Chapter V: Work and Energy

Summary

1. Introduction

Newton's laws make it possible to solve all the problems of classical mechanics, if we know the position $\vec{r}$, the initial velocity $\vec{V}_0$ as well as all the forces $\sum \vec{F}$ acting on a material point M over time t. But in practice, we do not always know all the forces, or the equations and their solution are too numerous or too complex. Therefore, we will use the concepts of work W and energy E.

2. Work of a force W

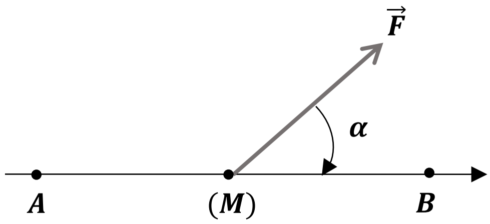
We say that the force $\vec{F}$ works when it moves an object along a path $\overrightarrow{AB}$, and we call it the work of a force $W_{AB}(\vec{F})$.

2.1. Work of a constant force on a rectilinear movement

Consider a constant force acting on a material point M which moves between points A and B. The work of the force $\vec{F}$ on the rectilinear displacement $\overrightarrow{AB}$ is given by the following scalar product:

$$W_{AB}(\vec{F}) = \vec{F}.\overrightarrow{AB}$$

$$W_{AB}(\vec{F}) = F.AB.\cos\alpha$$



Where :

- ✓ α is the angle that $\vec{F}$ makes with $\overrightarrow{AB}$.
- ✓ Joule (J) is the unit of work W: 1J = 1N.m.

This work can be positive, negative or null, depending on the direction of the force with respect to the displacement, and therefore it depends on the sign of cos α:

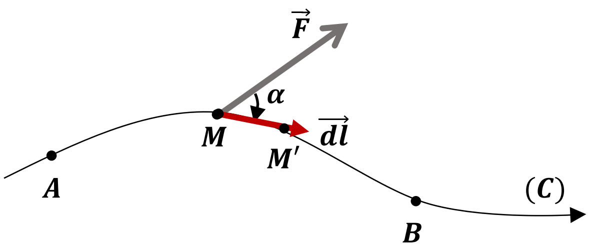
- ✓ if $cos\alpha > 0 \Longrightarrow W_{AB}(\vec{F}) > 0,$ it is a motor work.
- ✓ if $cos\alpha < 0 \Longrightarrow W_{AB}(\vec{F}) < 0,$ it is a resistant work.
- ✓ if $cos\alpha = 0 \Longrightarrow W_{AB}(\vec{F}) = 0$, it is a null work, because $\vec{F} \perp \overrightarrow{AB}$ and $\alpha = \frac{\pi}{2}$.

2.2. Work of variable force - elementary work

If the force $\vec{F}$ varies during the displacement which can be arbitrary (C), it is impossible to use directly the previous expression. We therefore decompose the path $\overrightarrow{AB}$ into an infinitesimal rectilinear succession which called the elementary displacements $\overrightarrow{dl}$: $\overrightarrow{dl} = \overrightarrow{MM'}$. We are talking about the elementary work dW that defined by:

$$dW = \vec{F}.\overrightarrow{dl}$$

$$dW = F.dl.\cos\alpha$$



The total work of the force $\vec{F}$ applied to M between the positions A and B is obtained by integration between A and B:

$$W_{AB}(\vec{F}) = \int_A^B dW = \int_A^B \vec{F}.\overrightarrow{dl}$$

a) Expression of work in the Cartesian coordinate system

$$\vec{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix};\ \overrightarrow{dl} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

$$W_{AB}(\vec{F}) = \int_A^B \vec{F}.\overrightarrow{dl} = \int_A^B [F_x.dx + F_y.dy + F_z.dz]$$

$$W_{AB}(\vec{F}) = \int_{x_A}^{x_B} F_x . dx + \int_{y_A}^{y_B} F_y . dy + \int_{z_A}^{z_B} F_z . dz$$

b) *Expression of work in the intrinsic coordinate system*

$$\vec{F} = \begin{pmatrix} F_T \\ F_N \end{pmatrix}; \ \overrightarrow{dl} = \begin{pmatrix} dS \\ 0 \end{pmatrix}$$

$$W_{AB}(\vec{F}) = \int_A^B \vec{F}.\overrightarrow{dl} = \int_A^B F_T . dS$$

$$W_{AB}(\vec{F}) = \int_{t_A}^{t_B} F_T . V . dt$$

c) *Expression of work in the polar coordinate system*

$$\vec{F} = \begin{pmatrix} F_r \\ F_\theta \end{pmatrix}; \ \overrightarrow{dl} = \begin{pmatrix} dr \\ rd\theta \end{pmatrix}$$

$$W_{AB}(\vec{F}) = \int_A^B \vec{F}.\overrightarrow{dl} = \int_{r_A}^{r_B} F_r . dr + \int_{\theta_A}^{\theta_B} F_\theta . r . d\theta$$

d) *Expression of work in the cylindrical coordinate system*

$$\vec{F} = \begin{pmatrix} F_\rho \\ F_\theta \\ F_Z \end{pmatrix}; \ \overrightarrow{dl} = \begin{pmatrix} d\rho \\ \rho d\theta \\ dz \end{pmatrix}$$

$$W_{AB}(\vec{F}) = \int_A^B \vec{F}.\overrightarrow{dl} = \int_{\rho_A}^{\rho_B} F_\rho . d\rho + \int_{\theta_A}^{\theta_B} F_\theta . \rho . d\theta + \int_{z_A}^{z_B} F_z . dz$$

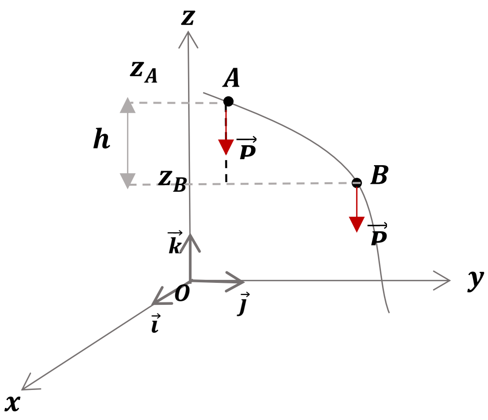
e) *Expression of work in the spherical coordinate system*

$$\vec{F} = \begin{pmatrix} F_r \\ F_\theta \\ F_\varphi \end{pmatrix}; \ \overrightarrow{dl} = \begin{pmatrix} dr \\ r\, d\theta \\ r \sin\theta\, d\varphi \end{pmatrix}$$

$$W_{AB}(\vec{F}) = \int_A^B \vec{F}.\overrightarrow{dl} = \int_{r_A}^{r_B} F_r . dr + \int_{\theta_A}^{\theta_B} F_\theta . r . d\theta + \int_{\varphi_A}^{\varphi_B} F_\varphi . r \sin\theta\, d\varphi$$

2.3. Examples of calculating the work of a force

2.3.1. Work of an weight force (the force of gravity)



$$\vec{P} = \begin{pmatrix} 0 \\ 0 \\ -P \end{pmatrix}; \ \overrightarrow{dl} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

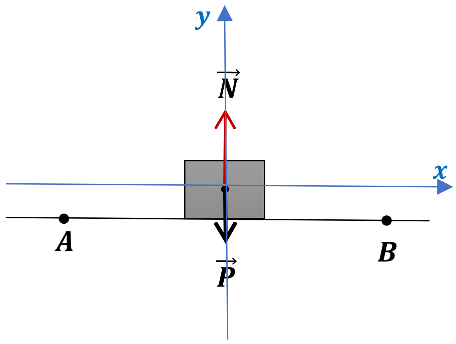
$$W_{AB}(\vec{p}) = \int_A^B \vec{P}.\overrightarrow{dl} = \int_{z_A}^{z_B} -P\, dz$$

$$W_{AB}(\vec{P}) = P\, [z_A - z_B]$$

$$W_{AB}(\vec{P}) = Ph = mgh$$

In conclusion, the work of an weight force does not depend on the trajectory of the material point M, but on the height difference between the transition points.

2.3.2. Work of a reaction force



Method 1

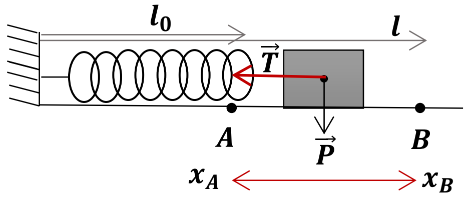
$$\vec{N} = \begin{pmatrix} 0 \\ N \\ 0 \end{pmatrix}; \ \vec{dl} = \begin{pmatrix} dx \\ 0 \\ 0 \end{pmatrix}$$

$$W_{AB}(\vec{N}) = \int_A^B \vec{N}.\vec{dl} = 0$$

Method 2

$$W_{AB}(\vec{N}) = \int_A^B \vec{N}.\vec{dl} = N.dl.\cos 90 = 0$$

2.3.3. Work of an elastic force



$$\vec{T} = \begin{pmatrix} T_x \\ 0 \\ 0 \end{pmatrix}; \ \vec{dl} = \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

$$W_{AB}(\vec{T}) = \int_A^B \vec{T}.\vec{dl} = \int_{x_A}^{x_B} T_x \, dx$$

Knowing that : $T_x = kx$

$$W_{AB}(\vec{T}) = \int_{x_A}^{x_B} k\,x\,dx = \frac{1}{2} k\,[{x_B}^2 - {x_A}^2]$$

$$W_{AB}(\vec{T}) = \frac{1}{2} k\,x^2$$

2.4. Power

The power of a force $\boldsymbol{p}$ is by definition the ratio of the work $\boldsymbol{W_{AB}(\vec{F})}$ over the time of its realization $\Delta \boldsymbol{t}$. Depending on the duration considered, this power is called average or instantaneous.

The average power is defined by :

$$p_{moy} = \frac{W_{AB}(\vec{F})}{\Delta t}$$

The instantaneous power is defined by the derivative of work with respect to time:

$$p = \frac{dW}{dt}$$

$$p = \frac{dW}{dt} = \frac{\vec{F}.\overrightarrow{dl}}{dt} = \vec{F}.\frac{\overrightarrow{dl}}{dt}$$

$$p = \vec{F}.\vec{V}$$

Where:

- $\vec{V}$: instantaneous velocity of the material point M.
- Power is expressed in Watts (W) , W = J. s^{-1} = Kg. m^2. s^{-3}

3. Kinetic, potential and mechanical energy

3.1. Kinetic energy

3.1.1. Definition

A material point of mass m moves in a Galilean reference frame under the action of external forces $\sum \vec{F}_{ext}$. According to Newton's 2nd law:

$$\sum \vec{F}_{ext} = m.\vec{\gamma} = m.\frac{d\vec{V}}{dt}$$

The elementary work of the resultant of the forces is given by:

$$dW = \sum \vec{F}_{ext}.\overrightarrow{dl} = m.\frac{d\vec{V}}{dt}.\overrightarrow{dl} = m\,\vec{V}\,d\vec{V}$$

The total work of the resultant of the forces $\sum \vec{F}_{ext}$ applied to M is obtained by integration between positions A and B:

$$W_{AB}\left(\sum \vec{F}_{ext}\right) = \int_A^B dW = \int_{V_A}^{V_B} m\,\vec{V}\,d\vec{V} = \left[\frac{1}{2}\,m\,V^2\right]_{V_A}^{V_B}$$

$$W_{AB}\left(\sum \vec{F}_{ext}\right) = \frac{1}{2}\,m\,({V^2}_B - {V^2}_A)$$

« The quantity $\frac{1}{2}\,m\,V^2$ is called Kinetic energy of the material point M of mass m and velocity V, denoted E_c».

$$\boldsymbol{E_c = \frac{1}{2}\,m\,v^2}$$

And we finally get: $\boldsymbol{W_{AB}(\sum \vec{F}_{ext}) = E_c(B) - E_c(A) = \Delta E_c}$

3.1.2. Kinetic energy theorem

«In a Galilean reference frame, the variation in kinetic energy of a material point subjected to an external forces between a position A and another position B is equal to the sum of the works of these forces between these two points » :

$$\Delta E_c = E_c(B) - E_c(A) = \sum_{i=1}^{n} W_{AB}(\vec{F}_i)$$

3.2. Potential energy

3.2.1. Non-conservative forces and conservative forces

The forces are called **conservative** $\vec{F}_C$ when their work does not depend on the followed path, but only on the starting point A and the arrival point B.

Examples :

- Weight force (gravity force) $\vec{P}$;
- Electrical force $\vec{F}_e$;
- Tension force $\vec{T}$.

The forces are called **non-conservative** $\vec{F}_{NC}$ when their work depends on the followed path.

Example :

- Friction force $\vec{f}$;

3.2.2. Potential energy

When a material point M is subjected to forces, it has different forms of energy. We have seen that the kinetic energy of a material point is associated with its movement, and we will now see another form of energy « **potential energy** » which is associated with its heigh, denoted E_p, where: $E_p = mgh$

The **work** of a **conservative force** $\vec{F}_C$ applied to a material point M, between positions A and B, can be expressed by the **potential energy**. This work $W_{AB}(\vec{F}_c)$ is equal to the difference of this potential energy E_p between the initial points A and final points B:

$$E_p(B) - E_p(A) = -W_{AB}(\vec{F_c})$$

$$\Delta E_p = -W_{AB}(\vec{F_c})$$

When the variation is very small, $\Delta E_p \implies dE_p$

Using the notion of elementary work, we have:

$$dE_p = -dW_{AB}(\vec{F_c})$$

$$\boldsymbol{dE_p = -\vec{F}_C \,.\, \overrightarrow{dl}} \quad : \text{(differential definition of } E_p\text{)}$$

This relation can be expressed differently by introducing the operator « **gradient** » : $\overrightarrow{grad}\, E_p$. To simplify, the expression of the differential of a potential energy function $E_p\,(x, y, z)$ with three variables in Cartesian coordinates is:

$$\boldsymbol{dE_p = \frac{\partial E_p}{\partial x}dx + \frac{\partial E_p}{\partial y}dy + \frac{\partial E_p}{\partial z}dz}$$

Knowing that :

$$\overrightarrow{grad}\, E_p = \vec{\nabla} E_p = \left[\frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}\right] E_p = \frac{\partial E_p}{\partial x}\vec{\imath} + \frac{\partial E_p}{\partial y}\vec{\jmath} + \frac{\partial E_p}{\partial z}\vec{k}$$

Therefore dE_p can be written in the following form:

$$dE_p = \left(\frac{\partial E_p}{\partial x}\vec{\imath} + \frac{\partial E_p}{\partial y}\vec{\jmath} + \frac{\partial E_p}{\partial z}\vec{k}\right).\left(dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k}\right)$$

$$dE_p = \left(\frac{\partial E_p}{\partial x}\vec{\imath} + \frac{\partial E_p}{\partial y}\vec{\jmath} + \frac{\partial E_p}{\partial z}\vec{k}\right).\overrightarrow{dl}$$

$$dE_p = \overrightarrow{grad}\, E_p . \overrightarrow{dl}$$

In comparison with: $dE_p = -\vec{F}_C \,.\, \overrightarrow{dl}$, we obtrain : $\vec{F}_C = -\overrightarrow{grad}\, E_p$

If the force $\overrightarrow{F_C}$ is conservative, it derives from a potential energy E_p and it can be written:

$$\boldsymbol{\vec{F}_C = -\overrightarrow{grad}\, E_p}$$

$\overrightarrow{grad}\, E_p$ is expressed in coordinate systems as follows:

- *Cartesian coordinates:*

$$\overrightarrow{grad}\, E_p = \frac{\partial E_p}{\partial x}\vec{\imath} + \frac{\partial E_p}{\partial y}\vec{\jmath} + \frac{\partial E_p}{\partial z}\vec{k}$$

-*Polar coordinates:*

$$\overrightarrow{grad}\, E_p = \frac{\partial E_p}{\partial r}\overrightarrow{u_r} + \frac{1}{r}\frac{\partial E_p}{\partial \theta}\overrightarrow{u_\theta}$$

-*Cylindrical coordinates:*

$$\overrightarrow{grad}\, E_p = \frac{\partial E_p}{\partial \rho}\overrightarrow{u_\rho} + \frac{1}{\rho}\frac{\partial E_p}{\partial \theta}\overrightarrow{u_\theta} + \frac{\partial E_p}{\partial z}\vec{k}$$

-*Spherical coordinates:*

$$\overrightarrow{grad}\, E_p = \frac{\partial E_p}{\partial r}\overrightarrow{u_r} + \frac{1}{r}\frac{\partial E_p}{\partial \theta}\overrightarrow{u_\theta} + \frac{1}{r\sin\theta}\frac{\partial E_p}{\partial \varphi}\overrightarrow{u_\varphi}$$

3.3. Mechanical energy

3.3.1. Definition

Let M a material point moves between points A and B under the effect of conservative forces $\vec{F}_C$ and non-conservative forces $\vec{F}_{NC}$. According to the kinetic energy theorem we have:

$$E_c(B) - E_c(A) = \sum W_{AB}(\vec{F}_C) + \sum W_{AB}(\vec{F}_{NC})$$

Therefore, $$E_c(B) - E_c(A) = -\left(E_p(B) - E_p(A)\right) + \sum W_{AB}(\vec{F}_{NC})$$

and, $$\sum W_{AB}(\vec{F}_C) = -\left(E_p(B) - E_p(A)\right)$$

He comes: $$\left(E_c(B) + E_p(B)\right) - \left(E_c(A) + E_P(A)\right) = \sum W_{AB}(\vec{F}_{NC})$$

The quantity $E = E_c + E_p$ is called mechanical energy denoted E_m or E_T, where :

$$E_m(B) - E_m(A) = \sum W_{AB}(\vec{F}_{NC})$$

3.3.2. Mechanical energy theorem

« In a Galilean reference frame, the variation of the mechanical energy E_m of a material point M between positions A and B, is equal to the work of the resultant of the non-conservative forces applied to this material point » :

$$E_m(B) - E_m(A) = \sum W_{AB}(\vec{F}_{NC})$$

However, when the system is isolated (it is not subject to any non-conservative external force) the total energy is conserved: $E_m(B) = E_m(A)$

4. Applications

☒ Exercise 1

A particle is subjected to a force defined by its Cartesian coordinates: :

$$\vec{F} = (x + 2y + \beta z)\,\vec{\imath} + (\beta x - 3y - z)\,\vec{\jmath} + (4x + \gamma y + 2z)\vec{k}$$

where α, β, γ are constants. x, y, z are in meter and $\vec{F}$ in Newton.

1/ Find the values of α, β, γ so that $\vec{F}$ derives from a potential.

2/ Find the expression of the potential $E_p\,(x, y, z)$ from which the force derives, where $E_p(0,0,0) = 2$.

☒ Solution

$$\boldsymbol{\alpha = 4, \qquad \beta = 2, \qquad \gamma = -1}$$

$$\boldsymbol{E_p = -\frac{1}{2}x^2 - 2xy - 4xz + \frac{3}{2}\,y^2 + yz - z^2 + 2}$$