Chapter 05 : Integrals

Introduction

The development of integral calculus arises out the effortd of solving the problems of the following types :

(a) the problem of finding a function whenever its derivative is given,

(b) the problem of finding the area bounded by the graph of a function under certain conditions.

These two problems lead to the two forms of the integrals, e.g., indefinite and definite integrals, which together constitute the ***Integral Calculus***.

Definition01 :

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Examples :

Find an antiderivative for each of the following functions.

(a) $f(x) = 2x$

(b) $g(x) = \cos x$

(a) $F(x) = x^2$

(b) $G(x) = \sin x$

Corollary 01 :

If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is

$$F(x) + C$$

where C is an arbitrary constant.

Thus, $\{F + C, C \in \mathbf{R}\}$ denotes a family of anti derivatives of f.

Example :

The function $F(x) = x^2$ is not the only function whose derivative is $2x$. The function $x^2 + 1$ has the same derivative. So does $x^2 + C$ for any constant C. Are there others?

Definition 02 :

We introduce a new symbol, namely, $\int f(x)\,dx$ which will represent the entire class of anti derivatives read as the indefinite integral of f with respect to x.
Symbolically, we write $\int f(x)\,dx = F(x) + C$.

Examples :

Find

1 $\int 1\,dx$ 2 $\int (3x-1)^4\,dx.$

1 We can observe directly that x is an antiderivative of 1, or we can use the above rule for the antiderivative of x^n when $n = 0$. Either way, the general antiderivative is $x + c$ and so $\int 1\,dx = x + c$, where c is a constant.

2 Using the above rule for the antiderivative of $(ax+b)^n$ with $a = 3$, $b = -1$ and $n = 4$, we obtain

$$\int (3x-1)^4\,dx = \frac{1}{15}(3x-1)^5 + c, \qquad \text{where } c \text{ is a constant.}$$

Theorem (Linearity of integration)

a *If f and g are continuous functions, then*

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$$

b *If f is a continuous function and k is a real constant, then*

$$\int kf(x)\,dx = k\int f(x)\,dx.$$

Examples :

$$\int\left(\frac{1}{2x}-\frac{2}{x^2}+\frac{3}{\sqrt{x}}\right)dx = \frac{1}{2}\int\frac{1}{x}dx-2\int x^{-2}dx+3\int x^{-\frac{1}{2}}dx =$$
$$= \frac{1}{2}\ln|x|-2\cdot(-1)x^{-1}+3\cdot 2x^{\frac{1}{2}}+C =$$
$$= \frac{\ln|x|}{2}+\frac{2}{x}+6\sqrt{x}+C.$$

$$\int(3x^3-4x^2+2)\,dx=\int 3x^3\,dx-\int 4x^2\,dx+\int 2\,dx$$
$$=3\int x^3\,dx-4\int x^2\,dx+2\int 1\,dx$$
$$=\frac{3x^4}{4}-\frac{4x^3}{3}+2x+c.$$

Theorem :

Let $f(x)$ be a continuous real-valued function on the interval $[a,b]$. Then

$$\int_a^b f(x)\,dx=\Big[F(x)\Big]_a^b=F(b)-F(a),$$

where $F(x)$ is any antiderivative of $f(x)$.

Example :

$$\int_0^5(x^2+1)\,dx=\left[\frac{1}{3}x^3+x\right]_0^5$$
$$=\left(\frac{1}{3}\cdot 5^3+5\right)-\left(\frac{1}{3}\cdot 0^3+0\right)$$
$$=\frac{125}{3}+5=\frac{140}{3}=46\frac{2}{3}.$$

Integration Rules and Techniques

1.Integration by parts

Theorem (Integration by Parts Formula)

$$\int f(x)g(x)\,dx = F(x)g(x) - \int F(x)g'(x)\,dx$$

where $F(x)$ is an anti-derivative of $f(x)$.

Examples : **1)**

$$g(x) = x \quad f(x) = \cos(x)$$
$$g'(x) = 1 \quad F(x) = \sin(x)$$

$$\int x\cos(x)\,dx = x\sin(x) - \int \sin(x)\,dx$$
$$= \boxed{x\sin(x) + \cos(x) + C}$$

2)

$$g(x) = x \quad f(x) = e^{-x}$$
$$g'(x) = 1 \quad F(x) = -e^{-x}$$

$$\int_0^4 xe^{-x}\,dx = -xe^{-x}\Big|_0^4 - \int_0^4 -e^{-x}\,dx$$
$$= -xe^{-x} - e^{-x}\Big|_0^4$$
$$= \left[-4e^{-4} - e^{-4}\right] - \left[0 - e^{-0}\right]$$
$$= -5e^{-4} + 1$$
$$= \boxed{1 - 5e^{-4}}$$

2) Integration by substitution

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))\, g'(x)\, dx = \int f(u)\, du$$

If we have a definite integral, then we can either change back to xs at the end and evaluate as usual; alternatively, we can leave the anti-derivative in terms of u, convert the limits of integration to us, and evaluate everything in terms of u without changing back to xs:

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du$$

Examples :

1. $\int (2x+6)^5 dx$

   Solution. Substituting $u = 2x+6$ and $\frac{1}{2}du = dx$, you get

$$\int (2x+6)^5 dx = \frac{1}{2}\int u^5 du = \frac{1}{12}u^6 + C = \frac{1}{12}(2x+6)^6 + C.$$

2.

$\int xe^{x^2} dx$

Solution. Substituting $u = x^2$ and $\frac{1}{2}du = xdx$, you get

$$\int xe^{x^2} dx = \frac{1}{2}\int e^u du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

3.

$\int \frac{2x^4}{x^5+1} dx$

Solution. Substituting $u = x^5 + 1$ and $\frac{2}{5}du = 2x^4 dx$, you get

$$\int \frac{2x^4}{x^5+1} dx = \frac{2}{5}\int \frac{1}{u} du = \frac{2}{5}\ln|u| + C = \frac{2}{5}\ln\left|x^5+1\right| + C.$$

<u>Table of Integrals :</u> Let $u : I \to IR,\ a,b \in IR,\ and\ F\ is\ antiderivative\ of\ f.$

$f(x) =$	$F(x) =$
a (a عدد حقيقي)	$ax + c$
x	$\frac{1}{2}x^2 + c$
$x^n / n \in \mathbb{Q} - \{-1\}$	$\frac{1}{n+1}x^{n+1} + c\ / n \in \mathbb{Q} - \{-1\}$
$\frac{1}{x^2}$	$-\frac{1}{x} + c$
$u'u$	$\frac{1}{2}u^2 + c$
$u'u^n$ ($n \in \mathbb{Q} - \{-1\}$)	$\frac{1}{n+1}u^{n+1} + c\ / n \in \mathbb{Q} - \{-1\}$
$\frac{u'}{u^2}$	$-\frac{1}{u} + c$
$\frac{u'}{u^n}$ ($n \in \mathbb{Q} - \{1\}$)	$\frac{1}{(n-1)u^{n-1}} + c\ / n \in \mathbb{Q} - \{1\}$
$\frac{u'}{\sqrt{u}}$	$2\sqrt{u} + c$
$\cos(ax + b)$	$\frac{1}{a}\sin(ax + b)$
$\sin(ax + b)$	$-\frac{1}{a}\cos(ax + b)$
$\frac{1}{x^n}\ / n \in \mathbb{Q} - \{1\}$	$\frac{1}{(n-1)x^{n-1}} + c\ / n \in \mathbb{Q} - \{1\}$
$\frac{1}{\sqrt{x}}$	$2\sqrt{x} + c$
$\sin x$	$-\cos x + c$
$\cos x$	$\sin x + c$
$1 + \tan^2 x = \frac{1}{\cos^2 x}$	$\tan x + c$