Mathematics for Engineering Sciences

Chapter 02: Matrix Analysis

Chapter Objective:

- Finding Eigenvalues and eigenvectors for an matrix A
- Solve linear systems $AX = B$ using direct methods:
  - **Gauss-Elimination Method** for any matrix A.
  - **Gauss-Jordan Method** for any matrix A.
  - **Cholesky Method** for symmetric positive-definite matrices.
- Solve linear systems $AX = B$ using iterative methods:
  - **Jacobi Method**
- Solve deferential linear systems

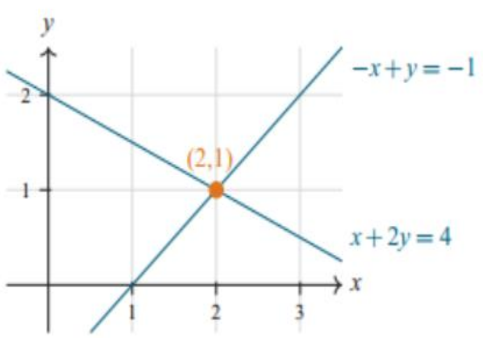
Introduction:

A large part of linear algebra revolves around solving and manipulating the simplest types of equations: linear equations.

For example, the following are linear equations:

$$x + 3y = 4, \qquad 2x - \pi y = 3, \qquad 4x + 3 = 6y,$$
$$\sqrt{3}x - y = \sqrt{5}, \qquad \cos(1)x + \sin(1)y = 2, \quad \text{and} \quad x + y - 2z = 7.$$

Often, we want to solve multiple linear equations simultaneously, meaning we seek values for variables $x1, x2, \ldots, xn$ such that several different linear equations are satisfied simultaneously. This leads us to **systems of linear equations**. Geometrically, a solution to a system of linear equations corresponds to the point of intersection of all the lines, planes or hyperplanes defined by the linear equations of the system. For example, consider the following linear system composed of two equations.



The lines defined by these equations are shown in Figure 2.2. Based on this point, these lines have a unique point of intersection, located at point (2; 1). The vector x = (2; 1) therefore appears to be the unique solution to this system of linear equations. To find this solution algebraically, we could add the two equations of the linear system to obtain the new equation 3y = 3, which tells us that y = 1. Putting y = 1 back into the original equation x + 2y = 4, we obtain x = 2.

I. Matrix Representation of Systems:

Using matrices provides a compact way to work with linear systems. For a system:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots\dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots\dots + a_{2n}x_n = b_2 \\ \vdots \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \dots\dots + a_{nn}x_n = b_n \end{cases}$$

we can write it as a single matrix equation: $AX = B$

where:

- $A \in M_{n\times n}$ is the **coefficient matrix** $\begin{bmatrix} a_{11} & 0 & \dots & \dots & 0 \\ a_{21} & a_{22} & 0 & \dots & 0 \\ & & \ddots & & \\ & & \ddots & & \\ a_{n1} & a_{n2} & \dots & \dots & a_{nn} \end{bmatrix}$.
- $\boldsymbol{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}$ is the variable vector.
- $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_n \end{bmatrix}$ is the constant vector.

II.1 Eigenvalues and Eigenvectors

If we allowed $\mathbf{v} = \mathbf{0}$ as an eigenvector then every scalar λ would be an eigenvalue corresponding to it.

If $A \in \mathcal{M}_n$, $v \neq \mathbf{0}$ is a vector, λ is a scalar, and $A\mathbf{v} = \lambda\mathbf{v}$, then we say that $\mathbf{v}$ is an **eigenvector** of A with corresponding **eigenvalue** λ. Geometrically, this means that A stretches $\mathbf{v}$ by a factor of λ, but does not rotate it at all (see Figure A.4). The set of all eigenvectors corresponding to a particular eigenvalue (together with the zero vector) is always a subspace of $\mathbb{R}^n$, which we call the **eigenspace** corresponding to that eigenvalue. The dimension of an eigenspace is called the **geometric multiplicity** of the corresponding eigenvalue.

This matrix *does* change the direction of any vector that is not on one of the two lines displayed here.

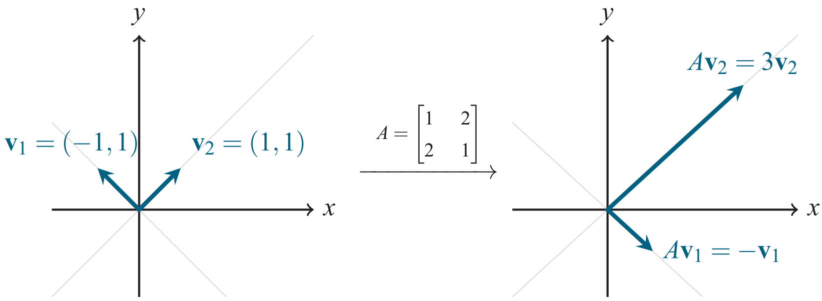


Figure: Matrices do not change the line on which any of their eigenvectors lie, but rather just scale them by the corresponding eigenvalue. The matrix displayed here has eigenvectors $\mathbf{v}_1 = (-1,1)$ and $\mathbf{v}_2 = (1,1)$ with corresponding eigenvalues -1 and 3, respectively.

The standard method of computing a matrix's eigenvalues is to construct its **characteristic polynomial** $p_A(\lambda) = \det(A - \lambda I)$. This function p_A really is a polynomial in λ by virtue of the permutation formula for the determinant (i.e., Theorem A.1.4). Furthermore, the degree of p_A is n (the size of A), so it has at most n real roots and exactly n complex roots counting multiplicity (see the upcoming Theorem A.3.1). These roots are the eigenvalues of A, and the eigenvectors $\mathbf{v}$ that they correspond to can be found by solving the linear system $(A - \lambda I)\mathbf{v} = \mathbf{0}$ for each eigenvalue λ.

Example. Computing the Eigenvalues and Eigenvectors of a Matrix

Compute all of the eigenvalues and corresponding eigenvectors of the matrix $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$.

Solution:

To find the eigenvalues of A, we first compute the characteristic polynomial $p_A(\lambda) = \det(A - \lambda I)$:

For a 2×2 matrix, the determinant is simply $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = ad - bc.$

$$\begin{aligned} p_A(\lambda) = \det(A - \lambda I) &= \det\left(\begin{bmatrix} 1-\lambda & 2 \\ 5 & 4-\lambda \end{bmatrix}\right) \\ &= (1-\lambda)(4-\lambda) - 10 = \lambda^2 - 5\lambda - 6. \end{aligned}$$

Setting this determinant equal to 0 then gives

$$\begin{aligned} \lambda^2 - 5\lambda - 6 = 0 \quad &\iff \quad (\lambda+1)(\lambda-6) = 0 \\ &\iff \quad \lambda = -1 \quad \text{or} \quad \lambda = 6, \end{aligned}$$

so the eigenvalues of A are $\lambda = -1$ and $\lambda = 6$. To find the eigenvectors corresponding to these eigenvalues, we solve the linear systems $(A+I)\mathbf{v} = \mathbf{0}$ and $(A-6I)\mathbf{v} = \mathbf{0}$, respectively:

$\lambda = -1$: In this case, we want to solve the linear system $(A-\lambda I)\mathbf{v} = (A+I)\mathbf{v} = \mathbf{0}$, which we can write explicitly as follows:

$$\begin{aligned} 2v_1 + 2v_2 &= 0 \\ 5v_1 + 5v_2 &= 0 \end{aligned}.$$

To solve this linear system, we use Gaussian elimination as usual:

$$\left[\begin{array}{cc|c} 2 & 2 & 0 \\ 5 & 5 & 0 \end{array}\right] \xrightarrow{R_2 - \frac{5}{2}R_1} \left[\begin{array}{cc|c} 2 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

It follows that v_2 is a free variable and v_1 is a leading variable with $v_1 = -v_2$. The eigenvectors corresponding to the eigenvalue $\lambda = -1$ are thus the non-zero vectors of the form $\mathbf{v} = (-v_2, v_2) = v_2(-1,1)$.

$\lambda = 6$: Similarly, we now want to solve the linear system $(A - \lambda I)\mathbf{v} = (A-6I)\mathbf{v} = \mathbf{0}$, which we can do as follows:

$$\left[\begin{array}{cc|c} -5 & 2 & 0 \\ 5 & -2 & 0 \end{array}\right] \xrightarrow{R_2 + R_1} \left[\begin{array}{cc|c} -5 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

It follows that v_2 is a free variable and v_1 is a leading variable with $v_1 = 2v_2/5$. The eigenvectors corresponding to the eigenvalue $\lambda = 6$ are thus the non-zero vectors of the form $\mathbf{v} = (2v_2/5, v_2) = v_2(2/5,1)$.

By multiplying $(2/5,1)$ by 5, we could also say that the eigenvectors here are the multiples of $(2,5)$, which is a slightly cleaner answer.

The multiplicity of an eigenvalue λ as a root of the characteristic polynomial is called its **algebraic multiplicity**, and the sum of all algebraic multiplicities of eigenvalues of an $n \times n$ matrix is no greater than n (and it exactly equals n if we consider complex eigenvalues). The following remarkable (and non-obvious) fact guarantees that the sum of geometric multiplicities is similarly no larger than n:

! The geometric multiplicity of an eigenvalue is never larger than its algebraic multiplicity.

II.3 Diagonalization

One of the primary reasons that eigenvalues and eigenvectors are of interest is that they let us **diagonalize** a matrix. That is, they give us a way of decomposing a matrix $A \in \mathcal{M}_n$ into the form $A = PDP^{-1}$, where P is invertible and D is diagonal. If the entries of P and D can be chosen to be real, we say that A is **diagonalizable over** $\mathbb{R}$. However, some real matrices A can only be diagonalized if we allow P and D to have complex entries (see the upcoming discussion of complex numbers in Appendix A.3). In that case, we say that A is **diagonalizable over** $\mathbb{C}$ (but not over $\mathbb{R}$).

Theorem
Characterization of Diagonalizability

Suppose $A \in \mathcal{M}_n$. The following are equivalent:

a) A is diagonalizable over $\mathbb{R}$ (or $\mathbb{C}$).

b) There exists a basis of $\mathbb{R}^n$ (or $\mathbb{C}^n$) consisting of eigenvectors of A.

c) The sum of geometric multiplicities of the real (or complex) eigenvalues of A is n.

Furthermore, in any diagonalization $A = PDP^{-1}$, the eigenvalues of A are the diagonal entries of D and the corresponding eigenvectors are the columns of P in the same order.

To get a feeling for why diagonalizations are useful, notice that computing a large power of a matrix directly is quite cumbersome, as matrix multiplication itself is an onerous process, and repeating it numerous times only makes it worse. However, once we have diagonalized a matrix we can compute an arbitrary power of it via just two matrix multiplications, since

$$A^k = \underbrace{(PD\overbrace{P^{-1})(P}^{P^{-1}P=I}D\overbrace{P^{-1})(P}^{P^{-1}P=I}D\overbrace{P^{-1})}^{\cdots}\qquad\qquad\qquad}_{k\text{times}} = PD^kP^{-1},$$

and D^k is trivial to compute (for diagonal matrices, matrix multiplication is the same as entrywise multiplication).

Example
Diagonalizing a Matrix

Diagonalize the matrix $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}$ and then compute A^{314}.

Solution:

We showed in Example A.1.1 that this matrix has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 6$ corresponding to the eigenvectors $\mathbf{v}_1 = (-1,1)$ and $\mathbf{v}_2 = (2,5)$, respectively. Following the suggestion of Theorem A.1.5, we stick these eigenvalues along the diagonal of a diagonal matrix D, and the corresponding eigenvectors as columns into a matrix P in the same order:

We could have also chosen $\mathbf{v}_2 = (2/5,1)$, but our choice here is prettier. Which multiple of each eigenvector we choose does not matter.

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 6 \end{bmatrix} \quad \text{and} \quad P = \left[\, \mathbf{v}_1 \mid \mathbf{v}_2 \,\right] = \begin{bmatrix} -1 & 2 \\ 1 & 5 \end{bmatrix}.$$

It is straightforward to check that P is invertible, so Theorem A.1.5 tells us that A is diagonalized by this D and P.

To compute A^{314}, we first compute P^{-1} to be

$$P^{-1} = \frac{1}{7}\begin{bmatrix} -5 & 2 \\ 1 & 1 \end{bmatrix}.$$

The inverse of a 2×2 matrix is simply
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det(A)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

We can then compute powers of A via powers of the diagonal matrix D in this diagonalization:

Since 314 is even, $(-1)^{314} = 1$.

$$A^{314} = PD^{314}P^{-1} = \frac{1}{7}\begin{bmatrix} -1 & 2 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} (-1)^{314} & 0 \\ 0 & 6^{314} \end{bmatrix}\begin{bmatrix} -5 & 2 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{7}\begin{bmatrix} 5+2\cdot 6^{314} & -2+2\cdot 6^{314} \\ -5+5\cdot 6^{314} & 2+5\cdot 6^{314} \end{bmatrix}.$$

We close this section with a reminder of a useful connection between diagonalizability of a matrix and the multiplicities of its eigenvalues:

! If a matrix is diagonalizable then, for each of its eigenvalues, the algebraic and geometric multiplicities coincide.

II.3 Gauss -Elimination Method

II 3.1 Principle:

The Gauss-Elimination method transforms the original system $AX = B$ into an equivalent system $A'X = B'$ where A' is an upper triangular matrix. This transformation is achieved using elementary transformations based on Perlis operators.

II 3.2 Method Description:

Consider the system:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

We form the **augmented matrix** $[A \mid B]$, which includes both the coefficients and the constants:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{14} \\ a_{24} \\ a_{34} \end{bmatrix} \qquad [A|\ B] = \left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{array}\right]$$

Step 1: Eliminate sub-diagonal elements in the first column

To make the elements below the pivot a_{11} zero, we use **row transformations**:

- For row 2:

$$R2 \leftarrow R2 - \frac{a_{21}}{a_{11}}R1$$

- For row 3:

$$R3 \leftarrow R3 - \frac{a_{31}}{a_{11}}R1$$

Step 2: Eliminate the element below the diagonal in the second column

- Modify row 3 using:

$$R3 \leftarrow R3 - \frac{{a_{32}}^{(1)}}{{a_{22}}^{(1)}} R2$$

At the end of these steps, the matrix will be in upper triangular form:

$$[A', B'] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}^{(1)} & a_{23}^{(1)} \\ 0 & 0 & a_{33}^{(2)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a_{14} \\ a_{24}^{(1)} \\ a_{34}^{(2)} \end{bmatrix}$$

Step 3: Back-substitution

Solve the triangular system from the bottom up:

$$\begin{cases} x_3 = \dfrac{a_{34}^{(2)}}{a_{33}^{(2)}} \\ x_2 = \dfrac{1}{a_{22}^{(1)}} \left[a_{24}^{(1)} - a_{23}^{(1)} x_3\right] \end{cases}$$

Par exemple, la matrice associée au système linéaire (A.1.1) est la suivante :

$$\begin{aligned} y + 3z &= 3 \\ 2x + y - z &= 1 \\ x + y + z &= 2. \end{aligned} \quad \xrightarrow[{[A|B]}]{\text{La matrice augmentée correspondante}} \quad \left[\begin{array}{ccc|c} 0 & 1 & 3 & 3 \\ 2 & 1 & -1 & 1 \\ 1 & 1 & 1 & 2 \end{array}\right].$$

Ensuite, nous utilisons une méthode appelée élimination de Gauss ou réduction de ligne, qui fonctionne en utilisant l'une des trois opérations élémentaires suivantes sur les lignes pour simplifier au maximum cette matrice :

- Échanger deux lignes que nous notons $R_i \mapsto R_j$.
- Multiplier une ligne par une constante non nulle nous notons cR_j.
- Ajouter ou soustraire une ligne à une autre ligne, multipliée par une constante que nous notons $R_i + cR_j$.

En particulier, nous pouvons utiliser ces trois opérations élémentaires sur les lignes pour mettre n'importe quelle matrice sous forme échelonnée réduite (RREF), ce qui signifie qu'elle présente les trois propriétés suivantes :

✓ Dans chaque ligne non nulle, la première entrée non nulle (appelée entrée principale) est à gauche de toutes les entrées principales en dessous d'elle.

✓ Chaque entrée principale est égale à 1 et est la seule entrée non nulle dans sa colonne. Par exemple, nous pouvons mettre la matrice sous forme échelonnée réduite en utilisant la séquence d'opérations élémentaires suivante :

$$\left[\begin{array}{ccc|c} 0 & 1 & 3 & 3 \\ 2 & 1 & -1 & 1 \\ 1 & 1 & 1 & 2 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & -1 & 1 \\ 0 & 1 & 3 & 3 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & -3 & -3 \\ 0 & 1 & 3 & 3 \end{array}\right]$$

$$\xrightarrow{-R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 1 & 3 & 3 \end{array}\right] \xrightarrow[R_3 - R_2]{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & -1 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

L'une des caractéristiques utiles de la forme échelonnée réduite est que les solutions du système linéaire correspondant peuvent être lues directement. Par exemple, si nous interprétons la forme échelonnée réduite ci-dessus comme un système linéaire, la dernière ligne dit simplement 0x + 0y + 0z = 0 (nous l'ignorons donc), la deuxième ligne dit que y + 3z = 3, et la première ligne dit que x - 2z = -1. Si nous déplaçons simplement le terme "z" dans chacune de ces équations de l'autre côté, nous voyons que chaque solution de ce système linéaire a x = 2z - 1 et y = 3 - 3z, où z est arbitraire (nous appelons donc z une variable libre et x et y des variables principales).

II. 3.3 LU Factorization

Theorem A matrix A is regular if and only if it can be factored

$$A = LU,$$

where L is a lower unitriangular matrix, having all 1's on the diagonal, and U is upper triangular with nonzero diagonal entries, which are the pivots of A. The nonzero off-diagonal entries l_{ij} for $i > j$ appearing in L prescribe the elementary row operations that bring A into upper triangular form; namely, one subtracts l_{ij} times row j from row i at the appropriate step of the Gaussian Elimination process.

In practice, to find the LU factorization of a square matrix A, one applies the regular Gaussian Elimination algorithm to reduce A to its upper triangular form U. The entries of L can be filled in during the course of the calculation with the negatives of the multiples used in the elementary row operations. If the algorithm fails to be completed, which happens whenever zero appears in any diagonal pivot position, then the original matrix is *not* regular, and does *not* have an LU factorization.

Example Let us compute the LU factorization of the matrix $A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 5 & 2 \\ 2 & -2 & 0 \end{pmatrix}$.

Applying the Gaussian Elimination algorithm, we begin by adding -2 times the first row to the second row, and then adding -1 times the first row to the third. The result is the matrix $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & -3 & -1 \end{pmatrix}$. The next step adds the second row to the third row, leading to the upper triangular matrix $U = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, whose diagonal entries are the pivots. The corresponding lower triangular matrix is $L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$; its entries lying below the main diagonal are the *negatives* of the multiples we used during the elimination procedure. For instance, the $(2, 1)$ entry indicates that we added -2 times the first row to the second row, and so on. The reader might wish to verify the resulting factorization

$$\begin{pmatrix} 2 & 1 & 1 \\ 4 & 5 & 2 \\ 2 & -2 & 0 \end{pmatrix} = A = LU = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

II.4 Gauss Jordan Method

The **Gauss-Jordan elimination** method extends the Gauss method by further transforming the matrix into **reduced row echelon form (RREF)**.

Steps of Gauss-Jordan Method:

1. **Form the augmented matrix** $[A|\ B]$.
2. **Perform row operations** to make each pivot equal to 1 and ensure all elements above and below the pivot are zero.
3. **Continue until the matrix is in RREF**.

The final form allows direct reading of the solutions. If the matrix is inconsistent (a row of zeros except for the last element), the system has no solutions.

Example

Résolvez le système suivant en utilisant la méthode d'élimination de Gauss-Jordan.

$$\begin{cases} x + y + z = 5 \\ 2x + 3y + 5z = 8 \\ 4x + 5z = 2 \end{cases}$$

La matrice augmentée du système est la suivante.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

Nous allons maintenant effectuer des opérations sur les lignes jusqu'à obtenir une matrice en forme échelonnée réduite.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 2 & 3 & 5 & 8 \\ 4 & 0 & 5 & 2 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 4 & 0 & 5 & 2 \end{array}\right]$$

$$\xrightarrow{R_3-4R_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & -4 & 1 & -18 \end{array}\right]$$

$$\xrightarrow{R_3+4R_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 13 & -26 \end{array}\right]$$

$$\xrightarrow{\frac{1}{13}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_2-3R_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

$$\xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

À partir de cette matrice finale, nous pouvons lire la solution du système. Elle est :

$$\boxed{x = 3, \quad y = 4, \quad z = -2.}$$

Example N01 :

Résoudre le système suivant en utilisant la méthode d'élimination de Gauss-Jordan.

$$\begin{cases} x + 2y - 3z = 2 \\ 6x + 3y - 9z = 6 \\ 7x + 14y - 21z = 13 \end{cases} \xrightarrow[{[A|B]}]{\text{La matrice augmentée correspondante}} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array}\right]$$

Effectuons maintenant des opérations sur les lignes de cette matrice augmentée.

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 6 & 3 & -9 & 6 \\ 7 & 14 & -21 & 13 \end{array}\right] \xrightarrow{R_2-6R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & -9 & 9 & -6 \\ 7 & 14 & -21 & 13 \end{array}\right]$$

$$\xrightarrow{R_3-7R_1} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & -9 & 9 & -6 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

Nous obtenons une ligne dont les éléments sont tous nuls à l'exception du dernier à droite. Par conséquent, nous concluons que le système d'équations est inconsistant, c'est-à-dire, il n'a pas de solutions.

I.5 Cholesky Decomposition

For symmetric positive-definite matrices A, the **Cholesky decomposition** is used.

II.5.1 Cholesky Theorem:

If A is symmetric and positive-definite, it can be decomposed as:

$$A = LL^T$$

where L is a lower triangular matrix.

II.5.2 Method:

1. Decompose A into LL^T

$$A = L\,L^t \rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ & & \vdots\ddots & & \\ & & \vdots\ddots & & \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & \cdots & \cdots & 0 \\ l_{21} & l_{22} & 0 & \cdots & 0 \\ & & \vdots\ddots & & \\ & & \vdots\ddots & & \\ l_{n1} & l_{n2} & \cdots & \cdots & l_{nn} \end{bmatrix} \begin{bmatrix} l_{11} & l_{12} & & \cdots & \cdots & l_{1n} \\ 0 & l_{22} & l_{23} & & \cdots & l_{2n} \\ & & \vdots\ddots & & & \\ & & \vdots\ddots & & & \\ 0 & 0 & \cdots & \cdots & \cdots & l_{nn} \end{bmatrix}$$

$$a_{ij} = \sum_{k=1}^{\min(i,j)} l_{ik}\, l_{jk} \qquad \begin{matrix} i = 1 \rightarrow n \\ j = 1 \rightarrow n \end{matrix}$$

2. Solve $LY = B$ for Y (using forward substitution).

$$y_i = \frac{1}{l_{ii}} \left[b_i - \sum_{j=1}^{i-1} l_{ij} y_j \right] \quad (i = 1, \ldots, n)$$

3. Solve $L^T X = Y$ for X (using backward substitution).

$$x_i = \frac{1}{l_{ii}} \left[y_i - \sum_{j=i+1}^{n} l_{ij} x_j \right] \quad (i = n,\ n-1, \ldots 1)$$

II.6 Iterative Methods

II.6.1 Jacobi Method

The **Jacobi method** iteratively finds the solution to $AX = B$. It constructs a sequence $X^{(k)}$ that converges to the solution X.

II.1.1 Convergence Condition:

The Jacobi method converges if the matrix A is **diagonally dominant** (i.e.,$|a_{ii}| > \sum_{i\neq j}|aij|$).

II.1.2 Jacobi Algorithm:

$$\boldsymbol{x_i^{(x+1)} = \frac{1}{a_{ii}}\left(b_i - \sum_{i\neq j} a_{ij} x_j^{(k)}\right)}.$$

II.1.3 Stopping Criteria:

The iteration stops when:

$$\frac{\left\| X^{(k+1)} - X^{(k)} \right\|}{X^{(k+1)}} < \epsilon,$$

where ϵ is a small tolerance value.

II.6 solving ODEs (ordinary differential equations) linear systems

The key to solving linear ODEs systems is to find eigenvalues and eigenvectors for the matrix.

If A is a square matrix, then we say λ is an *eigenvalue* for A if there is a non-zero vector $\vec{v}$ so that $A\vec{v} = \lambda\vec{v}$. We call $\vec{v}$ an *eigenvector* for A and λ.

How to find eigenvalues. Rewriting the equation above, $(A - \lambda I)\vec{v} = \vec{0}$, where I is the identity matrix. For any matrix B, $B\vec{v} = \vec{0}$ for a non-zero vector $\vec{v}$ if and only if the determinant of B is zero (this is the key fact about matrices mentioned earlier).

Find eigenvalues for A by solving $det(A - \lambda I) = 0$ for λ.

How to find eigenvectors. Given an eigenvalue for A, say λ_0, then we can plug in λ_0 into the matrix $A - \lambda_0 I$, and this is now a matrix of numbers.

Find eigenvectors for A and λ_0 by solving $(A - \lambda_0 I)\vec{v} = 0$ for non-zero $\vec{v}$.

There will be more than one solution, so pick a simple solution.

Solutions to linear systems. The solutions can have one of three forms, depending on the eigenvalues of the matrix. The three possibilities are 1) distinct real roots, 2) complex roots, and 3) a repeated real root.

For 1), if the eigenvalues are λ_1 and λ_2, with eigenvectors $\vec{v}_1$ and $\vec{v}_2$, then the solutions are

$$\vec{Y}_1(t) = \vec{v}_1 e^{\lambda_1 t}, \qquad \vec{Y}_2(t) = \vec{v}_2 e^{\lambda_2 t}.$$

For 2), if the eigenvalues are $a + bi$ and $a - bi$, then the eigenvectors are going to be complex numbers, say $\vec{v}_1 = \begin{bmatrix} \alpha + i\beta \\ \gamma + i\delta \end{bmatrix}$.

Then, by taking real and imaginary parts, the solutions are

$$\vec{Y}_1(t) = e^{at} \begin{bmatrix} \alpha\cos(bt) - \beta\sin(bt) \\ \gamma\cos(bt) - \delta\sin(bt) \end{bmatrix} \quad \vec{Y}_2(t) = e^{at} \begin{bmatrix} \beta\cos(bt) + \alpha\sin(bt) \\ \delta\cos(bt) + \gamma\sin(bt) \end{bmatrix}.$$

If you want to write the solutions in terms of complex numbers, you can always write $\vec{Y}_1(t) = \vec{v}_1 e^{(a+bi)t}$, $\vec{Y}_2(t) = \vec{v}_2 e^{(a-bi)t}$ but the first form of the solution is better, as it involves only real numbers.

For 3), if the eigenvalue is λ_1, with an eigenvector $\vec{v}_1$, then the solutions are

$$\vec{Y}_1(t) = \vec{v}_1 e^{\lambda_1 t}, \qquad \vec{Y}_2(t) = (\vec{v}_1 t + \vec{v}_2)\, e^{\lambda_1 t},$$

where $\vec{v}_2$ is a solution to $(A - \lambda_1 I)\vec{v}_2 = \vec{v}_1$.

Examples. Next, we carry out this process for three examples, to show how it works in each case. Consider the system

$$\frac{dx}{dt} = 2x + 2y, \qquad \frac{dy}{dt} = 3x + y.$$

To write this as a matrix, let $\vec{Y}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix}$. Then we have $\dfrac{d\vec{Y}}{dt} = A\vec{Y}$.

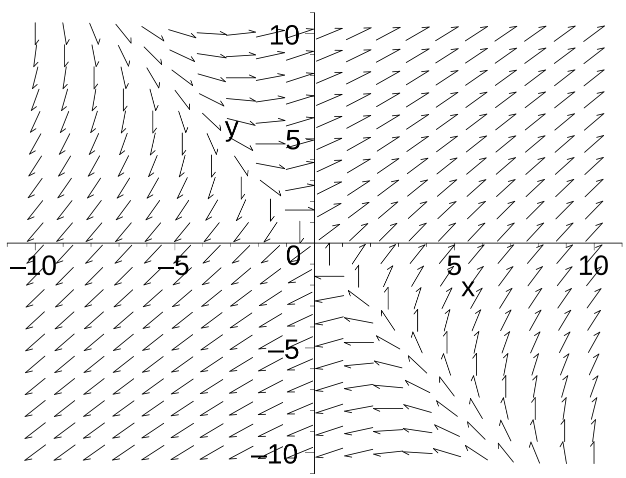
Eigenvalues. We solve $\begin{vmatrix} 2-\lambda & 2 \\ 3 & 1-\lambda \end{vmatrix} = 0$, which is $\lambda^2 - 3\lambda - 4 = 0$. The roots are $\lambda = -1$ and $\lambda = 4$.

Eigenvectors. First we find the eigenvector for $\lambda = 4$. Solve $\begin{bmatrix} 2-4 & 2 \\ 3 & 1-4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The two equations are $-2a + 2b = 0$ and $3a - 3b = 0$. **It will always be true that the two equations are multiplies of each other. If this does not happen, then you've made a mistake somewhere.** Pick a simple solution, like $a = 1$ and $b = 1$. So the eigenvector for $\lambda = 4$ is $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Next, we find the eigenvector for $\lambda = -1$. Solve $\begin{bmatrix} 2-(-1) & 2 \\ 3 & 1-(-1) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Here, the two equations are $3a + 2b = 0$ and $3a + 2b = 0$, so the equations are not just multiples, but are identical. A simple solution is $a = -2$ and $b = 3$. So the eigenvector for $\lambda = -1$ is $\vec{v}_2 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$.

Thus, the general solution is $\vec{Y}(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + C_2 \begin{bmatrix} -2 \\ 3 \end{bmatrix} e^{-t}$. In terms of the component functions, $x(t)$ and $y(t)$, we have $x(t) = C_1e^{4t} - 2C_2e^{-t}$ and $y(t) = -C_1e^{4t} + 3C_2e^{-t}$.

Phase plane. The phase plane of this system is



Notice the line that is multiples of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, that is, the line $y = x$. On this line, solutions move straight out, away from the origin. This is because the eigenvector gives us a straight line of solutions through the origin on the phase plane. Moreover, because the associated eigenvalue is positive, the solutions move away from the origin. (See the picture on the last page.)

The other eigenvector, $\begin{bmatrix} -2 \\ 3 \end{bmatrix}$, also gives us a straight line of solutions through the origin, now on the line $y = -3x/2$. Because the eigenvalue is negative, the solutions more towards the origin on this line.

If the two eigenvalues are positive and distinct, then solutions move away from the origin along both straightline solutions. If the two eigenvalues are negative and distinct, then solutions move towards the origin along both straightline solutions. (See the pictures at the end of the handout.)

Next, we consider a system which will turn out to have complex eigenvalues,

$$\frac{dx}{dt} = x + 5y, \qquad \frac{dy}{dt} = -x + 3y.$$

It has the form $\dfrac{d\vec{Y}}{dt} = A\vec{Y}$. with $A = \begin{bmatrix} 1 & 5 \\ -1 & 3 \end{bmatrix}$ and $\vec{Y}(t)$ in the last example.

Eigenvalues. We solve $\begin{vmatrix} 1-\lambda & 5 \\ -1 & 3-\lambda \end{vmatrix} = 0$, which is $\lambda^2 - 4\lambda + 8 = 0$. The roots are $\lambda = 2 \pm 2i$.

Eigenvectors. To find this for $\lambda = 2 - 2i$, solve $\begin{bmatrix} 1-(2-2i) & 5 \\ -1 & 3-(2-2i) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The two equations are $(-1+2i)a + 5b = 0$ and $-a + (1+2i)b = 0$. These equations are still multiples of each other (multiply the first by $(1+2i)/5$ to get the second) but it is maybe more trouble than it's worth to check this when the equations with complex numbers.

To pick a solution we set a equal to the coefficient of b in the equation and b equal to minus the coefficient of a. Thus, $a = 5$ and $b = 1 - 2i$ is a solution. So the eigenvector for $\lambda = 2 - 2i$ is $\begin{bmatrix} 5 \\ 1-2i \end{bmatrix}$.

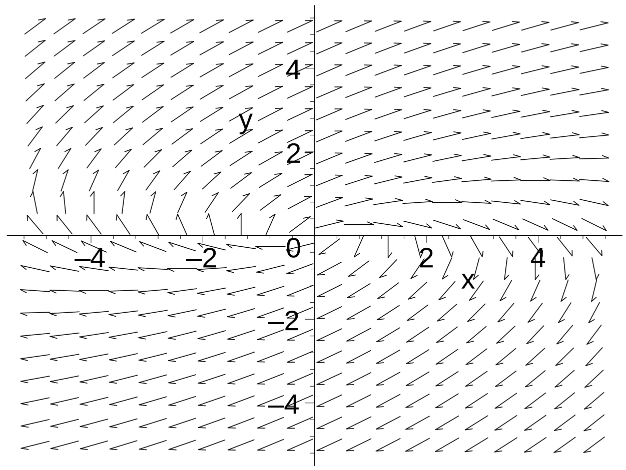
From this one eigenvector, we can find two solutions, using the formula given on the first page. The solutions are

$$\vec{Y}_1(t) = e^{-2t} \begin{bmatrix} 5\cos(2t) \\ \cos(2t) + 2\sin(2t) \end{bmatrix} \quad \vec{Y}_2(t) = e^{-2t} \begin{bmatrix} 5\sin(2t) \\ -2\cos(2t) + \sin(2t) \end{bmatrix}.$$

Notice that **we do not need to find the eigenvector for the second complex eigenvalue**. The general solution of the system is

$$\vec{Y}(t) = C_1 e^{-2t} \begin{bmatrix} 5\cos(2t) \\ \cos(2t) + 2\sin(2t) \end{bmatrix} + C_2 e^{-2t} \begin{bmatrix} 5\sin(2t) \\ -2\cos(2t) + \sin(2t) \end{bmatrix}.$$

Phase plane. The phase plane of this system is



Because the eigenvalues are complex instead of real, we get a spiral instead of straight lines of solutions. The real part of the eigenvalue tells us whether the solutions spiral inwards toward the origin or spiral outwards away from the origin. If the real part is negative, they spiral inward; if positive, they spiral outward; if zero, then the solutions loop, staying roughly the same distance from the origin. (Again, see the last page of the handout.)

For the final example, we consider a solution with a repeated eigenvalue,

$$\frac{dx}{dt} = x + y, \qquad \frac{dy}{dt} = -x + 3y.$$

It has the form $\dfrac{d\vec{Y}}{dt} = A\vec{Y}$. with $A = \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix}$ and $\vec{Y}(t)$ in the last example.

Eigenvalues. We solve $\begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = 0$, which is $\lambda^2 - 4\lambda + 4 = 0$. The roots $\lambda = 2$, repeated.

Eigenvectors. First we find the eigenvector for $\lambda = 2$. Solve $\begin{bmatrix} 1-2 & 1 \\ -1 & 3-2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The two equations are $-a + b = 0$ and $-a + b = 0$, so we happen to have exactly the same equation. Pick a solution $a = 1$ and $b = 1$. So the eigenvector for $\lambda = 2$ is $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and the first solution in the fundamental set is

$$\vec{Y}_1(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.$$

There is no second eigenvalue, so we have to find a second linearly independent solution by finding another vector. We solve the system the system given on the first page, namely

$$\begin{bmatrix} 1-2 & 1 \\ -1 & 3-2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

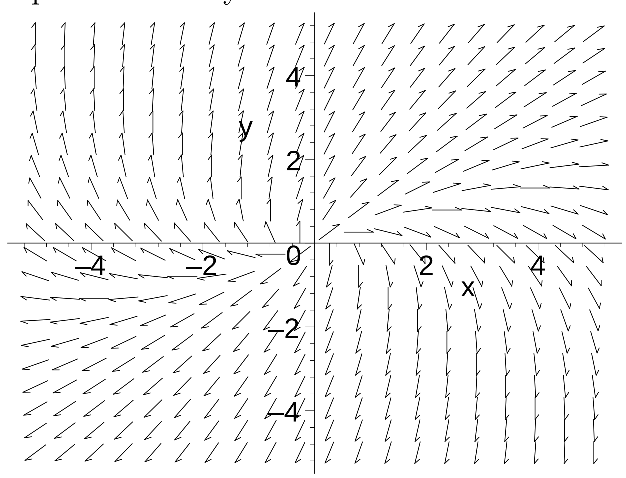
This system is the equations $-a + b = 1$ and $-a + b = 1$. A simple solution is $a = 0$ and $b = 1$. According to the forumla on the first page, the second solution is

$$\vec{Y}_2(t) = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{2t}.$$

The general solution is

$$\vec{Y}(t) = C_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + C_2 \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) e^{2t}.$$

Phase plane. The phase plane of this system is



Because we have only one eigenvalue and one eigenvector, we get a single straight-line solution; for this system, on the line $y = x$, which are multiples of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Notice that the system has a bit of spiral to it.