCl	CO_2	C_2H_5OH
$P\ (D)$	1.855	1.109	0	1.69

$1\,D\ (Debye) = 3.33564 \times 10^{-30}\ C.m\ (SI\ unit)$

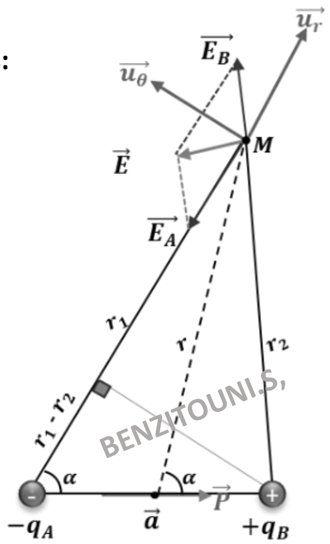
Dipole moment of certain polar molecules





6.2. The field $\vec{E}$ and the potential V of an electric dipole

a) The electrostatic potential V created by two charges q_A and q_B at point M is:



$principle\ of superposition$: $\quad V_M = V_A + V_B$

$$V_M = \frac{kq_A}{r_1} + \frac{kq_B}{r_2}$$

$$V_M = -\frac{kq}{r_1} + \frac{kq}{r_2} = kq\left(\frac{1}{r_2} - \frac{1}{r_1}\right)$$

$$V_M = kq\left(\frac{r_1 - r_2}{r_1}\right)$$

As long as $r >>> a$, we can consider: $\begin{cases} r_1 r_2 = r^2 \\ r_1 - r_2 = a\cos\theta \\ \vec{P} = q\,\vec{a} \end{cases} \Rightarrow V_M = \frac{kqa\cos\theta}{r^2}$

$$\Rightarrow \boldsymbol{V_M = \frac{k\,P\cos\theta}{r^2}}$$

b) The electrostatic field E_M created by two charges q_A and q_B at point M is:

We use the relation: $\vec{E} = -\overrightarrow{grad}\,V$ in the polar coordinate system:

$$\begin{cases} E_r = -\dfrac{\partial V}{\partial r} = \dfrac{\boldsymbol{2k\,P\cos\theta}}{\boldsymbol{r^3}} \\ E_\theta = -\dfrac{1}{r}\dfrac{\partial V}{\partial \theta} = \dfrac{\boldsymbol{k\,P\sin\theta}}{\boldsymbol{r^3}} \end{cases}$$

The field $E_M = \sqrt{{E_r}^2 + {E_\theta}^2}$

$$E_M = \frac{k\,P}{r^3}\sqrt{1 + 3\ \cos^2\theta}$$

Particular cases:

$$\textit{if}\ :\ \theta = 0 \Rightarrow \begin{cases} E_r = \frac{2k\,P}{r^3} \\ E_\theta = 0 \end{cases} \quad ; \quad \textit{if}: \theta = \frac{\pi}{2} \Rightarrow \begin{cases} E_r = 0 \\ E_\theta = \frac{k\,P}{r^3} \end{cases}$$

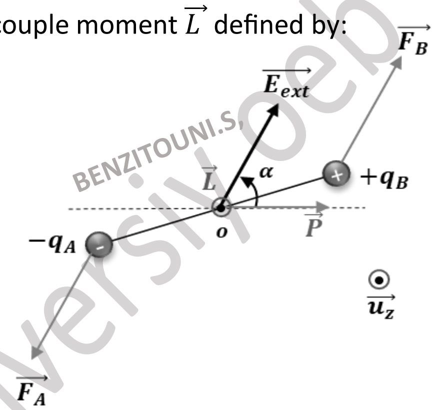
6.3. Electric dipole in an external electric field $\overrightarrow{E_{ext}}$

When we place an electric dipole in an external field $\overrightarrow{E_{ext}}$, the dipole is subjected to electrostatic forces applied to its charges. These forces (equal and opposite) cause a couple moment $\vec{L}$ defined by:

$$\vec{L} = \vec{P}\ \wedge\ \overrightarrow{E_{ext}}$$

$$L = P\,E_{ext}\sin\alpha$$

Knowing that: $\begin{cases} \overrightarrow{F_A} = -q\,\overrightarrow{E_{ext}} \\ \overrightarrow{F_B} = +q\,\overrightarrow{E_{ext}} \end{cases} \Rightarrow \overrightarrow{F_A} + \overrightarrow{F_B} \neq \vec{0}$



6.4. Potential energy E_p of an electric dipole located in an external field.

we have: $E_p = q\,V \Rightarrow dE_p = q\,dV \Rightarrow dE_p = q\left(-\overrightarrow{E_{ext}}\,.\,\overrightarrow{dl}\right) \Rightarrow E_p = -\,q\overrightarrow{E_{ext}}\,.\,\vec{a}$

$$\boldsymbol{E_p = -\,\overrightarrow{E_{ext}}\,.\,\vec{P}}$$

$$\begin{cases} if\ \alpha = 0 & \Rightarrow E_p = -E_{ext}.P & \Rightarrow E_{p,min}(stable) \\ if\ \alpha = \pi & \Rightarrow E_p = +E_{ext}.P & \Rightarrow E_{p,max}(unstable) \end{cases}$$

6.5. Field lines and equipotential surfaces

Equipotentials are closed lines that surround the charge and are perpendicular to the field lines.

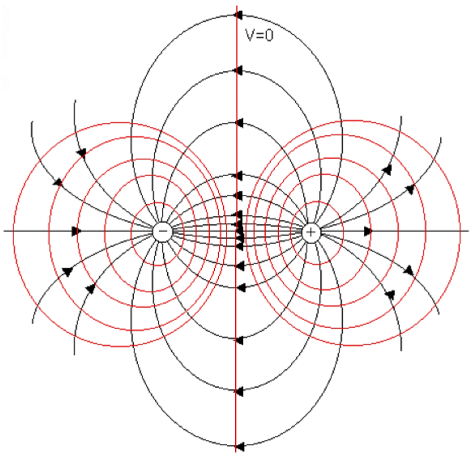


Figure: The field lines and equipotential lines of an electric dipole

7. Circulation of the electric field دوران الحقل الكهربائي

Consider a region of space where an electrostatic field $\vec{E}$ reigns, any charge q_0 moving in this field is subject to a force $\overrightarrow{F_e}$:

$$\overrightarrow{F_e} = q_0\vec{E}$$

The work done by $\overrightarrow{F_e}$, for the charge q_0 to move from A to B is given by:

$$dW = \overrightarrow{F_e}.\overrightarrow{dl}$$

$$W_{AB} = \int_A^B dW = \int_A^B \overrightarrow{F_e}.\overrightarrow{dl}$$

$$W_{AB} = q_0 \int_A^B \vec{E}.\overrightarrow{dl}$$

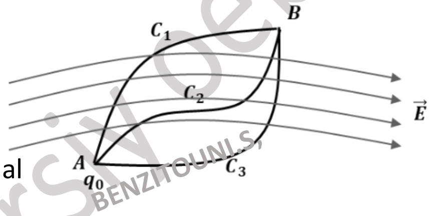
7.1. Definition

The integral $\int_A^B \vec{E}.\overrightarrow{dl}$ is called the circulation of the electric field along the curve $(A \to B)$, denoted "C", we write:

$$dC = \vec{E}.\overrightarrow{dl}$$

$$\boldsymbol{C} = \int_{\boldsymbol{A}}^{\boldsymbol{B}} \boldsymbol{\vec{E}}.\boldsymbol{\overrightarrow{dl}}$$

Remarks:



i. The circulation of the field is conservative, i.e.:

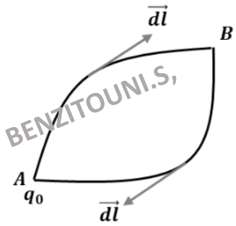
$$C_1 = C_2 = C_3$$

ii. The circulation of the electric field is equal to the difference in potential

$$dC = \vec{E}.\overrightarrow{dl} = -dV$$

$$\int_A^B dC = -\int_A^B dV$$

$$C = V_A - V_B$$



iii. The circulation of the electric field C along a closed curve is zero.

$$C = \oint \vec{E}.\overrightarrow{dl} = 0$$

We therefore say that the field E is irrotational ($\overrightarrow{rot}\,\vec{E} = \vec{0}$)

<u>8. Flux of the electrostatic field Ø تدفق الحقل الكهربائي</u>

8.1. Elementary surface vector $\overrightarrow{dS}$

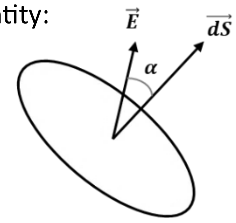
$\overrightarrow{dS}$ is an elementary surface vector; it is always normal to the surface and directed towards the exterior:

$$\overrightarrow{dS} = dS\,\vec{n}$$

$\vec{n}$: normal unit vector

8.2. Flux of the electrostatic field Ø

We call the flux of the electrostatic field Ø across the surface the following quantity:



$$\boldsymbol{d\emptyset} = \boldsymbol{\vec{E}}.\boldsymbol{\overrightarrow{dS}}$$

$$\emptyset = \iint \vec{E}.\overrightarrow{dS}$$

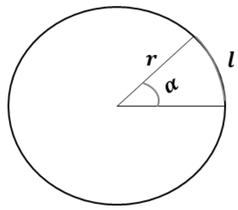
$$\emptyset = \iint E.dS.\cos\alpha$$

The unit of Ø is the weber $[Wb]$

- $if: \alpha = \frac{\pi}{2} \Rightarrow \emptyset = 0$
- $if: \alpha > \frac{\pi}{2} \Rightarrow \emptyset < 0$
- $if: 0 < \alpha < \frac{\pi}{2} \Rightarrow \emptyset > 0$

8.3. Definition of solid angle Ω

a. Plane angle α الزاوية المستوية



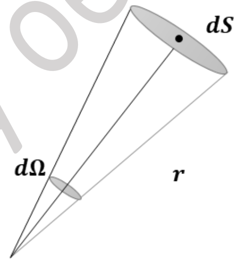
In the plane, we can define the elementary plane angle $d\alpha$, by:

$$d\alpha = tg\ \alpha = \frac{\overrightarrow{AB}}{r} = \frac{dl}{r}$$

$$d\alpha = \frac{dl}{r}$$

b. Solid angle Ω الزاوية الصلبة

In space ($\boldsymbol{3D}$), elementary solid angle $d\Omega$ is the space contained in an elementary conic surface dS of radius r. We write:



$$d\Omega = \frac{dS}{r^2}$$

$$\Omega = \frac{S}{r^2}$$

In spherical coordinate system: $dS = r^2 \sin\theta\, d\theta\, d\phi$

$$dS = r^2 \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\varphi$$

$$d\Omega = \frac{dS}{r^2}$$

$$\Omega = 4\pi \quad (sr)\text{: Stéradian}$$

8.4. Relation between the flux ϕ and the solid angle Ω

We have:

$$d\emptyset = \vec{E}.\overrightarrow{dS} \qquad (\vec{E} \parallel \overrightarrow{dS})$$

$$d\emptyset = E\ dS = E d\Omega\, r^2$$

$$d\emptyset = kq d\Omega$$

Example: the flux created by $(+q)$ through space ($\Omega = 4\pi$) is equal to:

$$d\emptyset = kq d\Omega$$

$$\emptyset = kq\Omega$$

$$\boldsymbol{d\emptyset = \frac{q}{\varepsilon_0}}$$

<u>9. Gauss's theorem</u>

Let *"q"* be a positive electric charge, it produces a radial electric field, directed outwards, where:

$$E = \frac{kq}{r^2}$$

To calculate the flux $\emptyset$ we must choose a spherical surface of radius r as a closed imaginary surface, called the Gaussian surface *"$\boldsymbol{S_G}$"*.

$$d\emptyset = \vec{E}.\overrightarrow{dS}$$

$$d\emptyset = E.dS.\cos\alpha$$

$$\emptyset = E.S \quad : \quad S = 4\pi r^2; \qquad E = \frac{kq}{r^2}$$

$$\emptyset = \frac{q}{\varepsilon_0}$$

Note : $\forall\, r \quad : \quad \emptyset = \frac{q}{\varepsilon_0} \quad (the\ flux\ does\ not\ depend\ on\ r)$

<u>Generalization:</u>

If we consider $''n''$ electric charges whatever their signs, the elementary electric flux $''\emptyset''$ across the surface is given by:

$$\emptyset = \frac{\sum q_i}{\varepsilon_0} = \frac{Q}{\varepsilon_0}$$

9.1. Gauss's theorem statement

The flux of an electric field through a closed surface is equal to the algebraic sum of the charges found within the volume limited by this surface, divided by the vacuum permittivity:

$$\emptyset = \oiint \vec{E}.\overrightarrow{dS_G} = \frac{\sum q_{int}}{\varepsilon_0} = \frac{Q_{int}}{\varepsilon_0}$$

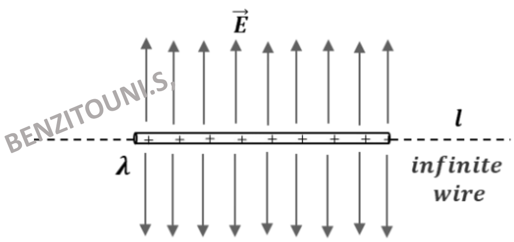
9.2. Conditions for applying Gauss's theorem

1) In practice, Gauss's theorem allows us to calculate the field $''E''$ for a symmetric distribution of charges (i.e., a system with a high degree of symmetry).
2) Gauss's theorem allows us to calculate the field $''E''$ in all points in space.
3) When we use the theorem, we must define an imaginary and closed surface on which the field $\vec{E}$ is constant and radial.
4) The chosen surface must depend on the symmetry of the problem (distribution of field $\vec{E}$).

10. Applications on Gauss's theorem

10.1. An infinite wire uniformly charged by a linear charge density (λ) (exercise)

By a symmetry, the field $\vec{E}$ at any point (r) in space is radial, perpendicular to the axis of the wire, and directed towards the exterior.



a) **<u>The total charge Q_T</u>**

$Q_T = \int dq = \int_0^l \lambda\, dl = \lambda l \qquad \Rightarrow \boldsymbol{Q_T = \lambda l}$

b) **<u>The field E in all points of space</u>**

According to Gauss's theorem:

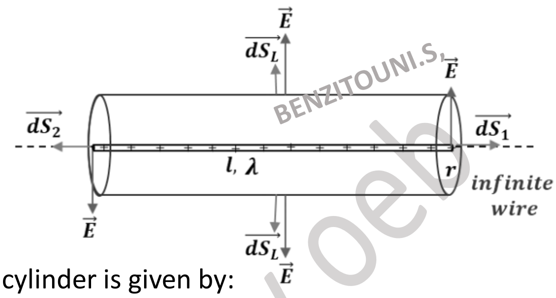
$$\phi = \oiint \vec{E}\,.\,\overrightarrow{dS_G} = \frac{Q_{int}}{\varepsilon_0}$$

The Gaussian surface which is suitable here is a surface of a cylinder (r, h), where:

- The axis of the cylinder coincides with the wire.
- The two bases of the cylinder cover the extremities of the wire.

According to the diagram, there are three surfaces:

1- Base surface S_1 (disk) of $\overrightarrow{dS_1}$
2- Base surface S_2 (disk) of $\overrightarrow{dS_2}$
3- Lateral surface S_L (rectangular) of $\overrightarrow{dS_L}$



The total flux through all surfaces constituting the Gaussian cylinder is given by:

$$\emptyset_T = \emptyset_1 + \emptyset_2 + \emptyset_L$$

Where:

$$\begin{cases} \emptyset_1 = \iint \vec{E}.\overrightarrow{dS_1} = \iint E.dS_1.\cos\theta = 0; & (\vec{E} \perp \overrightarrow{dS_1}) \\ \emptyset_2 = \iint \vec{E}.\overrightarrow{dS_2} = \iint E.dS_2.\cos\theta = 0; & (\vec{E} \perp \overrightarrow{dS_2}) \\ \emptyset_L = \iint \vec{E}.\overrightarrow{dS_L} = \iint E.dS_L.\cos\theta = E.S_L; & (\vec{E} \parallel \overrightarrow{dS_L}) \end{cases}$$

Therefore:

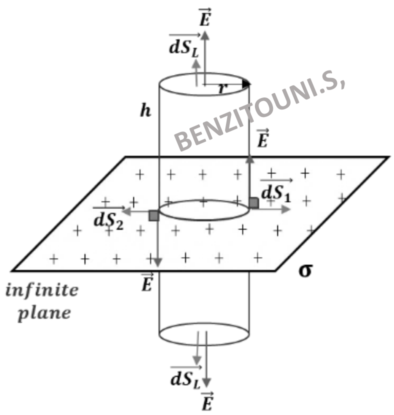
$$\emptyset_T = E.S_L = \frac{Q}{\varepsilon_0}$$

Knowing that: $\begin{cases} S_L = 2\pi r h \\ Q_T = \lambda\, l \\ h = l \end{cases}$

$$\boldsymbol{E(r) = \frac{\lambda}{2\pi\,\varepsilon_0\, r}}$$

10.2. An infinite plane uniformly charged by a surface charge density (σ) (exercise)

By a symmetry, the field E at all points in space is: radial, perpendicular to the plane and directed towards the exterior.



c) <u>**The total charge Q_T**</u>

$Q_T = \int dq = \int \sigma\, dS = \sigma\, S \quad \Rightarrow \boldsymbol{Q_T = \sigma\, S}$

d) <u>**The field E at all points in the space**</u>

According to Gauss's theorem:

$$\phi = \oiint \vec{E}\,.\,\overrightarrow{dS_G} = \frac{Q_{int}}{\varepsilon_0}$$

The most suitable Gauss surface $\boldsymbol{S_G}$ which gives a high degree of symmetry is a cylinder of (r, h).

The total flux through all surfaces constituting the Gaussian cylinder is given by:

$$\emptyset_T = \emptyset_1 + \emptyset_2 + \emptyset_L$$

Where:

$$\begin{cases} \emptyset_1 = \iint \vec{E}.\overrightarrow{dS_1} = \iint E.dS_1.\cos\theta = E.S; \quad (\vec{E} \parallel \overrightarrow{dS_1}) \\ \emptyset_2 = \iint \vec{E}.\overrightarrow{dS_2} = \iint E.dS_2.\cos\theta = E.S; \quad (\vec{E} \parallel \overrightarrow{dS_2}) \\ \emptyset_L = \iint \vec{E}.\overrightarrow{dS_L} = \iint E.dS_L.\cos 90° = 0; \quad (\vec{E} \perp \overrightarrow{dS_L}) \end{cases}$$

Therefore:

$$\emptyset_T = 2E.S = \frac{Q_{int}}{\varepsilon_0}$$

Q_{int} : here presents the charge of the plane inside the cylinder (a disk of radius r).

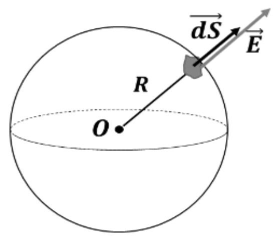
$$\begin{cases} Q_{int} = \sigma\,\pi\,r^2 \\ S = \pi\,r^2 \end{cases}$$

$$\boldsymbol{E(r) = \frac{\sigma}{2\,\varepsilon_0}}$$

We see that the field created by an infinite charged plane is constant at every point in space.

10.1. A full sphere uniformly charged with a volume charge density (ρ)

A full sphere with center O and radius R is uniformly charged by a positive and constant volume charge density ρ ($\rho > 0$).



a) The total charge Q of the sphere.

$$Q_T = \int dq = \iiint \rho dV = \rho V \rightarrow \boldsymbol{Q_T = \frac{4}{3}\pi R^3 \rho}$$

b) The expression of the electric field $E(r)$ at any point M in space.

To determine the field at any point M in space, we use Gauss' theorem:

$$\boldsymbol{\phi = \oiint \vec{E}\,.\,\overrightarrow{dS_G} = \frac{Q_{int}}{\varepsilon_0}}$$

As the electrostatic field is radial, $\vec{E}$ and $\overrightarrow{dS}$ are collinear ($\vec{E} \parallel \overrightarrow{dS} \Rightarrow \theta = 0°$), we can then write:

$$\Rightarrow \oiint E.dS_G\;\cos\theta = \frac{Q_{int}}{\varepsilon_0}; (\vec{E} \parallel \overrightarrow{dS} \Rightarrow \theta = 0°)$$

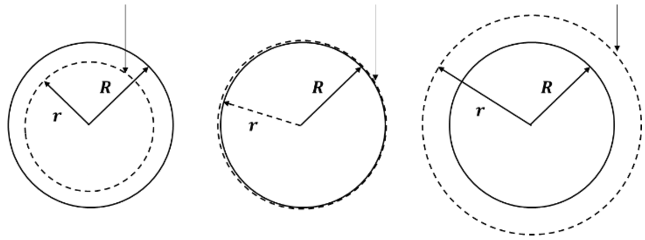
$$\Rightarrow \boldsymbol{E.S_G} = \frac{\boldsymbol{Q_{int}}}{\boldsymbol{\varepsilon_0}} \ldots (*)$$

Due to spherical symmetry, the suitable closed Gauss surface S_G here is a surface of sphere with center O and radius r. ($S_G = 4\pi r^2$).

To determine the field at any point M in space, we distinguish 03-regions:

Region-1 $r < R$:

$$\begin{cases} S_G = 4\pi r^2 \\ Q_{int} = \frac{4}{3}\pi r^3 \rho \end{cases} \Rightarrow l'eq(*) \Rightarrow \boldsymbol{E_1} = \frac{\boldsymbol{\rho}}{\boldsymbol{3\,\varepsilon_0}}\boldsymbol{r}$$



Region-2 $r = R$:

$$\begin{cases} S_G = 4\pi r^2 \\ Q_{int} = \frac{4}{3}\pi R^3 \rho \end{cases} \Rightarrow l'eq(*) \Rightarrow \boldsymbol{E_2} = \frac{\boldsymbol{\rho}}{\boldsymbol{3\,\varepsilon_0}}\frac{\boldsymbol{R^3}}{\boldsymbol{r^2}}$$

$$r = R \Rightarrow \boldsymbol{E_2} = \frac{\boldsymbol{\rho R}}{\boldsymbol{3\,\varepsilon_0}}$$

Region-3 $r > R$:

$$\begin{cases} S_G = 4\pi r^2 \\ Q_{int} = \frac{4}{3}\pi R^3 \rho \end{cases} \Rightarrow l'eq(*) \Rightarrow E_3 = \frac{\rho}{3\,\varepsilon_0}\frac{R^3}{r^2} \Rightarrow \boldsymbol{E_3} = \frac{\boldsymbol{\rho}}{\boldsymbol{3\,\varepsilon_0}}\frac{\boldsymbol{R^3}}{\boldsymbol{r^2}}$$

Remarks:

We can obtain the expression of the field $\boldsymbol{E_2}$ on the surface of the sphere $(r = R)$, by replacing $\boldsymbol{r}$ with R in the expression of $\boldsymbol{E_3}$ found outside the sphere or in $\boldsymbol{E_1}$ found inside the sphere.

c) The electric potential $V(r)$

To determine the potential, we use the relation: $\vec{E} = -\overrightarrow{grad}\,V$

E is radial $\Rightarrow E = -\frac{dV}{dr} \Rightarrow V = -\int E\,dr$

Region-1 $r < R$: $V_3 = -\int E_3\,dr = V = -\int \frac{\rho}{3\,\varepsilon_0}\frac{R^3}{r^2}\,dr = \frac{\rho}{3\,\varepsilon_0}\frac{R^3}{r} + C_3$

To determine C_3, we use the limit conditions of the potential: $\lim_{r\to\infty} V = 0$

$$\Rightarrow \lim_{r\to\infty} V_3 = 0 + C_3 = 0 \Rightarrow \boldsymbol{C_3 = 0} \Rightarrow \boldsymbol{V_3} = \frac{\boldsymbol{\rho}}{\boldsymbol{3\,\varepsilon_0}}\frac{\boldsymbol{R^3}}{\boldsymbol{r}}$$

Region-2 $r = R$: $V_2 = -\int E_2\,dr = -\int \frac{\rho R}{3\,\varepsilon_0}\,dr = -\frac{\rho R}{3\,\varepsilon_0}r + C_2$

To determine C_2 , we use the condition of continuity of the potential at the interface $(r = R)$:

$$V_3(r = R) = V_2(r = R)$$

$$\begin{cases} V_2(R) = -\frac{\rho R^2}{3\,\varepsilon_0} + C_2 \\ V_3(R) = \frac{\rho R^2}{3\,\varepsilon_0} \end{cases} \Rightarrow C_2 = 2\frac{\rho R^2}{3\,\varepsilon_0} \Rightarrow \boldsymbol{V_2} = \frac{\boldsymbol{\rho R^2}}{\boldsymbol{3\,\varepsilon_0}}$$

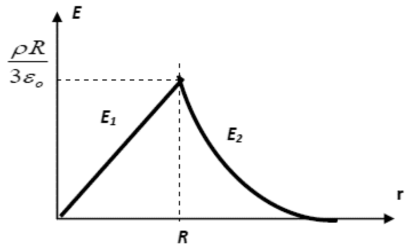
Region-2 $r > R$: $V_1 = -\int E_1\,dr = -\int \frac{\rho}{3\,\varepsilon_0}r\,dr = -\frac{\rho}{3\,\varepsilon_0}\frac{r^2}{2} + C_2$

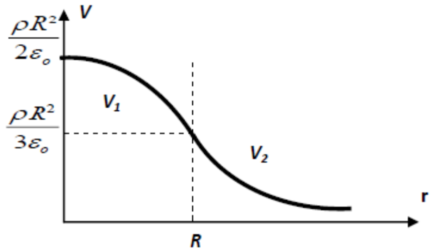
To determine C_1 , we use the condition of continuity of the potential at the interface $(r = R)$:

$$V_1(R) = V_2(R)$$

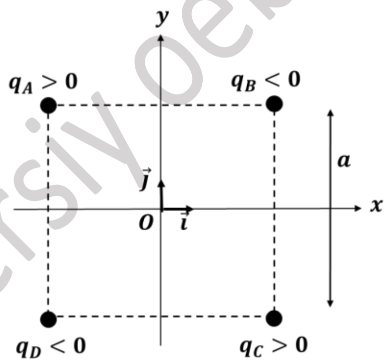
$$\begin{cases} V_2(R) = \frac{\rho R^2}{3\,\varepsilon_0} \\ V_1(R) = -\frac{\rho}{3\,\varepsilon_0}\frac{R^2}{2} + C_2 \end{cases} \Rightarrow C_1 = \frac{\rho R^2}{2\,\varepsilon_0} \Rightarrow \boldsymbol{V_1} = \frac{\boldsymbol{\rho R^2}}{\boldsymbol{3\,\varepsilon_0}}\left[\frac{\boldsymbol{3}}{\boldsymbol{2}} - \frac{\boldsymbol{r^2}}{\boldsymbol{2R^2}}\right]$$

d) The variation curve of $E(r)$ and $V(r)$.





11. Exercises



<u>Exercise 1</u> (*The solution can be found in the **series #2***)

We place four electrical charges q_A, q_B, q_C and q_D at the tops ABCD of a square, with side a and center O, origin of an orthonormal base $(O, \vec{\imath}, \vec{\jmath})$.

1- Give the expression of the electric field $\overrightarrow{E_O}$ at point O.
2- Give the expression of electric potential V_O at point O.

We give: $q_A = q,\ q_B = -2q,\ q_C = 2q\ et\ q_D = -q$

<u>Exercise 2</u> (*The solution can be found in the* course **page -16**)

An electrostatic dipole is made up of two identical charges with different signs: $(q_A = -q)$ and $(q_B = +q)$ separated by a small distance $\boldsymbol{a}$, where $(r >> a)$. The dipole moment is defined by: $\vec{P} = q\,\overrightarrow{AB}$.

1- Find the potential V_M of this dipole at the point M distant r from its center.
2- Deduce the components E_r and E_θ of the electrostatic field E and its module $\|\vec{E}\|$ in the polar coordinate system.
3- We now assume that the dipole is subject to a uniform external field $\overrightarrow{E_{ext}}$. Give the expression for the electrostatic moment $\vec{L}$ and the potential energy of the dipole E_p.