Chapter 02

1 Optimization with equality constraints

Optimization with equality constraints refers to the process of finding the maximum or minimum value of a function subject to certain constraints, where these constraints are expressed as equalities. This type of optimization problem is commonly encountered in various fields such as engineering, economics, and physics.

Mathematically, an optimization problem with equality constraints can be formulated as follows:

Minimize or maximize:

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Mathematically, an optimization problem with equality constraints can be formulated as follows:

Minimize or maximize: $f(x)$

Subject to equality constraints: $g_i(x) = 0,\ \ i = 1...m$

Where

$f(x)$ is the objective function or cost function to be optimized.

x is a vector of decision variables.

$g_i(x)$ are the equality constraint functions.

m is the number of equality constraints.

- Other way:

$$\boxed{\begin{cases} \min f(x) \\ s.t \\ \Omega = \{x \in \mathbb{R}^n \diagup g_i(x) = 0,\ \ i = 1...m\} \end{cases}}$$

Let's consider a simple examples of optimization with equality constraints:

$$\begin{cases} \min f(x,y) = \min xy \\ s.t \\ \Omega = \left\{(x,y) \in \mathbb{R}^2 \diagup g(x,y) = x^2 + y^2 - 1 = 0, \right\} \end{cases}$$

$$\begin{cases} \min f(x,y) = \min x^2 + y^2 \\ s.t \\ \Omega = \left\{(x,y) \in \mathbb{R}^2 \diagup g(x,y) = x + y - 1 = 0 \ \right\} \end{cases}$$

$$\begin{cases} \min f(x,y,z) = \min x + y + z \\ s.t \\ \Omega = \left\{(x,y,z) \in \mathbb{R}^3 \diagup g(x,y) = x^2 + y^2 + z^2 - 1 = 0, \ \right\} \end{cases}$$

$$\begin{cases} \max f(x,y) = \max xy \\ s.t \\ \Omega = \left\{(x,y) \in \mathbb{R}^2 \diagup g(x,y) = x + y - 6 = 0, \ \right\} \end{cases}$$

1.1 Constraint set:

Constraint qualification (CQ) is a fundamental concept in mathematical optimization, particularly in the context of constrained optimization problems. It ensures that certain conditions are satisfied at a feasible solution, which is essential for the validity of optimality conditions and the convergence of optimization algorithms.

Example 1 *Lets consider the following constraints*

$$\Omega = \{x \in \mathbb{R}^n \diagup g_i(x) = 0,\ i = 1...m\}$$

The feasible region or the constraints set called Constraint qualification with equality constraints.

1.2 Local Maximum:

A point $x^* \in \Omega$ is said to be a point of local maximum of f subject to the constraints $g(x) = 0$; if there exists an open ball around x^*; $B_\varepsilon(x^*)$; such that $f(x^*) \geq f(x)$ for all $x \in B_\varepsilon(x^*) \cap \Omega$.

1.3 Global Maximum:

A point $x^* \in \Omega$ is said to be a point of global maximum of f subject to the constraints $g(x) = 0$; if $f(x^*) \geq f(x)$ for all $x \in \Omega$.

Remark 2 *Local minimum and global minimum can be defined similarly by just reverting the inequalities.*

1.4 The Constraint Qualification:

The condition

$$\nabla g(x) \neq 0$$

is known as the constraint **qualification**.

Notation 3 *It is important to check the constraint qualification before applying the theorem of resolution.*

1.5 Active and Inactive Constraints

An optimal solution that lies at the intersection point of two constraints causes both of those constraints to be considered active.

At the stationary point

$x = x^*$, some of the constraints $g_i(x^*) = 0$. These constraints are called active constraints.

or

The ith constraint is said to be active (at a solution y) if $g_i(y) = 0$

1.6 The method of Lagrange multipliers

In optimization, the method of **Lagrange multipliers** is a powerful technique used to solve constrained optimization problems with equality constraints, which can be represented by functions of the form $g(x) = 0$.

Consider an optimization problem with an objective function $f(x)$ subject to equality constraints $g(x) = 0$.

Minimize $f(x)$subject to $g(x) = 0$

The Lagrangian function

$L(x, \lambda)$ is defined as:

$$L(x, \lambda) = f(x) + \lambda g(x)$$

where

λ is a Lagrange multiplier associated with the constraint

$$g(x) = 0.$$

The critical points of

$L(x, \lambda)$ are found by taking partial derivatives with respect to each variable x_i and λ, and setting them equal to zero:

$$\begin{cases} \frac{\partial L}{\partial x_i} = 0 \\ \frac{\partial L}{\partial \lambda} = 0 \end{cases}$$

Solving these equations simultaneously gives the critical points (x^*, λ^*)

These critical points correspond to The solutions of the constrained optimization problem.

However, not all critical points are valid solutions. The critical points must satisfy the original equality constraint $g(x^*) = 0$ and, in some cases, second-order conditions may need to be checked for optimality.

In summary, the method of Lagrange multipliers extends optimization to problems with equality constraints by introducing Lagrange multipliers, which allow us to find critical points that satisfy both the objective function and the equality constraints.

Problem 4 *Minimize $f(x, y) = x^2 + 2y^2$ under the constraint*

$$g(x, y) = x + y^2 - 1.$$

Problem 5 *Find the shortest distance from the origin to the curve*

$$x^6 + 2y^2 = 4.$$

Problem 6 *Which cylindrical soda cans of height h and radius r has minimal surface for fixed volume?*

Problem 7 *Find the extrema of $f(x, y, z) = z$ on the sphere*

$$g(x, y, z) = x^2 + y^2 + z^2 = 1.$$

Problem 8 *Find the dimensions of the box with the largest volume if the total surface area is* 64 cm^2

Remark 9 *The Lagrange Multiplier Theorem is a fundamental result in mathematical optimization that provides necessary conditions for constrained optimization problems. It helps identify critical points where the objective function is optimized subject to equality constraints.*

The method of Lagrange multipliers is a technique in mathematics to find the local maxima or minima of a function $f(x)$ subject to constraints $g(x) = 0$.

The theorem can be stated as follows:

Theorem 10 *(*Lagrange Multiplier Theorem*):*

Under the conditions of the establishment of the problem where the constraints are qualified, if in addition $f \in C^1(\Omega)$*,and the function* f *to have an extremum relative conditioned at the point ,*x^**, then, there exist* m *real numbers such that*

$$\nabla L(x^*) = 0.$$

where,

$$L(x) = f(x) + \sum_{i=0}^{M} \lambda_i g_i(x)$$

$(\lambda_1, ..., \lambda_m)$ *are called Lagrange multipliers.*

Proof. For $n = 2$, we have $f(x, y) \in C^1(\Omega)$, (x^*, y^*) extremum subject to $g(x, y) = 0$, $g(x, y) \in C^1$ and $\nabla g(x^*, y^*) \neq 0$, then $\exists \lambda$ st

$$\begin{cases} \nabla_{x,y} L(x^*, y^*) = \nabla f(x^*, y^*) + \lambda \nabla g(x^*, y^*) = 0 \\ g(x, y) = 0 \end{cases}$$

- $\nabla g(x^*, y^*) \neq 0$,assume that $\frac{\partial g}{\partial y}(x^*, y^*) \neq 0$, also $g(x^*, y^*) = 0$ and $g \in C^1$

By the implicit function theorem there is a function $y = y(x)$ such that $g(x, y(x)) = 0$ and furthermore

$$y'(x) = -\frac{g_x}{g_y}$$

- At (x^*, y^*) the function $f(x, y)$ has a local extremum

$\Rightarrow f(x, y(x))$ has a local extremum

$$\begin{cases} f_x + f_y y'(x) = 0 \\ y'(x) = -\frac{g_x}{g_y} \end{cases}$$

$\Rightarrow$ at (x^*, y^*)

$$f_x - f_y \frac{g_x}{g_y} = 0$$

Denote

$$\frac{f_y}{g_y} = -\lambda$$

So, we have

$$\begin{cases} f_y + \lambda g_y = 0 \\ f_x + \lambda g_x = 0 \end{cases}$$

Then

$$\nabla_{x,y} L(x^*, y^*) = \nabla f(x^*, y^*) + \lambda \nabla g\,(x^*, y^*) = 0$$

■

Corollary 11 For $n = 3$ and $m = 2$, *we have* $f(x, y, z) \in C^1(\Omega)$, (x^*, y^*, z^*) *extremum subject to* $g(x, y, z) = 0$ *and* $h(x, y, z) = 0$,

$h(x, y, z) \in C^1$, $g(x, y, z) \in C^1$ *and* $\nabla h(x, y, z)$ *and* $\nabla g(x, y, z)$ *are linearly independant at* (x^*, y^*, z^*).

Then, $\exists \lambda$ *and* μ *st*

$$\begin{cases} \nabla_{x,y,z} L(x^*, y^*, z^*) = \nabla f(x^*, y^*, z^*) + \lambda \nabla g\,(x^*, y^*, z^*) + \mu \nabla h(x^*, y^*, z^*) = 0 \\ g(x, y, z) = 0 \\ h(x, y, z) = 0 \end{cases}$$

Example 12 *Lets the following problem*

$$\begin{cases} opt\ f(x, y, z) = opt\ x + y + z \\ s.t \\ g(x, y, z) = x^2 + y^2 - 4 = 0 \\ h(x, y, z) = x + z - 2 = 0 \end{cases}$$

We have $f, g, h \in C^1$ *and* $\nabla g(x, y, z) = \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix}$, $\nabla h(x, y, z) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\nabla h(x, y, z)$ *and* $\nabla g(x, y, z)$ *are linearly independant unless* $x = y = 0$
but $x = y = 0 \notin \Omega$ *with* Ω *is a feasible set*
Then, $\exists \lambda$ *and* μ *st*

$$\begin{cases} f_x + \lambda g_x + \mu g_x = 0 \\ f_y + \lambda g_y + \mu g_y = 0 \\ f_z + \lambda g_z + \mu g_z = 0 \\ g(x, y, z) = 0 \\ h(x, y, z) = 0 \end{cases}$$

$\Rightarrow$

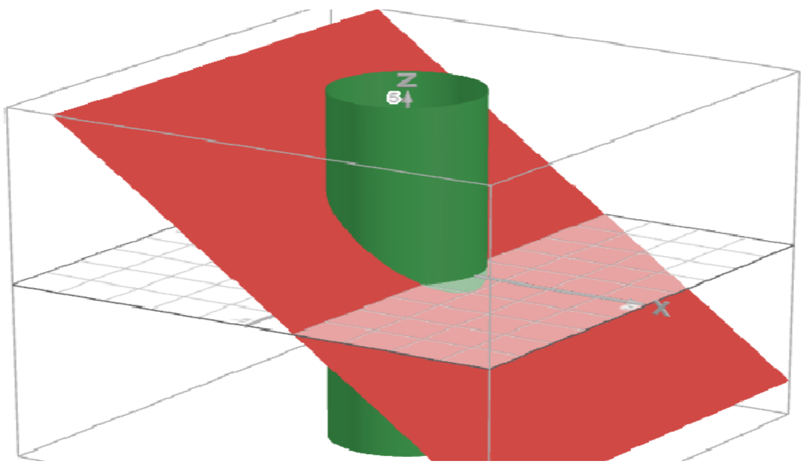
$$\begin{cases} 1 + 2\lambda x + \mu = 0 \\ 1 + 2\lambda y = 0 \\ 1 + \mu = 0 \\ g(x, y, z) = x^2 + y^2 - 4 = 0 \\ h(x, y, z) = x + z - 2 = 0 \end{cases} \Rightarrow \begin{cases} \lambda x = 0 \\ \lambda y = -\frac{1}{2} \\ \mu = -1 \\ x^2 + y^2 = 4 \\ x + z = 2 \end{cases}$$

Then

$$\begin{cases} x = 0 \\ \lambda y = -\frac{1}{2} \\ \mu = -1 \\ y = \pm 2 \\ z = 2 \end{cases}$$

So, we have two critical points

$$\begin{aligned} (x_1, y_1, z_1, \lambda_1, \mu_1) &= (0, 2, 2, -\frac{1}{4}, -1) \\ & \textit{and} \\ (x_2, y_2, z_2, \lambda_2, \mu_2) &= (0, -2, 2, \frac{1}{4}, -1) \end{aligned}$$



g(x,y,z) in green and h(x,y,z) in red

Finally, $f(x_1, y_1, z_1) = 4$ *and* $f(x_2, y_2, z_2) = 0$

The intersection between $g = 0$ *and* $h = 0$ *is the intersection between cyllinder and plane and the result is closed and bounded.*

- The functin $f \in C^1(\Omega)$ *such that* Ω *is a compact set, hence by Weiestrasse* f *has a min and max subject to this constraint*

$$\begin{aligned} \max f(x, y, z) &= f(x_1, y_1, z_1) = 4 \\ & \textit{and} \\ \min f(x, y, z) &= f(x_2, y_2, z_2) = 0 \end{aligned}$$