University of Oum El Bouaghi

Faculty of Economic Sciences, Commercial Sciences and Management Sciences

First year, common trunk

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Mathematics 1 Module
Chapter 04 : Derivatives.

Definition 01:

Let $f: D \to IR, x_0 \in D$. The derivative of f at a point x_0 , writtren $f'(x_0)$, is given by :

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$$

if this limit exists.

Graphically, the derivative of a function corresponds to **the slope of its tangent line at one specific point.**

Note

If f is derived at x_0, it continues at this point and the opposite is not true in general.

Definition 02 :

The limites

$$\lim_{\substack{> \\ x \to x_0}} \frac{f(x) - f(x_0)}{x - x_0} \qquad \lim_{\substack{< \\ x \to x_0}} \frac{f(x) - f(x_0)}{x - x_0}$$

are called right-hand derivative and left-hand derivative of f at x_0 respectively.

Note

The function f has a derivative at a point x_0 if and only if the function's right-hand derivative and left-hand derivative are defined and equal at that point.

Examples :1) Let

$$h(x) = |x| \qquad D_h = R$$

$$h(x) = \begin{cases} x \ si & x \geq 0 \\ -x \ si & x \leq 0 \end{cases}$$

$$\lim_{\substack{x \to 0 \\ >}} \frac{h(x) - h(0)}{x - 0} = \lim_{\substack{x \to 0 \\ >}} \frac{x}{x} = 1 \qquad \lim_{\substack{x \to 0 \\ <}} \frac{h(x) - h(0)}{x - 0} = \lim_{\substack{x \to 0 \\ <}} \frac{-x}{x} = -1$$

Then the left-hand derivative and right-hand derivative of h at zero are not equal. Therefore, h does not have a derivative at 0.

1) Let :

$$g(x) = \sqrt{x} \qquad D_g = [0, +\infty[$$

$$\lim_{\substack{x \to 0 \\ >}} \frac{g(x) - g(0)}{x - 0} = \lim_{\substack{x \to 0 \\ >}} \frac{\sqrt{x}}{x} = \frac{0}{0}$$

$$= \lim_{\substack{x \to 0 \\ >}} \frac{\sqrt{x}}{\sqrt{x}\sqrt{x}} = \lim_{\substack{x \to 0 \\ >}} \frac{1}{\sqrt{x}} = +\infty$$

Then the right-hand derivative of g at zero does not exist. Therefore, g does not have a derivative at 0.

<u>Definition 3 :</u>

Afunction f is differentiable on a closed interval $[a, b]$ if it has a derivative at every interior point on the interval and if left-hand derivative of f at b and right-hand derivative of f at a are exists.

<u>List of derivative rules</u>

- **Constant Rule:** $f(x) = c$ then $f'(x) = 0$
- **Constant Multiple Rule:** $g(x) = c \cdot f(x)$ then $g'(x) = c \cdot f'(x)$
- **Power Rule:** $f(x) = x^n$ then $f'(x) = nx^{n-1}$
- **Sum and Difference Rule:** $h(x) = f(x) \pm g(x)$ then $h'(x) = f'(x) \pm g'(x)$
- **Product Rule:** $h(x) = f(x)g(x)$ then $h'(x) = f'(x)g(x) + f(x)g'(x)$
- **Quotient Rule:** $h(x) = \frac{f(x)}{g(x)}$ then $h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$
- **Chain Rule:** $h(x) = f(g(x))$ then $h'(x) = f'(g(x))g'(x)$
- **Power function :** $h(x) = (f(x))^n, \text{then } h'(x) = nf'(x)(f(x))^{n-1}$

- **Trig Derivatives:**
  - $f(x) = \sin(x)$ then $f'(x) = \cos(x)$
  - $f(x) = \cos(x)$ then $f'(x) = -\sin(x)$

- Exponential Derivatives

- $f(x) = e^x$ then $f'(x) = e^x$

- $f(x) = e^{g(x)}$ then $f'(x) = e^{g(x)}g'(x)$

- Logarithm Derivatives

- $f(x) = \ln(x)$ then $f'(x) = \dfrac{1}{x}$

- $f(x) = \ln(g(x))$ then $f'(x) = \dfrac{g'(x)}{g(x)}$

Higher-Order Derivatives

$f''(x)$, read as "**f double prime** of (or at) x," is the second derivative (or the second-order derivative) of f with respect to x.

- It is the [first] derivative of $f'(x)$ with respect to x.

$f'''(x)$, read as "**f triple prime** of (or at) x," is the third derivative (or the third-order derivative) of f with respect to x.

- It is the [first] derivative of $f''(x)$ with respect to x.

Higher-order derivatives are denoted by $f^{(4)}(x)$, $f^{(5)}(x)$, etc.

Note : $\forall n \in IN$, **the nth derivative of f is giving by**

$$f^{(n)}(x) = \left(f^{(n-1)}(x)\right)'$$

$$f^{(0)}(x) = f(x)$$

Example :

$$f(x) = x^{-1} = \frac{1}{x}$$

$$f'(x) = (x^{-1})' = -\frac{1}{x^2} = -x^{-2}$$

$$f^{(2)(x)} = (-1)(-2)x^{-3}$$

$$f^{(3)(x)} = (-1)(-2)(-3)(x^{-4}$$

$$f^{(4)(x)} = (-1)(-2)(-3)(-4)x^{-5}$$

$$f^{(n)(x)} = (-1)(-2)(-3)\ldots(-n)x^{-(n+1)} = \frac{(-1)^n n!}{x^{n+1}}$$

L'Hôpital's Rule

If $\lim_{x\to c} \frac{f(x)}{g(x)}$ exists and has indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then $\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$.

Examples

$$\lim_{x\to 4} \frac{\sqrt{x}-\sqrt{4}}{x-4} = \lim_{x\to 4} \frac{\frac{1}{2\sqrt{x}}-0}{1-0} = \lim_{x\to 4} \frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0} \frac{\cos(x)}{1} = \cos(0) = 1$$

$$\lim_{x\to 0} \frac{1-\cos(x)}{x} = \lim_{x\to 0} \frac{0+\sin(x)}{1} = \frac{0+0}{1} = 0$$

$$\lim_{x\to 0} \frac{1+1/x}{2+1/x} = \lim_{x\to 0} \frac{0-1/x^2}{0-1/x^2} = \lim_{x\to 0} \frac{-1/x^2}{-1/x^2} = 1$$