Corrigé type de l'examen: Complément de la Théorie des Probabilités (2023/2024)

Exercice 1 (14 points).

1. On a

$$\int_0^{\frac{9}{5}}\int_x^{\frac{9}{5}}\frac{\alpha}{\sqrt{xy}}dydx=\alpha\int_0^{\frac{9}{5}}\frac{1}{\sqrt{x}}\left(\int_x^{\frac{9}{5}}\frac{1}{\sqrt{y}}dy\right)dx=1. \quad \textbf{(0.5)}$$

Donc $\alpha=\dfrac{5}{18}$ **(1)** et

$$f_{XY}(x,y)=\frac{5}{18\sqrt{xy}}\mathbf{1}_{]0,\frac{9}{5}[}(x)\,\mathbf{1}_{[x,\frac{9}{5}[}(y).$$

2. La densité marginale de X est

$$\begin{aligned} f_X(x) &= \frac{5}{18}\int_x^{\frac{9}{5}}\frac{1}{\sqrt{x}\sqrt{y}}dy \quad \textbf{(0.5)}\\ &= \frac{\sqrt{5}}{3\sqrt{x}}-\frac{5}{9}. \quad \textbf{(1)}\end{aligned}$$

. La densité marginale de Y est

$$\begin{aligned} f_Y(y) &= \frac{5}{18}\int_0^{y}\frac{1}{\sqrt{x}\sqrt{y}}dx \quad \textbf{(0.5)}\\ &= \frac{5}{9}. \quad \textbf{(0.5)}\end{aligned}$$

3. La covariance est

$$cov(X,Y)=E(XY)-E(X)E(Y). \quad \textbf{(0.25)}$$

On a

$$\begin{aligned} E(X) &= \int_0^{\frac{9}{5}} x\left(\frac{\sqrt{5}}{3\sqrt{x}}-\frac{5}{9}\right)dx \quad \textbf{(0.5)}\\ &= \frac{3}{10}, \quad \textbf{(0.5)}\end{aligned}$$

$$\begin{aligned} E(Y) &= \frac{5}{9}\int_0^{\frac{9}{5}} ydy \quad \textbf{(0.5)}\\ &= \frac{9}{10}, \quad \textbf{(0.5)}\end{aligned}$$

et

$$\begin{aligned} E(XY) &= \frac{5}{18}\int_0^{\frac{9}{5}}\frac{x}{\sqrt{x}}\left(\int_x^{\frac{9}{5}}\frac{y}{\sqrt{y}}dy\right)dx \quad \textbf{(0.5)}\\ &= \frac{9}{25}. \quad \textbf{(0.5)}\end{aligned}$$

Donc

$$cov(X,Y)=\frac{9}{25}-\frac{27}{100}=\frac{9}{100}. \quad \textbf{(0.25)}$$

On a

$$E\left(X \mid Y=y\right)=\int_0^y x f_{X|Y}\left(x,y\right)dx, \qquad \textbf{(0.5)}$$

où

$$f_{X|Y}\left(x,y\right)=\frac{\frac{5}{18\sqrt{x}\sqrt{y}}}{\frac{5}{9}}=\frac{1}{2\sqrt{x}\sqrt{y}}. \qquad \textbf{(0.5)}$$

Donc

$$E\left(X \mid Y=y\right)=\int_0^y \frac{x}{2\sqrt{x}\sqrt{y}}dx=\frac{1}{3}y. \qquad \textbf{(0.5)}$$

4. Méthode 1. On a

$$Z=2E\left(X \mid Y\right)-3=\frac{2}{3}Y-3, \qquad \textbf{(0.25)}$$

et

$$E\left(Z\right)=g_Z'\left(s\right)\big|_{s=0}, \qquad \textbf{(0.5)}$$

où

$$g_Z\left(s\right)=e^{-3s}g_Y\left(\frac{2}{3}s\right) \quad \textbf{(0.25)} \Longrightarrow g_Z'\left(s\right)=-3e^{-3s}g_Y\left(\frac{2}{3}s\right)+e^{-3s}g_Y'\left(\frac{2}{3}s\right), \qquad \textbf{(0.5)}$$

avec

$$g_Y\left(\frac{2}{3}s\right)=\int_0^{9/5}\frac{5}{9}e^{\frac{2}{3}sy}dy=\frac{5e^{\frac{6}{5}s}-5}{6s}, \qquad \textbf{(0.5)}$$

alors

$$g_Y'\left(\frac{2}{3}s\right)=\frac{\left(6s-5\right)e^{\frac{6}{5}s}+5}{6s^2}. \qquad \textbf{(0.5)}$$

En utilisant la règle de l'Hôpital, on obtient

$$\begin{aligned} g_Z'\left(s\right)\big|_{s=0} &= -3e^{-3s}\, g_Y\left(\frac{2}{3}s\right)\Big|_{s=0}+e^{-3s}\, g_Y'\left(\frac{2}{3}s\right)\Big|_{s=0} \\ &= -3+\frac{3}{5} \qquad \textbf{(0.5)} \\ &= -\frac{12}{5}, \qquad \textbf{(0.25)} \end{aligned}$$

et

$$E\left(Z\right)=-\frac{12}{5}. \qquad \textbf{(0.25)}$$

Méthode 2. On a

$$Z=2E\left(X \mid Y\right)-3=\frac{2}{3}Y-3=h\left(Y\right), \qquad \textbf{(0.25)}$$

et

$$E\left(Z\right)=g_Z'\left(s\right)\big|_{s=0}, \qquad \textbf{(0.25)}$$

où

$$g_Z\left(s\right)=\int_{-3}^{-9/5} e^{sz} f_Z\left(z\right)dz, \qquad \textbf{(0.25)}$$

avec

$$\begin{aligned} f_Z\left(z\right) &= \left|\left(h^{-1}\left(z\right)\right)'\right| f_Y\left(h^{-1}\left(z\right)\right) \qquad \textbf{(0.25)} \\ &= \left|\left(\frac{3z+9}{2}\right)'\right|\frac{5}{9} \qquad \textbf{(0.25)} \\ &= \frac{5}{6}, \qquad \textbf{(0.25)} \end{aligned}$$

alors

$$\begin{aligned} g_Z(s) &= \int_{-3}^{-9/5} \frac{5}{6} e^{sz} dz \\ &= \frac{5}{6s}\left(e^{-\frac{9}{5}s} - e^{-3s}\right), \quad \textcolor{red}{(\mathbf{0.5})} \end{aligned}$$

et

$$g'_Z(s) = \frac{-5}{6s^2}\left(e^{-\frac{9}{5}s} - e^{-3s}\right) + \frac{5}{6s}\left(3e^{-3s} - \frac{9}{5}e^{-\frac{9}{5}s}\right). \quad \textcolor{red}{(\mathbf{0.5})}$$

En utilisant la règle de l'Hôpital, on obtient

$$\begin{aligned} g'_Z(s)\big|_{s=0} &= \frac{-5}{12}\left(\frac{81}{25}e^{-\frac{9}{5}s} - 9e^{-3s}\right)\Bigg|_{s=0} + \frac{5}{6}\left(\frac{81}{25}e^{-\frac{9}{5}s} - 9e^{-3s}\right)\Bigg|_{s=0} \\ &= -\frac{12}{5}. \quad \textcolor{red}{(\mathbf{0.25})} \end{aligned}$$

Donc

$$E(Z) = -\frac{12}{5}. \quad \textcolor{red}{(\mathbf{0.25})}$$

5. On a

$$f_{UV}(u,v) = |J|\, f_{XY}\left(h^{-1}(u,v), g^{-1}(u,v)\right). \quad \textcolor{red}{(\mathbf{0.5})}$$

Faisons le changement de variables $(u,v) = \left(h(x,y) = \frac{x}{y},\ g(x,y) = x+y\right)$. La réciproque est

$$(x,y) = \left(h^{-1}(u,v) = \frac{uv}{1+u}, g^{-1}(u,v) = \frac{v}{1+u}\right). \quad \textcolor{red}{(\mathbf{0.5})}$$

Le Jacobien J est

$$J = \det\begin{pmatrix} \frac{v}{(1+u)^2} & \frac{u}{1+u} \\ \frac{-v}{(1+u)^2} & \frac{1}{1+u} \end{pmatrix} = \frac{v}{(1+u)^2}. \quad \textcolor{red}{(\mathbf{0.5})}$$

Donc

$$\begin{aligned} f_{UV}(u,v) &= \frac{v}{(1+u)^2}\frac{5}{18}\frac{1}{\sqrt{\frac{uv}{1+u}}\sqrt{\frac{v}{1+u}}} \\ &= \frac{5}{18(u+1)\sqrt{u}}.\ (u,v) \in D. \quad \textcolor{red}{(\mathbf{0.5})} \end{aligned}$$

Exercice 2 (03.5 points).

1. On a

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\theta^n C_n^k p\left(1-p\right)^n = 1, \qquad \mathbf{(0.5)}$$

où

$$\begin{aligned}\sum_{n=0}^{\infty}\sum_{k=0}^{n} C_n^k p\left[\theta\left(1-p\right)\right]^n &= \sum_{n=0}^{\infty} p\left[\theta\left(1-p\right)\right]^n \sum_{k=0}^{n} C_n^k \\ &= \sum_{n=0}^{\infty} p\left[\theta\left(1-p\right)\right]^n 2^n \qquad \mathbf{(0.25)} \\ &= \sum_{n=0}^{\infty} p\left[2\theta\left(1-p\right)\right]^n \\ &= \frac{p}{1-2\theta\left(1-p\right)} = 1. \qquad \mathbf{(0.25)}\end{aligned}$$

Donc

$$\theta = \frac{1}{2}. \qquad \mathbf{(0.25)}$$

2. On a

$$\begin{aligned}P\left(X=k\right) &= \sum_{n=k}^{\infty} P\left(X=k, Y=n\right) \qquad \mathbf{(0.25)} \\ &= \sum_{n=k}^{\infty} C_n^k p\left(\frac{1-p}{2}\right)^n \\ &= \frac{2p}{p+1}\left(\frac{1-p}{1+p}\right)^k. \qquad \mathbf{(0.25)}\end{aligned}$$

Donc

$$\begin{aligned}\varphi_X\left(t\right) &= \sum_{k=0}^{\infty} e^{itk} P\left(X=k\right) \qquad \mathbf{(0.25)} \\ &= \frac{2p}{p+1}\sum_{k=0}^{\infty} e^{itk}\left(\frac{1-p}{1+p}\right)^k. \\ &= \frac{2p}{p+1}\sum_{k=0}^{\infty}\left[\frac{1-p}{1+p}e^{it}\right]^k \\ &= \left(\frac{2p}{p+1}\right)\frac{1}{1-\frac{1-p}{1+p}e^{it}} \qquad \mathbf{(0.25)} \\ &= \frac{2p}{1+p+\left(p-1\right)e^{it}}. \qquad \mathbf{(0.25)}\end{aligned}$$

On a

$$E\left(X\right) = \frac{\varphi_X'\left(0\right)}{i}, \qquad \mathbf{(0.25)}$$

où

$$\varphi_X'\left(t\right) = \frac{-2i\left(p-1\right)pe^{it}}{\left(1+p+\left(p-1\right)e^{it}\right)^2} \qquad \mathbf{(0.25)} \Longrightarrow \varphi_X'\left(0\right) = \frac{-i\left(p-1\right)}{2p}, \qquad \mathbf{(0.25)}$$

alors

$$E\left(X\right) = \frac{1-p}{2p}. \qquad \mathbf{(0.25)}$$

Exercice 3 (02.5 points).

On a

$$\begin{aligned} G_T(u) &= \sum_{k=0}^{\infty} u^k P(T=k) \qquad \textbf{(0.25)} \\ &= \sum_{k=0}^{\infty} u^k \frac{k^2+1}{\lambda k!} = \frac{1}{\lambda}\left(\sum_{k=0}^{\infty} \frac{k^2}{k!} u^k + \sum_{k=0}^{\infty} \frac{u^k}{k!}\right) \\ &= \frac{1}{\lambda}\left(u^2 e^u + u e^u + e^u\right) \qquad \textbf{(0.25)} \\ &= \frac{1}{\lambda}\left(u^2+u+1\right) e^u, \qquad \textbf{(0.25)} \end{aligned}$$

alors

$$G_T(1) = \frac{1}{\lambda} 3e = 1 \qquad \textbf{(0.25)} \Longrightarrow \lambda = 3e. \qquad \textbf{(0.25)}$$

c'est-à-dire

$$G_T(u) = \frac{\left(u^2+u+1\right)}{3} e^{u-1}.$$

On a

$$E(T) = G_T'(1), \qquad \textbf{(0.25)}$$

où

$$G_T'(u) = \frac{(2u+1)}{3} e^{u-1} + \frac{\left(u^2+u+1\right)}{3} e^{u-1} \qquad \textbf{(0.5)} \Longrightarrow G_T'(1) = 2. \qquad \textbf{(0.25)}$$

Donc

$$E(T) = 2. \qquad \textbf{(0.25)}$$