Module: Optimization with constraints
Master 1
Lecturer: Pr. Taieb Hamaizia

1 Chapter 01:

1.1 Introduction in optimization with constraints

Optimization with constraints is a fundamental concept in mathematics, engineering, economics, and various other fields. It involves the process of maximizing or minimizing a function subject to certain constraints. These constraints can represent limitations on the variables involved in the optimization problem.

The general form of an optimization problem with constraints can be expressed as:

Minimize (or maximize)

$$f(x_1, x_2, ..., x_n)$$

Subject to:

$g_1(x_1, x_2, ..., x_n) \leq 0$

$g_2(x_1, x_2, ..., x_n) \leq 0$

...

$g_m(x_1, x_2, ..., x_n) \leq 0$

Where:

f is the objective function that you want to minimize or maximize.

$x_1, x_2, ..., x_n$ are the variables that you can adjust to optimize the objective function.

$g_1, g_2, ..., g_m$ are the constraint functions that define the limitations or conditions that the decision variables must satisfy.

The goal is to find the values of the decision variables $(x_1, x_2, ..., x_n)$ that optimize the objective function (f) while still satisfying all the constraints $(g_1, g_2, ..., g_m)$.

1.2 Types of problems

Optimization problems with constraints can be classified into several types based on various characteristics.

1.2.1 Linear Programming (LP):

In linear programming, both the objective function and the constraints are linear.

Example:

$$\begin{cases} \min f(x,y,z) = \min x + y - z \\ st \\ g_1(x,y,z) = 2x + y = 0 \\ g_2(x,y,z) = x + 3z = 1 \end{cases}$$

1.2.2 Nonlinear Programming (NLP):

In nonlinear programming, either the objective function or the constraints (or both) are nonlinear.

Example:

$$\begin{cases} \min f(x) = \min 10x \;-\; (0.01x^3 + 50) \\ st \\ x \leq 1000 \\ x \leq 800 \end{cases}$$

1.2.3 Quadratic Programming (QP):

In quadratic programming, the objective function is quadratic, and the constraints can be linear or quadratic.

Example:

$$\begin{cases} \min f(x,y) = \min x^2 - y^2 \\ st \\ x + y \leq 100 \\ 2x + y \leq 240 \end{cases}$$

1.2.4 Convex Optimization:

Convex optimization deals with optimization problems where the objective function and the feasible region (defined by constraints) are convex.

Example:

$$\begin{cases} \min f(x,y) = \min 100x + 120y - 0.5x^2 - 0.8y^2 \\ st \\ x + y \leq 100 \\ 2x + 3y \leq 240 \end{cases}$$

1.2.5 Stochastic Optimization:

Stochastic optimization deals with optimization problems where some parameters are uncertain or subject to randomness.

1.3 Constraints

Constraints in optimization refer to the conditions or limitations that must be satisfied when finding the optimal solution to a problem. These constraints define the feasible region, which is the set of all possible solutions that meet the problem's requirements.

Constraints can be classified into several types, including:

1.3.1 Equality Constraints:

Constraints that require certain expressions to be equal.

$$g(x) = 0$$

1.3.2 Inequality Constraints:

Constraints that require certain expressions to be less than or equal to or greater than or equal to some values.

$$\begin{aligned} h(x) &\leq 0 \\ & or \\ g(x) &\geq 0 \end{aligned}$$

1.3.3 Bound Constraints:

Constraints that limit the variables to specific ranges.

$$x_{min} \leq x \leq x_{max}$$

1.3.4 Linear Constraints:

Constraints that involve linear combinations of variables.

$$\sum_{i=0}^{n} a_i x_i \leq b$$

1.3.5 Nonlinear Constraints:

Constraints that involve nonlinear functions of the variables.

$$f(x) \leq 0$$

1.3.6 Integer Constraints:

Constraints that restrict the variables to integer values.

$$x_i \in Z$$

1.3.7 Discrete Constraints:

Constraints that restrict the variables to discrete values from a finite set.

$$x_i \in \{a_1, a_2, ..., a_k\}$$

Remark 1 *Constraints play a crucial role in optimization problems as they define the boundaries within which the optimal solution must lie.*

1.4 How to determine the constraints

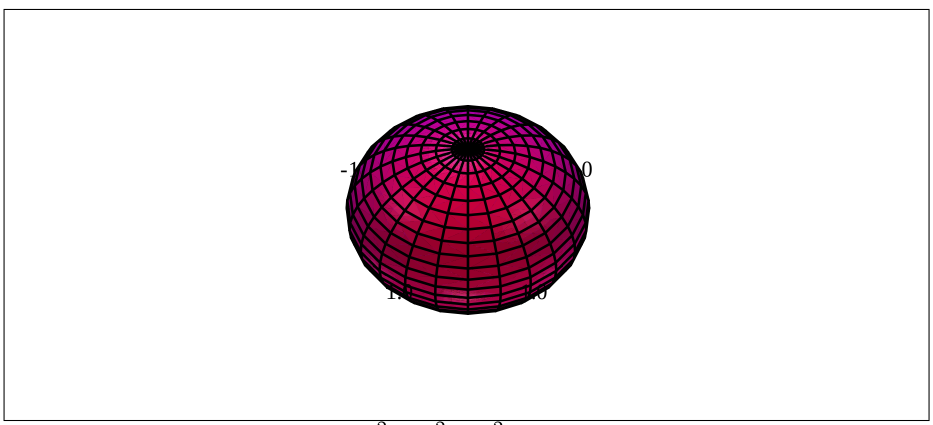
Determining the constraints in an optimization problem involves identifying the limitations or conditions that the solution must satisfy

- Determine the feasible region by considering the intersection of all constraints.

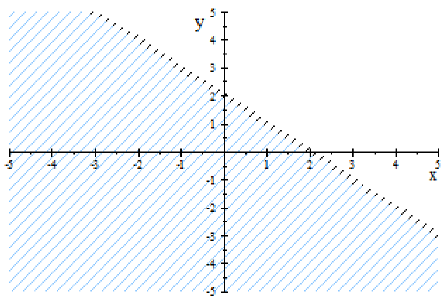
In higher dimensions, this may involve finding the intersection of planes, surfaces, or hyperplanes defined by the constraints.

If the problem is in two dimensions, find the intersection points of the constraint lines or curves. These intersection points represent potential vertices of the feasible set.

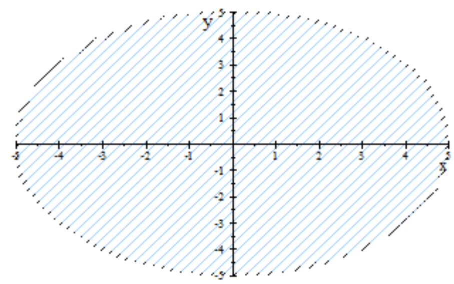
Examples



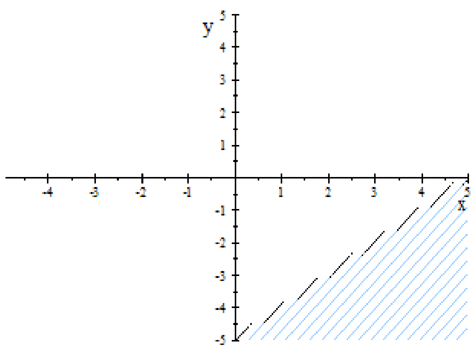
$x^2 + y^2 + z^2 \leq 1$



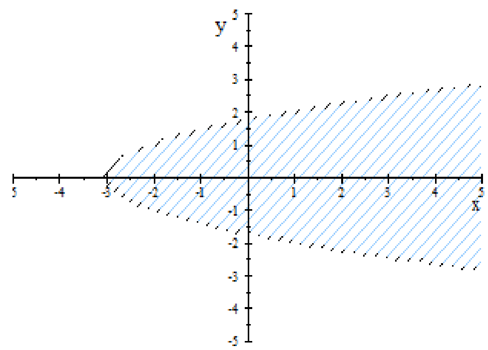
$x + y \leq 2$



$x^2 + y^2 \leq 25$



$x - y \geq 5$



$x - y^2 + 3 \geq 0$