Chapter 1: Riemann integral and primitives

1.1 Riemann integral

Definition.1.1 (partition)
A *partition* P of $[a, b]$ is a finite set of numbers $\{x_0, x_1, x_2, \dots\dots\dots\dots, x_{n-1}, x_n\}$ such that $a = x_0 < x_1 < x_2 < \dots\dots\dots\dots < x_{n-1} < x_n = b$
We write $\Delta x_i = x_i - x_{i-1}$.
We define the norm of *partition* P is the positive number $\|P\| = \max_{1\le i\le n}(x_i - x_{i-1})$.

Remak:
When the n subintervals have equal length $\Delta x_i = \frac{b-a}{n}$
The i^{th} term of the partition is $x_i = a + i\frac{b-a}{n}$ (This makes $x_n = b$.)

Definition 1.2 (Darboux sums)
Suppose $f: [a, b] \to \mathbb{R}$ is bounded and P is a partition of $[a, b]$. Define
$m_i = \inf\{f(x): x_{i-1} < x < x_i\}$ $\qquad M_i = \sup\{f(x): x_{i-1} < x < x_i\}$

$s(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$ $\qquad S(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$.

We call $s(P, f)$ the *lower Darboux sum* and $S(P, f)$ the *upper Darboux sum*.

Lemma.1.1. *Let P and Q be two partitions of* [a, b] *such that $P \subset Q$ Then*

$$s(P, f) \le s(Q, f)$$
$$S(P, f) \ge S(Q, f).$$

(The partition Q is called a refinement of P.)

Proof
First let us consider a particular case. Let P' be a partition formed from P by adding one extra point, say $c \in [x_{i-1}, x_i]$. Let $m_i' = \sup_{x_{i-1}\le x\le c} f(x)$, $m_i'' = \sup_{c\le x\le x_i} f(x)$.
Then $m_i' \ge m_i$, $m_i'' \ge m_i$, and we have

$$s(P', f) = \sum_{i=1}^{i-1} m_i \Delta x_{i-1} + m_i'(\mathrm{c} - x_{i-1}) + m_i''(x_i - \mathrm{c}) + \sum_{i=i+1}^{n} m_i \Delta x_{i+1}$$
$$\ge \sum_{i=1}^{i-1} m_i \Delta x_{i-1} + m_i(x_i - x_{i-1}) + \sum_{i=i+1}^{n} m_i \Delta x_{i+1} = s(P, f).$$

Similarly one obtains that

$S(P', f) \le S(P, f)$.
Now to prove the assertion one has to P consequetly a finite number of points in order to form Q.

Lemma.1.2
Let P and Q be arbitrary partitions of [a, b]. *Then*

$$s(P, f) \le S(Q, f).$$

Proof
Consider the partition $P \cup Q$. By Lemma 1 1 we have
$s(P, f) \le s(P \cup Q, f) \le S(P \cup Q, f) \le S(Q, f)$.

Proposition 1.1
Let $f: [a, b] \to \mathbb{R}$ be a bounded function. Let $m, M \in \mathbb{R}$ be such that for all $x \in [a, b]$, we have $m < x < M$. Then for every partition P of $[a, b]$,

$$m(b - a) \le s(P, f) \le S(P, f) \le M(b - a)$$

Proof

Let P be a partition of [*a*, *b*]. Note that $m \le m_i \le M_i \le M$ for all i and $\sum_{i=1}^{n} \Delta x_i = (b-a)$. Therefore,

$$m(b-a) = \sum_{i=1}^{n} m\,\Delta x_i \le \sum_{i=1}^{n} m_i\,\Delta x_i \le \sum_{i=1}^{n} M_i\,\Delta x_i \le \sum_{i=1}^{n} M\,\Delta x_i = M(b-a)$$

Definition 1.3

As the sets of lower and upper Darboux sums are bounded, we define

Lower Darboux integral $\underline{\int_a^b} f = \sup s(P,f) \colon P \text{ a partition of } [\mathrm{a},\mathrm{b}].$

Upper Darboux integral $\overline{\int_a^b} f = \inf S(P,f) : P \text{ a partition of } [\mathrm{a},\mathrm{b}].$

Lemma.1.3

$$\underline{\int_a^b} f \le \overline{\int_a^b} f.$$

Proof

Fix a partition Q*. Then by Lemma 1 2*

$$\forall P \colon s(P,f) \le S(Q,f).$$

Therefore

$$\underline{\int_a^b} f = \sup_P s(P,f) \le S(Q,f).$$

And from the above

$$\forall Q \colon \underline{\int_a^b} f \le S(Q,f).$$

Hence

$$\underline{\int_a^b} f \le \inf_Q S(Q,f) = \overline{\int_a^b} f.$$

Proposition 1.2. *Let* $f : [a,b] \to \mathbb{R}$ *be a bounded function. Let* $m, M \in \mathbb{R}$ *be such that for all* $x \in [a,b]$, *we have* $m \le f(x) \le M$. *Then*

$$m(b-a) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le M(b-a)$$

Proof. *By Proposition 1.1 , for every partition* P,

$$m(b-a) \le s(P,f) \le S(P,f) \le M(b-a)$$

The inequality $m(b-a) \le s(P,f)$ *implies* $m(b-a) \le \underline{\int_a^b} f$. *The inequality* $S(P,f) \le M(b-a)$ *implies* $\overline{\int_a^b} f \le M(b-a)$.

Definition 1.4. A function $f \colon [a,b] \to \mathbb{R}$ is called Riemann integrable if

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

The common value is called integral of f over $[a,b]$ and is denoted by $\int_a^b f(x)\,dx$.

Proposition 1.3. Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function. Let $m, M \in \mathbb{R}$ be such that $m \le f(x) \le M$ for all $x \in [a,b]$. Then

$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a).$$

Proof Is a direct consequence of Proposition 1.2.

Example 1 1
We integrate constant functions. If $f(x) = c$ for some constant c, then we *take* $m = M = c$. In Proposition 1.3. Thus f is integrable on $[a, b]$ and

$$\int_a^b f(x)\,dx = c(b - a).$$

Theorem 1.1
A function $f: [a, b] \to \mathbb{R}$ is Riemann integrable if and only if for any $\varepsilon > 0$ there exists a partition P of [a, b] such that $S(P, f) - s(P, f) < \varepsilon$.
Proof
1 Necessity: Let $\underline{\int_a^b} f = \overline{\int_a^b} f$, i.e. let us assume that f is integrable.
$\exists P_1, P_2$: $s(P_1, f) > \underline{\int_a^b} f - \frac{\varepsilon}{2}$ and $S(P_2, f) < \overline{\int_a^b} f + \frac{\varepsilon}{2}$.
Let $Q = P_1 \cup P_2$. Then

$$\underline{\int_a^b} f - \frac{\varepsilon}{2} < s(P_1, f) \le S(P_1 \cup P_2, f) \le S(P_1 \cup P_2, f) \le S(P_2, f) < \overline{\int_a^b} f + \frac{\varepsilon}{2}.$$

Therefore (since $\underline{\int_a^b} f = \overline{\int_a^b} f$)

$$S(Q, f) - s(Q, f) < \varepsilon.$$

2 sufficiency: Fix $\varepsilon > 0$. Let P be a partition such that $S(P, f) - s(P, f) < \varepsilon$.
Note that

$$\overline{\int_a^b} f - \underline{\int_a^b} f = S(P, f) - s(P, f) < \varepsilon.$$

Therefore it follows that

$$\forall \varepsilon > 0: \overline{\int_a^b} f - \underline{\int_a^b} f < \varepsilon.$$

This implies that

$$\overline{\int_a^b} f = \underline{\int_a^b} f.$$

Example 1.2
Let us show $f(x) = x^2$ is integrable on $[a, b]$ for all $b > a > 0$. We will see later that continuous functions are integrable, but let us demonstrate how we do it directly.
Let ε be given. Take $n \in \mathbb{N}$ and let $x_i = a + i\frac{b-a}{n}$ form the partition $P = \{x_0, x_1, x_2, \dots\dots\dots\dots, x_{n-1}, x_n\}$ of $[a, b]$. Then $\Delta x_i = \frac{b-a}{n}$ for all i. As f is increasing, for every subinterval $[x_{i-1}, x_i]$,

$$m_i = \inf\{f(x): x_{i-1} < x < x_i\} = \left(a + (i-1)\frac{b-a}{n}\right)^2$$
$$M_i = \sup\{f(x): x_{i-1} < x < x_i\} = \left(a + i\frac{b-a}{n}\right)^2$$

Then

$$S(P, f) - s(P, f) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i$$

$$= \frac{b-a}{n}\sum_{i=1}^{n}\left(\left(a+i\frac{b-a}{n}\right)^2-\left(a+(i-1)\frac{b-a}{n}\right)^2\right)$$
$$= \frac{b-a}{n}\sum_{i=1}^{n}\left(\left(a+n\frac{b-a}{n}\right)^2-\left(a+0\frac{b-a}{n}\right)^2\right) = \frac{\mathrm{b}^3}{n}$$
$$= \frac{b-a}{n}(b^2-a^2).$$

Picking n to be such that, $\frac{b-a}{n}(b^2-a^2) < \varepsilon$ the proposition is satisfied, and the function is integrable.

On the other hand, as we know from algebra (or can be proven by induction):

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}$$

So

$$S(P,f) = \frac{b-a}{n}\sum_{i=1}^{n}\left(a+i\frac{b-a}{n}\right)^2$$

$$= \frac{b-a}{n}\left[\sum_{i=1}^{n} a^2+2a\frac{b-a}{n}i+\left(\frac{b-a}{n}\right)^2 i^2\right]$$

$$= \frac{b-a}{n}\left[a^2n+2a\frac{b-a}{n}\sum_{i=1}^{n} i+\left(\frac{b-a}{n}\right)^2\sum_{i=1}^{n} i^2\right]$$

$$= \frac{b-a}{n}\left[a^2n+2a\frac{b-a}{n}\frac{n(n+1)}{2}+\left(\frac{b-a}{n}\right)^2\frac{n(2n+1)(n+1)}{6}\right]$$

$$= (b-a)\left[a^2+a(b-a)\frac{(n+1)}{n}+(b-a)^2\frac{(2n+1)(n+1)}{6n^2}\right].$$

Similarly one obtains that

$$s(P,f) = (b-a)\left[a^2+a(b-a)\frac{(n-1)}{n}+(b-a)^2\frac{(2n-1)(n-1)}{6n^2}\right].$$

So

$$\lim_{n\to\infty} S(P,f) = \lim_{n\to\infty} s(P,f) = \frac{(b-a)}{3}(b^2+ab+a^2) = \frac{1}{3}(b^2-a^2).$$

Finally we obtain

$$\int_a^b x^2dx = \frac{1}{3}(b^2-a^2).$$

Definition 1.5 (Riemann sums)
Let $f\colon [a,b] \to \mathbb{R}$ be defined on the interval $[a,b]$ and let $P = \{x_0, x_1, x_2, \dots\dots\dots\dots, x_{n-1}, x_n\}$ be a partition of $[a,b]$.

Let $C = \{c_1, c_2, \dots\dots c_{i-1}, c_i, \dots. \quad c_{n-1}, c_n\}$ where c_i denote any value in the i^{th} subinterval ($c_i \in [x_{i-1}, x_i]$). The Riemann sum of a function f on $[a,b]$ that corresponds to P and the point system C is

$$R(P,f,\mathrm{C}) = \sum_{i=1}^{n} f(c_i)\,\Delta x_i.$$

Theorem 1.2
A function f is Riemann integrable on $[a,b]$ if there is a number L such that for each $\varepsilon > 0$ there is $\delta > 0$ such that if P is any partition of $[a,b]$ with $\|P\| < \delta$ then $|R(P,f,\mathrm{C}) - L| < \varepsilon$. (In other words $\lim\limits_{\|P\|\to 0} R(P,f,\mathrm{C}) = L$).And we have $L = \int_a^b f(x)dx$.
The set of all Riemann integrable functions in $[a,b]$ is denoted by $\mathcal{R}([a,b])$.
Proof
1 Necessity: using $|R(P,f,\mathrm{C}) - L| \le S(P,f) - s(P,f)$.

2 sufficiency: To do this, we will first show that

$$S(P,f) = \sup_{\mathrm{C}} R(P,f,\mathrm{C}), \quad s(P,f) \inf_{\mathrm{C}} R(P,f,\mathrm{C}).$$

Remark
If the function f is Riemann integrable on $[a,b]$ then the number $\int_a^b f(x)dx$ is the common limit of the two sequences $u_n = \frac{b-a}{n}\sum_{i=0}^{n-1} f(a + \frac{b-a}{n} i\,)$ and $v_n = \frac{b-a}{n}\sum_{i=1}^{n} f(a + \frac{b-a}{n} i\,)$.
Example
Calculate the limit of the sum $v_n = \sum_{i=0}^{n} \frac{n}{(n+i)^2}$.
We have

$$v_n = \sum_{i=0}^{n-1} \frac{n}{(n+i)^2} + \frac{1}{4n} = \frac{1}{n}\sum_{i=0}^{n-1} \frac{1}{\left(1+\frac{i}{n}\right)^2} + \frac{1}{4n}$$

by putting $[a,b] = [1,2]$; $f(x) = \frac{1}{x^2}$ then

$$v_n = \frac{b-a}{n}\sum_{i=0}^{n-1} f(a + \frac{b-a}{n} i\,) + \frac{1}{4n}.$$

So

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n} f(1 + \frac{1}{n} i\,)\right) + 0 = \int_a^b f(x)dx = \int_1^2 \frac{1}{x^2}dx = \frac{1}{2}$$

1.2 Integrable functions

Theorem 1.3

Let f $f\colon [\mathrm{a},\mathrm{b}] \to \mathbb{R}$ be monotone. Then f is Riemann integrable.

Proof

Suppose that f *is increasing so that* $f(a) \le f(b)$.

If $f(a) = f(b)$ then f is constant, so f *is Riemann integrable and* $\int_a^b f(x)dx = f(a)(b-a)$.

If $f(a) < f(b)$ let $\varepsilon > 0$ and P a partition of $[a, b]$ such that $\|P\| < \delta = \frac{\varepsilon}{f(b)-f(a)}$.

For this partition we obtain

$$S(P,f) - s(P,f) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\,\Delta x_i$$

$$< \delta \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = \delta\big(f(b) - f(a)\big) = \varepsilon.$$

Theorem 1.4

Let $f\colon [a, b] \to \mathbb{R}$ be continuous. Then f is Riemann integrable.

Proof

Let $P = \{x_0, x_1, x_2, \dots\dots\dots\dots, x_{n-1}, x_n\}$ be a partition of $[a, b]$ and $m_i = \inf_{x_{i-1} \le x \le x_i} f(x)$; $M_i = \sup_{x_{i-1} \le x \le x_i} f(x)$.

Let $\varepsilon > 0$. A continuous function on closed interval $[x_{i-1}, x_i]$ is uniformly continuous and reaches its upper and lower bounds at least once, so there exists $\delta > 0$ such that $\forall x_{i-1}, x_i \in [a, b]\colon |x_i - x_{i-1}| < \delta \Longrightarrow |f(x_i) - f(x_{i-1})| < \frac{\varepsilon}{b-a}$ and there is at least x_i' ; x_i'' are from the subinterval $[x_{i-1}, x_i]$ where $m_i = f(x_i')$; $M_i = f(x_i'')$.

Choose a partition P such that $\|P\| < \delta$ so

$$S(P,f) - s(P,f) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\,\Delta x_i$$

$$< \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \Delta x_i = \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

Theorem 1.5

If $f, g\colon [a, b] \to \mathbb{R}$ are integrable, then $fg\colon [a, b] \to \mathbb{R}$ is integrable. If, in addition, $g \ne 0$ and $\frac{1}{g}$ is bounded, then $\frac{f}{g}\colon [a, b] \to \mathbb{R}$ is integrable.

1.3. Properties of the Riemann integral

Let f, g: $[a, b] \to \mathbb{R}$ *Riemann integrable functions on* $[a, b]$. The integral has the following three basic properties.

1) Linearity:

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b f(x)dx \quad , \quad \int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx$$

2) Monotonicity:

If $\forall x \in [a, b]: f(x) \le g(x), \text{then } \int_a^b f(x)dx \le \int_a^b g(x)dx.$

3) Additivity:If $\alpha, \beta, \gamma \in [a, b]$, then

a) $\int_\alpha^\beta f(x)dx = \int_\alpha^\gamma f(x)dx + \int_\gamma^\beta f(x)dx.$

b). $\int_\alpha^\alpha f(x)dx = 0.$

c) $\int_\alpha^\beta f(x)dx = -\int_\beta^\alpha f(x)dx.$

4) If f is continuous on $[a, b]$ and $\forall x \in [a, b]: f(x) \ge 0$ then

$$\left(\int_a^b f(x)dx = 0\right) \Longrightarrow (\forall x \in [a, b]: f(x) = 0).$$

5) $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx.$

6) If f is continuous on $[a, b]$, then there exists $c \in [a, b]$, where

$$\int_a^b f(x)dx = f(c)(b - a).$$

1.4 Integrals and primitives

Definition 1.6

Let f: $[a, b] \to \mathbb{R}$ function, we say that function F is a primitive function of f over $[a, b]$ if and only if F is differentiable over $[a, b]$ and $\forall x \in [a, b]: F'(x) = f(x)$.

Proposition 1.4

If F_1 and F_2 are primitive functions of f on $[a, b]$ then$\forall x \in [a, b]: F_1(x) - F_2(x) = C$ where C is a real constant.

Example

The function $F(x) = \frac{1}{3}x^3$ is a primitive of the function $f(x) = x^2$ over $\mathbb{R}$ because

$$\forall x \in \mathbb{R}: F'(x) = \left(\frac{1}{3}x^3\right)' = x^2 = f(x).$$

Theorem 1.6 (The fundamental theorem of calculus 1)

if $F: [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable in $]a, b[$ with $F' = f$ where $f: [a, b] \to \mathbb{R}$ is Riemann integrable, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof

Let $P = \{x_0, x_1, x_2, \ldots\ldots\ldots\ldots, x_{n-1}, x_n\}$ be a partition of $[a, b]$.

The function F is continuous on the closed interval $[x_{i-1}, x_i]$ and differentiable in the open interval $]x_{i-1}, x_i[$ with $F' = f$. By the mean value theorem, there exists

$c_i \in\,]x_{i-1}, x_i[$ such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1})$$
$$= f(c_i)(x_i - x_{i-1})$$

Since f is Riemann integrable, it is bounded and it follows that

$$m_i(x_i - x_{i-1}) \le f(c_i)(x_i - x_{i-1}) \le M_i(x_i - x_{i-1})$$

or

$$m_i(x_i - x_{i-1}) \le F(x_i) - F(x_{i-1}) \le M_i(x_i - x_{i-1})$$

where

$$M_i = \sup_{x_{i-1} \le x \le x_i} f(x) \quad \text{and} \quad m_i = \inf_{x_{i-1} \le x \le x_i} f(x).$$

So

$$\sum_{i=1}^{n} m_i\,\Delta x_i \le \sum_{i=1}^{n} \big(F(x_i) - F(x_{i-1})\big)\,\Delta x_i \le \sum_{i=1}^{n} M_i\,\Delta x_i$$

Hence $s(P, f) \le F(b) - F(a) \le S(P, f)$ of every partition of $[a, b]$ wich implies that $\underline{\int_a^b} f \le F(b) - F(a) \le \overline{\int_a^b} f$. Since f is integrable i.e. $\underline{\int_a^b} f = \overline{\int_a^b} f$ we obtain

$$F(b) - F(a) = \int_a^b f(x)\,dx.$$

Theorem 1.7 (The fundamental theorem of calculus 2)

Suppose that $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ and $F:[a,b]\to\mathbb{R}$ is defined by

$\forall x\in[a,b]: F(x)=\int_a^x f(t)\,dt$.Then F is differentiable over $[a,b]$ and

$\forall x\in[a,b]: F^{'}(x)=f(x)$ (that is, F is a primitive function of f over $[a,b]$).

Proof

Let $x,h\in[a,b]$ and $h>0$. Then

$$\frac{F(x+h)-F(x)}{h}=\frac{\int_a^{x+h} f(t)\,dt-\int_a^x f(t)\,dt}{h}=\frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

Let $\varepsilon>0$. Since f is continuous at x there exists $\delta>0$ such that

$$|f(t)-f(x)|<\varepsilon \quad \text{for} \quad |t-x|<\delta.$$

It follows that if $0<h<\delta$ then

$$\left|\frac{F(x+h)-F(x)}{h}-f(x)\right|=\left|\frac{1}{h}\int_x^{x+h} f(t)\,dt-f(x)\right|$$

$$=\left|\frac{1}{h}\int_x^{x+h}\big(f(t)-f(x)\big)dt\right|$$

$$\leq\frac{1}{h}\int_x^{x+h}|f(t)-f(x)|\,dt$$

$$\leq\frac{1}{h}\sup_{x<t<x+h}|f(t)-f(x)|\left|\int_x^{x+h}\right|dt$$

$$\leq\frac{1}{h}\varepsilon h=\varepsilon.$$

So

$$\lim_{\substack{h\to 0\\ >}}\frac{F(x+h)-F(x)}{h}=f(x).$$

In the same way, we obtain

$$\lim_{\substack{h\to 0\\ <}}\frac{F(x+h)-F(x)}{h}=f(x).$$

Which proves the result.

Corollary 1.1

Let $f: [a, b] \to \mathbb{R}$ be continuous in $[a, b]$ and F is a primitive function of f over $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof Proof is a direct consequence of Theorem 1.7.

Example

Since $F(x) = \frac{1}{3}x^3$ is primitive function of $f(x) = x^2$ over $\mathbb{R}$. Then

$$\forall a, b \in \mathbb{R}: \int_a^b f(x)\,dx = F(b) - F(a) = \frac{1}{3}b^3 - \frac{1}{3}a^3.$$

Theorem 1.8 (Change of variables)

Let $\varphi :\ [a, b] \to$ ☒ *be a continuously differentiable function, let* f *be continuous over* $\varphi([a, b])$, *Then* $\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$, *where* $b = \varphi(\beta)$; $a = \varphi(\alpha)$ *and* $x = \varphi(t)$; $dx = \varphi'(t)dt$.

Proof

The function $f(\varphi)\varphi'$ *is continuous and therefore integrable.* Let F be a primitive of f and then $F(\varphi)$ is a primitive of $f(\varphi(t))\varphi'(t)$. So according to the Corollary 1.1,

$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt = F\big(\varphi(\beta)\big) - F\big(\varphi(\alpha)\big) = F(b) - F(a) = \int_a^b f(x)dx.$$

Example 1

Calculate the integral $J = \int_0^1 \sqrt{1 - x^2}\,dx$. (Put $x = \varphi(t) = \sin t$).

$$x = \varphi(t) = \sin t \Longrightarrow dx = \cos t\,dt$$

$\sin\alpha = 0 \iff \alpha = 0, \pi, -\pi, 2\pi, -2\pi, \dots$ (The value of α can be chosen from among the values $0, \pi, -\pi, 2\pi, -2\pi \dots$).

$\sin\beta = 1 \iff \beta = \frac{\pi}{2}, \frac{-3\pi}{2}, \frac{5\pi}{2} \dots$ (The value of β can be chosen from among the values $\frac{\pi}{2}, \frac{-3\pi}{2}, \frac{5\pi}{2} \dots$).

So

$$J = \int_0^{\frac{\pi}{2}} \sqrt{1 - (\sin t)^2}\,\cos t\,dt = \int_0^{\frac{\pi}{2}} \sqrt{(\cos t)^2}\,\cos t\,dt.$$

Since $\forall x \in \left[0, \frac{\pi}{2}\right] : \cos t \geq 0$. Then

$$J = \int_0^{\frac{\pi}{2}} (\cos t)^2 \, dt = \frac{1}{2}\int_0^{\frac{\pi}{2}} (1 + \cos 2t) \, dt$$

$$= \frac{1}{2}\left[t + \frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

Example 2

Calculate the integral $K = \int_0^4 \frac{\sqrt{x}}{\sqrt{x}+1} dx$. (Put $x = \varphi(t) = t^2$).

$$x = \varphi(t) = t^2 \Longrightarrow dx = 2tdt.$$

$\varphi(\alpha) = a \quad \Leftrightarrow \alpha^2 = 0 \;\; \Leftrightarrow \alpha = 0$ (The value of α can be chosen from among the values

φ $\varphi(\beta) = b \;\; \Leftrightarrow \beta^2 = 4 \;\; \Leftrightarrow \beta = -2, \beta = 2$ (The value of β can be chosen from among the values $-2\,,2$). So

$$K = \int_0^{-2} \frac{\sqrt{t^2}}{\sqrt{t^2}+1} 2tdt$$

Since $\forall t \in \; [-2{,}0]: t \leq 0$. Then

$$K = 2\int_{-2}^{0} \frac{t^2}{-t+1} dt = 2\int_{-2}^{0} (-t - 1 - \frac{1}{t-1}) \, dt$$

$$= \; 2\left[-\frac{1}{2}t^2 - t - ln|t-1|\right]_{-2}^{0} = 2ln3.$$

Theorem 1.9 (Integration by parts)*. Suppose that* $u, v : [a, b] \; \rightarrow \; \mathbb{R}$ *are continuous on* $[a, b]$ *and differentiable in* (a, b)*, and* u', v' *are integrable on* $[a, b]$*. Then*

$$\int_a^b uv' \, dx = [uv]_a^b - \int_a^b u'v \, dx.$$

Proof. *The function* $u \; v$ *is continuous on* $[a, b]$ *and, by the product rule, differentiable in* (a, b) *with derivative* $(uv)' = u'v + uv'$*. Since* u, v, u' and v' *are integrable on* $[a, b]$*. Theorem 1.4 implies that* $u'v,\; uv'$ *and* $(uv)'$*, are integrable. From Theorem 1.5, we get that* $\int_a^b (uv' + u'v) \, dx = \int_a^b uv' \, dx + \int_a^b u'v \, dx = [uv]_a^b$*, which proves the result.*

Example 1

calculate the integral $I = \int_0^1 Arc \tan x \, dx$.

$$\begin{cases} v' = 1 \\ u = Arc\tan x \end{cases} \Rightarrow \begin{cases} v = x \\ u' = \dfrac{1}{x^2+1} \end{cases}$$

$$I = [uv]_0^1 - \int_0^1 u'v\,dx = [xArc\tan x]_0^1 - \int_0^1 \frac{1}{x^2+1} x\,dx$$

$$I = \left[xArc\tan x - \frac{1}{2} ln(x^2+1)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2} ln2.$$

Example 2

calculate the integral $I = \int_1^2 x\ln\frac{x}{x+1} dx$.

$$\begin{cases} v' = x \\ u = \ln\dfrac{x}{x+1} \end{cases} \Rightarrow \begin{cases} v = \dfrac{1}{2}x^2 \\ u' = \dfrac{1}{x(x+1)} \end{cases}$$

$$I = [uv]_1^2 - \int_1^2 u'v\,dx$$

$$I = \left[\frac{1}{2}x^2\ln\frac{x}{x+1}\right]_1^2 - \int_1^2 \frac{1}{2}x^2 \frac{1}{x(x+1)} dx$$

$$I = \left[\frac{1}{2}x^2\ln\frac{x}{x+1}\right]_1^2 - \int_1^2 \frac{1}{2}x^2 \frac{1}{x(x+1)} dx$$

$$I = \frac{5}{2}\ln2 - 2\ln3 - \int_1^2 \frac{1}{2}\frac{x}{(x+1)} dx$$

$$I = \frac{5}{2}\ln2 - 2\ln3 - \frac{1}{2}\int_1^2 1 - \frac{1}{(x+1)} dx$$

$$I = \frac{5}{2}\ln2 - 2\ln3 - \frac{1}{2}[x - \ln(x+1)]_1^2$$

$$I = \frac{5}{2}\ln2 - 2\ln3 - \frac{1}{2}[1 + \ln2 - \ln3] = 2\ln2 - \frac{3}{2}\ln3 - \frac{1}{2}.$$

Definition 1.7.(The Indefinite Integral)
The set of all primitive functions of f is the ***indefinite integral*** of f with respect to x and denoted by $\int f(x)\,dx$ where
$\int f(x)\,dx$ is read " the integral of f w.r.t x ".
Note: The above definition says that if a function F is an primitive of f, then

$$\int f(x)\,dx = F(x) + C \quad \text{where C is a real constant.}$$

Example

$$\int x^3\,dx = \frac{1}{4}x^4 + C$$

primitives of usual functions

$\int f(x)\,dx$	f
$\frac{1}{\alpha+1}x^{\alpha+1} + C$	$\left(\alpha \in \mathbb{R}^* - \{-1\}\text{حيث}\right)x^{\alpha}$
$\ln\lvert x\rvert + C$	$\frac{1}{x}$
$e^x + C$	e^x
$-\cos x + C$	$\sin x$
$\sin x + C$	$\cos x$
$-\ln\lvert\cos x\rvert + C$	$\tan x$
$\tan x + C$	$\frac{1}{\cos^2 x}$
$-\text{cotan}\, x + C$	$\frac{1}{\sin^2 x}$
$\cosh x + C$	$\sinh x$
$\sinh \text{x} + C$	$\cosh x$
$\frac{1}{a}\text{Arctan}\ \frac{x}{a} + C$	$\frac{1}{x^2+a^2}$
$\text{Arcsin}\ \frac{x}{a} + C$	$\frac{1}{\sqrt{a^2-x^2}}$
$\frac{1}{2a}\ln\left\lvert\frac{x+a}{x-a}\right\rvert + C$	$\frac{1}{x^2-a^2}$
$\frac{1}{\alpha+1}\left(u(x)\right)^{\alpha+1} + C$	$\left(u(x)\right)^{\alpha}u'(x)$ $\left(u \in C^1(I) \text{ و } \alpha \in \mathbb{R}^* - \{-1\}\text{حيث}\right)$
$\ln\lvert u(x)\rvert + C$	$\frac{u'(x)}{u(x)}$ $\left(u \in C^1(I) \text{ و } \forall x \in I: u(x) \neq 0 \text{ حيث}\right)$
$e^{u(x)} + C$	$u'(x)e^{u(x)}$ $\left(u \in C^1(I)\text{حيث}\right)$
$G\left(u(x)\right) + C$ G is a primitive of g over I	$g\left(u(x)\right)u'(x)$ Where $u \in C^1(I)$ and g *is continuous over* $u(I)$

Theorem 1.10 (change the variable)

Let $h: I \longrightarrow J$ C^1-diffeomorphism. We put $x = h(t)$ and $dx = h'(t)dt$ then

$$\int f(x)\,dx = \int f\left(h(t)\right)h'(t)dt \quad \text{and } t = h^{-1}(x).$$

Note: A function $h: I \longrightarrow J$ is called C^1-diffeomorphism if

a) h is a bejiction of I on J;

b) h and h^{-1} admit derivatives of order 1, continuous, respectively on I and J.

Example 1

Calculate $I = \int \sqrt{1-x^2}\, dx$.

We put $x = h(t) = \sin t$ where h: $\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\longrightarrow]-1,1[$ (h is C^1-diffeomorphism), and $dx = \cos t\, dt$.

So

$$I = \int \sqrt{1-\sin^2 t} \cos t\, dt = \int \sqrt{\cos^2 t} \cos t\, dt.$$

Since $\forall t \in \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[: \cos t \geq 0$ we get

$$I = \int \cos^2 t\, dt = \frac{1}{2}\int (1+\cos 2t)\, dt$$

$$= \frac{1}{2}\left(t + \frac{1}{2}\sin 2t\right) + C = \frac{1}{2}t + \frac{1}{2}\cos t \sin t + C$$

$$= \frac{1}{2}t + \frac{1}{2}\sqrt{1-\sin^2 t}\sin t + C.$$

Substituting $t = h^{-1}(x) = \mathrm{Arcsin} x$ we get the following result:

$$I = \frac{1}{2}\mathrm{Arcsin} x + \frac{1}{2}x\sqrt{1-x^2} + C.$$

Example 2

Calculate $J = \int \frac{x}{\sqrt{x}+1} dx$.

We put $x = h(t) = t^2$ where h: $]0, +\infty[\longrightarrow]0, +\infty[$ (h is C^1-diffeomorphism), and $dx = 2tdt$.

So

$$J = \int \frac{t^2}{\sqrt{t^2}+1} 2tdt$$

Since $\forall t \in\ ]0, +\infty[: t > 0$ we get

$$J = \int \frac{t^2}{\sqrt{t^2}+1} 2tdt$$

$$= 2\int \frac{t^3}{\mathrm{t}+1} dt$$

$$= 2\int \left(t^2 - \frac{1}{\mathrm{t}+1} - \mathrm{t} + 1 \right) dt$$

$$= 2\left(\frac{1}{3}t^3 - \ln(\mathrm{t}+1) - \frac{1}{2}t^2 + \mathrm{t}\right)$$

$$J = \frac{2}{3}t^3 - 2\ln(\mathrm{t}+1) - t^2 + 2\mathrm{t} + C.$$

Substituting $t = h^{-1}(x) = \sqrt{x}$ we get the following result:

$$J = \frac{2}{3}x\sqrt{x} - x + 2\sqrt{x} - 2\ln(\sqrt{x}+1) + C.$$

***Theorem 1.11 (Integration by parts)**.*

Let I be a interval for $\mathbb{R}$ and v, u are functions of class C^1 on the interval I then

$$\int uv'\,dx = uv - \int u'v\,dx.$$

Example 1

Calculate $I = \int xe^{2x}\,dx$. By putting:

$$\begin{cases} v' = e^{2x} \\ u = x \end{cases} \Longrightarrow \begin{cases} v = \frac{1}{2}e^{2x} \\ u' = 1 \end{cases}$$

$$I = \int xe^{2x}\,dx = uv - \int u'v\,dx$$

$$= x\frac{1}{2}e^{2x} - \int \frac{1}{2}e^{2x}\,dx$$

$$= \left(\frac{1}{2}x - \frac{1}{4}\right)e^{2x} + C.$$

Example 2

Calculate $J = \int e^x \sin x\,dx$. By putting:

$$\begin{cases} v' = \sin x \\ u = e^x \end{cases} \Longrightarrow \begin{cases} v = -\cos x \\ u' = e^x \end{cases}.$$

We get

$$J = -e^x \cos x + \int e^x \cos x\,dx.$$

Again we put

$$\begin{cases} v' = \cos x \\ u = e^x \end{cases} \Longrightarrow \begin{cases} v = \sin x \\ u' = e^x \end{cases},$$

so

$$J = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

$$J = -e^x \cos x + e^x \sin x - J$$

$$2J = -e^x \cos x + e^x \sin x$$

we obtain

$$= -e^x \cos x + e^x \sin x + C.$$

Example 3*

Calculate $J = \int x\sqrt{x}\, dx$. By putting:

$$\begin{cases} v' = x \\ u = \sqrt{x} \end{cases} \Longrightarrow \begin{cases} v = \dfrac{1}{2}x^2 \\ u' = \dfrac{1}{2\sqrt{x}} \end{cases}.$$

We get

$$J = \frac{1}{2}x^2\sqrt{x} - \int \frac{1}{2\sqrt{x}}\frac{1}{2}x^2 \, dx$$

$$= \frac{1}{2}x^2\sqrt{x} - \frac{1}{4}\int x\sqrt{x}$$

$$= \frac{1}{2}x^2\sqrt{x} - \frac{1}{4}J$$

$$J + \frac{1}{4}J = \frac{1}{2}x^2\sqrt{x}$$

we obtain

$$J = \frac{2}{5}x^2\sqrt{x} + C.$$

1.5 Special integration methods:

1.5.1 Integration of a rational function

Definition 1.8

Let P, Q be two real polynomials, $Q(x) \neq 0$. Function $x \to \frac{P(x)}{Q(x)}$ is called rational function or rational fraction

Definition 1.9

The functions $x \to \frac{A}{(x-a)^k}$, $x \to \frac{Mx+N}{(x^2+px+q)^k}$ where $k \in \mathbb{N}^*, a, A, M, N, p, q \in \mathbb{R}, p^2 - 4q < 0$, are called simple elements, of the first and second species respectively.

Theorem 1.12

Any rational fraction $\frac{P}{Q}$ is represented in the unique form $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$

the polynomials S, R being respectively the quotient and the remainder of the division of P by Q.

Theorem 1.13

Let $\deg P < \deg Q$,

$Q(x) = (x-a_1)^{m_1}(x-a_2)^{m_2} \ldots\ldots (x-a_k)^{m_k}.(x^2+p_1x+q_1)^{n_1}$

$(x^2+p_2x+q_2)^{n_2} \ldots\ldots (x^2+p_sx+q_s)^{n_s}$, $p_j^2 - 4q_j < 0, \forall\, 1 \le j \le s$.

Then the fraction $\frac{P}{Q}$ is represented in the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{(x-a_1)^2} + \cdots + \frac{A_{m_1}}{(x-a_1)^{m_1}}$$

$$+\frac{B_1}{x-a_2} + \frac{B_2}{(x-a_2)^2} + \cdots + \frac{B_{m_2}}{(x-a_2)^{m_2}}$$

$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{C_1}{x-a_k} + \frac{C_2}{(x-a_k)^2} + \cdots + \frac{C_{m_k}}{(x-a_k)^{m_k}}$$

$$+\frac{M_1^1x+N_1^1}{x^2+p_1x+q_1} + \frac{M_2^1x+N_2^1}{(x^2+p_1x+q_1)^2} + \cdots + \frac{M_{n_1}^1x+N_{n_1}^1}{(x^2+p_1x+q_1)^{n_1}}$$

$$+\frac{M_1^2x+N_1^2}{x^2+p_2x+q_2} + \frac{M_2^2x+N_2^2}{(x^2+p_2x+q_2)^2} + \cdots + \frac{M_{n_2}^2x+N_{n_2}^2}{(x^2+p_2x+q_2)^{n_2}}$$

$$+\cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{M_1^sx+N_1^s}{x^2+p_sx+q_s} + \frac{M_2^sx+N_2^s}{(x^2+p_sx+q_s)^2} + \cdots + \frac{M_{n_s}^sx+N_{n_s}^s}{(x^2+p_sx+q_s)^{n_2}}.$$

where A, B, C, M, N, p, q, a, are real canstants.

Examples

1) $\frac{P(x)}{Q(x)} = \frac{4x^4-4x^3-3x^2-12x+13}{(2x-1)^2(x-2)} = x + 2 + \frac{1}{x-2} + \frac{4}{2x-1} + \frac{-4}{(2x-1)^2}$.

$\frac{P(x)}{Q(x)} == x + 2 + \frac{12x^2-28x+17}{(2x-1)^2(x-2)} = S(x) + \frac{R(x)}{Q(x)}$. Where $\deg R < \deg Q$.So

$\frac{R(x)}{Q(x)} = \frac{1}{x-2} + \frac{4}{2x-1} + \frac{-4}{(2x-1)^2}$.

2) $\frac{P(x)}{Q(x)} = \frac{5x^7-x^6+6x^5+11x^4+29x^3+66x^2+29x+27}{(x-1)^3(x+2)^2(x^2+x+1)^2}$ where $\deg P < \deg Q$. So

$\frac{P(x)}{Q(x)} = \frac{1}{x-1} + \frac{-1}{(x-1)^2} + \frac{2}{(x-1)^3} + \frac{-2}{x+2} + \frac{3}{(x+2)^2} + \frac{x-1}{x^2+x+1} + \frac{2x+1}{(x^2+x+1)^2}$.

Integration of a rational fraction

To calculate the integral of a fraction $\frac{P(x)}{Q(x)}$, we first write this fraction as the sum of a polynomial and a finite number of rational fractions in the form $\frac{A}{(x-a)^k}$ or $\frac{Mx+N}{((x-\alpha)^2+\beta^2)^k}$

where k is a non zero natural number and $\beta, \alpha, N, M, A, a$ are real numbers, so the integral rational fractions returns to calculate integrals of the type $\int \frac{A}{(x-a)^k} dx$ and $\int \frac{Mx+N}{((x-\alpha)^2+\beta^2)^k} dx$.

Calculate the integral $\int \frac{A}{(x-a)^k} dx$

$$\int \frac{1}{x-a} dx = ln|x-a| + C$$

$$\forall\, k > 1: \int \frac{1}{(x-a)^k} dx = \frac{-1}{(k-1)(x-a)^{k-1}} + C.$$

Calculate the integral $\int \frac{Mx+N}{((x-\alpha)^2+\beta^2)^k} dx$.

Calculating this integral after changing the variable $x = \alpha + \beta t$ leads to calculating integrals of two types: $I_k = \int \frac{t}{(1+t^2)^k} dt$ And $J_k = \int \frac{1}{(1+t^2)^k} dt$, where we have:

$$I_1 = \frac{1}{2} ln(1+t^2) + C \quad \text{and} \quad \forall\, k > 1: \; I_k = \frac{-1}{2(k-1)(1+t^2)^{k-1}} + C.$$

As for the integration $J_k = \int \frac{1}{(1+t^2)^k} dt$, we use integration by parts and obtain the following recurrence relation:

$$J_1 = Arctanx + C \quad \text{and} \quad \forall\, k \geq 1: \; 2kJ_{k+1} = (2k-1)J_k + \frac{t}{(1+t^2)^k} \ldots\ldots(*)$$

Example 1

Calculate the integral $I = \int \frac{2x^4-x^3+2x^2-1}{(x^2+1)(x-1)} dx$.

By Euclidean division we get:

$$\frac{2x^4-x^3+2x^2-1}{x^3-x^2+x-1} = 2x+1+\frac{x^2+x}{(x^2+1)(x-1)}.$$

We put $\frac{x^2+x}{(x^2+1)(x-1)} = \frac{Mx+N}{x^2+1} + \frac{A}{x-1}$ we get $M = 0, N = 1, A = 1$.

So

$$I = \int \left(2x+1+\frac{1}{x^2+1}+\frac{1}{x-1}\right) dx$$

$$= x^2 + x + Arctanx + ln|x-1| + C.$$

Example 2

Calculate the integral $J = \int \frac{x^2-6x+11}{(x+1)(x-2)^2} dx$.

We put $\frac{x^2-6x+11}{(x+1)(x-2)^2} = \frac{a}{x+1} + \frac{b}{x-2} + \frac{c}{(x-2)^2}$ we get $a = 2, b = -1, c = 1$.

So

$$J = \int \left(\frac{2}{x+1} - \frac{1}{x-2} + \frac{1}{(x-2)^2}\right) dx$$

$$= 2\ln(x+1) - \ln(x-2) - \frac{1}{x-2} + C.$$

Example 3*

Calculate the integral $J = \int \frac{8x^6-8x^5+2x^4+23x^3-15x^2+7x+2}{(x+1)^2(2x^2-2x+1)^2} dx$

By Euclidean division we get:

$$\frac{8x^6-8x^5+2x^4+23x^3-15x^2+7x+2}{(x+1)^2(2x^2-2x+1)^2} = 2 + \frac{-8x^5+10x^4+15x^3-17x^2+11x}{(x+1)^2(2x^2-2x+1)^2}.$$

We put

$$\frac{-8x^5+10x^4+15x^3-17x^2+11x}{(x+1)^2(2x^2-2x+1)^2} = \frac{a}{x+1} + \frac{b}{(x+1)^2} + \frac{cx+d}{2x^2-2x+1} + \frac{ex+f}{(2x^2-2x+1)^2}$$

we get:

$$a = -2, b = -1, c = 0, d = 3, e = 1, f = 0.$$

So

$$I = \int \left(2 + \frac{-2}{x+1} + \frac{-1}{(x+1)^2} + \frac{3}{2x^2-2x+1} + \frac{x}{(2x^2-2x+1)^2}\right) dx$$

$$I = 2x - 2ln|x+1| + \frac{1}{x+1} + \int \left(\frac{3}{2x^2-2x+1} + \frac{x}{(2x^2-2x+1)^2}\right) dx.$$

Since $2x^2 - 2x + 1 = 2\left(\left(x-\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2\right)$, to calculate the integral on the second side we put $x = \frac{1}{2} + \frac{1}{2}t$.

So

$$\int \left(\frac{3}{2x^2-2x+1} + \frac{x}{(2x^2-2x+1)^2}\right) dx = 3\int \frac{1}{t^2+1} dx + \int \frac{t+1}{(t^2+1)^2} dx$$

$$= 3Arc\,tan\,x + \int \frac{t}{(t^2+1)^2} dx + \int \frac{1}{(t^2+1)^2} dx$$

$$\int \left(\frac{3}{2x^2-2x+1} + \frac{x}{(2x^2-2x+1)^2}\right) dx = 3\mathrm{Arc}\tan t - \frac{1}{2}\frac{1}{t^2+1} + \underbrace{\int \frac{1}{(t^2+1)^2} dx}_{J_2}$$

Substituting $k = 1$ in the regressive relationship $(*)$ we get: $2J_2 = J_1 + \frac{t}{(1+t^2)^1} = \mathrm{Arc}\tan t + \frac{t}{1+t^2}$ and from there $J_2 = \frac{1}{2}\mathrm{Arc}\tan t + \frac{t}{2(t^2+1)}$.

So

$$\int \left(\frac{3}{2x^2-2x+1} + \frac{x}{(2x^2-2x+1)^2}\right) dx = \frac{7}{2}\mathrm{Arc}\tan t + \frac{t-1}{2(t^2+1)}.$$

Substituting $t = 2x - 1$ we get

$$\int \left(\frac{3}{2x^2-2x+1} + \frac{x}{(2x^2-2x+1)^2}\right) dx = \frac{7}{2}\mathrm{Arc}\tan(2x-1) + \frac{x-1}{2(2x^2-2x+1)}.$$

So

$$I = 2x - 2ln|x+1| + \frac{1}{x+1} + \frac{7}{2}\mathrm{Arc}\tan(2x-1) + \frac{x-1}{2(2x^2-2x+1)} + C.$$

1.5.2 Integration of the type $\int \boldsymbol{R(sinx, cosx)\, dx}$:

Where $\boldsymbol{R(sinx, cosx)}$ is a rational fraction in the variables x and y.

This integral can be converted to a rational fractional integral using the change in the variable $t = \tan\frac{x}{2}$, where:

$$\cos x = \frac{1-t^2}{1+t^2} \;\; ; \;\; \sin x = \frac{2t}{1+t^2} \;\; ; \;\; dx = \frac{2}{1+t^2}dt.$$

Example 1 Calculate the integral $J = \int \frac{cos^2 x}{5-4sin\ x}dx$

by putting $t = tan\frac{x}{2}$ we get

$$J = \int \frac{\left(\frac{1-t^2}{1+t^2}\right)^2}{5-4\left(\frac{2t}{1+t^2}\right)}\left(\frac{2}{1+t^2}\right)dt = \int \frac{2(t^2-1)^2}{(5t^2-8t+5)(t^2+1)^2}dt.$$

we put

$$\frac{2(t^2-1)^2}{(5t^2-8t+5)(t^2+1)^2} = \frac{at+b}{t^2+1} + \frac{ct+d}{(t^2+1)^2} + \frac{et+f}{5t^2-8t+5}.$$

on obtain $a = 0\ ,\ b = \frac{5}{8}\ ,\ c = 1\ ,\ d = 0\ ,\ e = 0\ ,\ f = -\frac{9}{8}$.

So

$$J = \int \frac{\frac{5}{8}}{t^2+1} + \frac{t}{(t^2+1)^2} + \frac{-\frac{9}{8}}{5t^2-8t+5}dt = \frac{5}{8}\text{Arc}\tan t - \frac{1}{2(t^2+1)} + \underbrace{\int \frac{-\frac{9}{8}}{5t^2-8t+5}dt}_{I}.$$

Calculate the integral I:

Since $5t^2 - 8t + 5 = 5\left((\text{t} - (\frac{4}{5}))^2 + (\frac{3}{5})^2\right)$, we put $t = \frac{4}{5} + \frac{3}{5}y$ so

$$I = -\frac{3}{8}\int \frac{1}{y^2+1}dy = -\frac{3}{8}\text{Arc}\tan y = -\frac{3}{8}\text{Arc}\tan\left(\frac{5}{3}t - \frac{4}{3}\right).$$

And

$$J = \frac{5}{8}\text{Arc}\tan t - \frac{1}{2(t^2+1)} - \frac{3}{8}\text{Arc}\tan\left(\frac{5}{3}t - \frac{4}{3}\right).$$

Substituting $t = tan\frac{x}{2}$ we get

$$J = \frac{5}{16}x - \frac{1}{2}\cos^2\frac{x}{2} - \frac{3}{8}\text{Arc}\tan\left(\frac{5}{3}\tan\frac{x}{2} - \frac{4}{3}\right) + C.$$

1.5.3 Integration of the type $\int \boldsymbol{R}\left(\boldsymbol{x}, \left(\frac{ax+b}{cx+d}\right)^{\frac{m}{n}}, \left(\frac{ax+b}{cx+d}\right)^{\frac{p}{q}}, \dots\dots, \left(\frac{ax+b}{cx+d}\right)^{\frac{r}{s}}\right)\boldsymbol{dx}$

Where $\boldsymbol{R}(\boldsymbol{x}, y, \dots\dots, z)$ is a rational fraction in the variables $x\ , y, \dots\dots, z$ and $\frac{m}{n}, \frac{p}{q}, \dots\dots, \frac{r}{s}$ are rational numbers.To calculate this type of integration, we use a change in the variable $t = \left(\frac{ax+b}{cx+d}\right)^{\frac{1}{k}}$, where k is the Least Common Multiple (LCM) of the numbers $n, q, \dots\dots\dots, s$.

Example 1 calculate the integral $I = \int \frac{1+\sqrt{x+1}}{\sqrt[3]{x+1}} dx$.

We put $t = (x+1)^{\frac{1}{6}}$ and from it $x = t^6 - 1$ and $dx = 6t^5 dt$ so

$$I = \int \frac{1+t^3}{t^2} 6t^5 dt = 6\int t^6 + t^3 dt = \frac{6}{7}t^7 + \frac{3}{2}t^4 + C.$$

So

$$I = \frac{6}{7}(x+1)^{\frac{7}{6}} + \frac{3}{2}(x+1)^{\frac{4}{6}} + C.$$

Example 2 calculate the integral $J = \int x\sqrt{\frac{x-1}{x+1}} dx$.

We put $t = \sqrt{\frac{x-1}{x+1}}$ and from it $x = \frac{-t^2-1}{t^2-1}$ and $dx = \frac{4t}{(t^2-1)^2} dt$ so

$$J = \frac{-t^2-1}{t^2-1} t \frac{4t}{(t^2-1)^2} dt = \int \frac{-4(t^4+t^2)}{(t^2-1)^3} dx.$$

We put

$$\frac{-4(t^4+t^2)}{(t^2-1)^3} = \frac{a}{t-1} + \frac{b}{(t-1)^2} + \frac{c}{(t-1)^3} + \frac{d}{t+1} + \frac{e}{(t+1)^2} + \frac{f}{(t+1)^3}$$

we get

$$a = -\frac{1}{2}, \quad b = -\frac{3}{2}, \quad c = -1, \quad d = \frac{1}{2}, \quad e = -\frac{3}{2}, \quad f = 1$$

so

$$J = \int \frac{-\frac{1}{2}}{t-1} + \frac{-\frac{3}{2}}{(t-1)^2} + \frac{-1}{(t-1)^3} + \frac{\frac{1}{2}}{t+1} + \frac{-\frac{3}{2}}{(t+1)^2} + \frac{1}{(t+1)^3} dt$$

$$= -\frac{1}{2}\ln|t-1| + \frac{3}{2(t-1)} + \frac{1}{2(t-1)^2} + \frac{1}{2}\ln|t+1| + \frac{3}{2(t+1)} - \frac{1}{2(t+1)^2} + C.$$

Substituting $t = \sqrt{\frac{x-1}{x+1}}$ we get

$$J=\frac{1}{2}\ln\left|\frac{\sqrt{\frac{x-1}{x+1}}+1}{\sqrt{\frac{x-1}{x+1}}-1}\right|+\left(\frac{1}{2}x^2-\frac{1}{2}x\ -1\right)\sqrt{\frac{x-1}{x+1}}+C.$$

1.5.4 Integration of the type $\int\sqrt{\boldsymbol{ax^2+bx+c}}\,\boldsymbol{dx}$

After writing the trinomial $\boldsymbol{ax^2+bx+c}$ in canonical form, this integral takes one of the following forms:

$$\int\sqrt{(x-\alpha)^2+\beta^2}\,dx,\int\sqrt{(x-\alpha)^2-\beta^2}\,dx \text{ and } \int\sqrt{\beta^2-(x-\alpha)^2}\,dx.$$

To calculate the integral $\int\sqrt{(x-\alpha)^2+\beta^2}\,dx$, we use a change in the variable $x-\alpha=\beta\sinh t$.

To calculate the integral $\int\sqrt{(x-\alpha)^2-\beta^2}\,dx$, we use a change in the variable $x-\alpha=\pm\beta\cosh t$.(According to the interval of integration).

To calculate the integral $\int\sqrt{\beta^2-(x-\alpha)^2}\,dx$, we use a change in the variable $x-\alpha=\beta\cos t$.(or $x-\alpha=\beta\sin t$).

Example 1 Calculate the integral $L=\int\sqrt{x^2+4x+3}\,dx$.

We have $x^2+4x+3=(x+2)^2-1$ and from there

If $x+2\leq-1$ (i.e. if $x\in\,]-\infty,-3]$) we put $x+2=-\cosh t$ where $t\in[0,+\infty[$.

If $x+2\geq1$ (i.e. if $x\in[-1,+\infty[$) we put $x+2=\cosh t$ where $t\in[0,+\infty[$.

For $x\in\,]-\infty,-3]\cup[-1,+\infty[$ then $x+2=\mp\cosh t$ and $dx=\mp\sinh t\,dt$.

So

$$L=\int\sqrt{\cosh^2 t-1}\,(\mp\sinh t)dt=\int\sqrt{\sinh^2 t}\,(\mp\sinh t)dt=\int\mp\sinh^2 t\,dt$$

$$=\frac{1}{2}\int\mp(-\cosh 2t+1)\,dt=\mp\left(-\frac{1}{4}\sinh 2t+\frac{1}{2}t\right)=\mp\left(-\frac{1}{2}\cosh t\sinh t+\frac{1}{2}t\right)$$

$$=\mp\frac{1}{2}\left[\mp(x+2)\sqrt{(x+2)^2-1}\right]\mp\frac{1}{2}\operatorname{Arg}\cosh[\mp(x+2)]$$

$$=\frac{1}{2}(x+2)\sqrt{x^2+4x+3}\pm\frac{1}{2}\ln\left|\mp(x+2)+\sqrt{x^2+4x+3}\right|+C.$$

$$=\frac{1}{2}(x+2)\sqrt{x^2+4x+3}-\frac{1}{2}\ln\left|x+2+\sqrt{x^2+4x+3}\right|+C.$$

(Note that $-\frac{1}{2}\ln\left|(x+2)+\sqrt{x^2+4x+3}\right|=\frac{1}{2}\ln\left|-(x+2)+\sqrt{x^2+4x+3}\right|$).