Chapter IV: Dynamics of material point

Summary

1. Definitions

1.1. Dynamics

Kinematics is the description of the movement of a material point (its position, its velocity and its acceleration), without taking into account to their causes. While dynamics allows us to study the movement of the material point with their causes or actions. These actions are called “forces”.

Furthermore, dynamics is based on Newton's laws to relate the effect of forces to kinematic quantities ($\vec{r}, \vec{V}, \vec{\gamma}$). However, these laws can only be applied in the Galilean reference frame.

1.2. Mass (m)

Mass (inert) is an intrinsic characteristic of a material. In classical mechanics, mass is a **positive, conservative** (invariant in time and independent of reference) and **extensive** physical quantity, that is to say that the mass of a body is the sum of the masses of its constituents.

I.3. Isolated system

An isolated (closed) system is a system which does not interact with the environment, that is to say it does not exchange matter or energy with the outside world over time.

I.4. Force (F)

A material point of mass m is rarely isolated; it is most often subject to external interactions. These interactions, modeled by **force vectors** ($\vec{F}$), induce the modification of its velocity vector $\vec{V}$.

Forces can be classified according to their distance of action into **the contact forces** and **the distance forces**. In modern physics, there are **04 types of fundamental interaction forces** in the universe which are:

1. **Gravitational interaction**: responsible for the cohesion of space.
2. **Electromagnetic interaction**: responsible for the cohesion of the atom.
3. **Weak nuclear interaction**: responsible for the radioactive decay of subatomic particles and nuclear fusion.
4. **Strong nuclear interaction**: responsible for the cohesion of nuclei (protons, neutrons).

When a material point is subjected to several interaction forces from several material points, the resultant of the forces represents the sum of all these interaction forces:

$$\vec{F}_{tot} = \sum \vec{F}$$

2. Newton's laws

2.1. Principle of inertia PI (Newton's 1st law)

a. Statement of Newton's 1st law

The principle of inertia is based on various experiments carried out firstly by **Galileo**. In the modern form the principle of inertia is stated by Newton as follows:

> « ***Every object perseveres in its state of rest, or of uniform rectilinear movement, if no external force has intervened to change its state*** »
>
> An object is not subjected to any force, that is to say the resultant of external forces is zero:
>
> $$\sum \vec{F}_{ext} = \vec{0}$$

b. Galilean Reference

A reference frame is defined as a Galilean reference, or inertial reference, if the principle of inertia is verified, that is to say that:

« ***Any free body in uniform rectilinear movement or at rest is considered as a Galilean reference frame*** », (velocity is constant or zero).

« ***Any reference frame in uniform rectilinear translation movement or at rest with respect to a Galilean reference is itself Galilean***. »

There is no absolute Galilean reference in space. We can only propose certain **approximate reference frames** which are considered Galilean to different degrees such as:

1- **Copernicus reference frame**: Its origin is the center of gravity of the solar system and the axes directed towards three very distant stars which appear fixed in space.

2- **Geocentric reference frame**: Its origin is the center of gravity of the earth and its axes are always directed towards the same three stars of Copernicus.

3- **Kepler reference frame** (or **heliocentric reference frame**): Its origin is the center of gravity of the sun and the axes are parallel to those of the Copernicus reference frame.

4- **Terrestrial reference frame**: Its origin is any point on the earth and its axes are linked to the earth's rotation.

c. Quantity of movement ($\vec{p}$)

Every moving object has a quantity of movement. The quantity of movement of a material point M with respect to a reference frame R is the product of its mass m by its velocity $\vec{V}$. This vector quantity is defined by:

$$\vec{p} = m\,\vec{V}$$

- ✓ When the velocity is constant $\vec{V} = C^t$, the quantity of movement is constant $\vec{p} = C^t$
- ✓ When the velocity is varying $\vec{V} \neq C^t$, the quantity of movement is varying $\vec{p} \neq C^t$, which means that the object is subject to external forces.

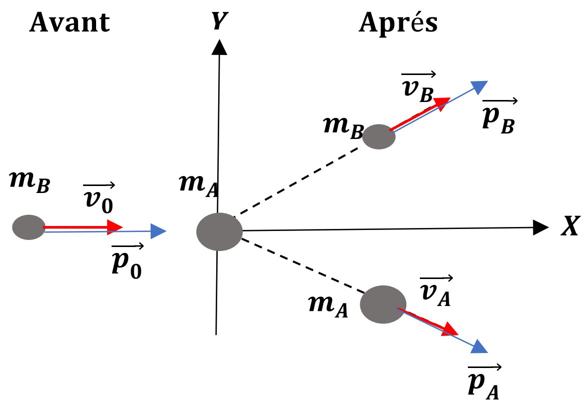
d. Principle of conservation of the quantity of movement

« In a Galilean reference frame, the quantity of movement vector of an isolated system is constant»

The law of conservation of the quantity of movement helps to explain the collisions or breakups of an isolated system. The total quantity of movement before the collision is equal to the total quantity of movement after the collision. From where:

$$\sum \vec{p}_i = \sum \vec{p}_f$$

Example



We consider two particles: A (at rest $\vec{v_0} = \vec{0}$) and B (in motion $\vec{v_0}$). The two particles coalesce and move with two different velocities $\vec{v_A}$ and $\vec{v_B}$, respectively (**Figure opposite**).

If the system is isolated, we apply the principle of conservation of $\vec{p}$ before and after the shock, we find:

The quantity of movement before the shock: $\vec{p_0} = m_B\,\vec{v_0}$

The quantity of movement after the shock: $\begin{cases} \vec{p_A} = m_A\,\vec{v_A} \\ et \\ \vec{p_B} = m_B\,\vec{v_B} \end{cases} \Longrightarrow \begin{cases} \Delta\vec{p_A} = \vec{p_A} - \vec{p_0} = \vec{p_A} - 0 \\ \text{et} \\ \Delta\vec{p_B} = \vec{p_B} - \vec{p_0} \end{cases}$

According to the principle, $\vec{p_0} = \vec{p_A} + \vec{p_B} \Longrightarrow |\Delta\vec{\boldsymbol{p_B}}| = |\Delta\vec{\boldsymbol{p_A}}| = C^t$

2.2. Fundamental principle of dynamics (Newton's 2nd law)

When forces are applied to a material point, they lead to the variation of its quantity of movement $\vec{p}$ as follows:

$$\sum \vec{F}_{ext} = \frac{d\vec{p}}{dt}$$

It comes:

$$\sum \vec{F}_{ext} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{v}) = m\,\frac{d\vec{v}}{dt} = m\,\vec{\gamma}$$

Therefore, the resultant of the external forces applied to the material point is characterized by its acceleration $\vec{\gamma}$. The fundamental principle of dynamics is therefore stated by Newton as follows:

« ***In a Galilean reference frame (R), the resultant of the external forces applied to a material point of mass m is proportional to its acceleration vector*** $\vec{\gamma}$ ».

$$\sum \vec{F}_{ext} = m\,\vec{\gamma}$$

2.3. Principle of action and reaction (Newton's 3rd law)

The principle of action and reaction, or principle of reciprocal reactions, was stated by Newton as follows:

« ***Let us consider two material points*** (A) ***and*** (B) ***interacting with each other; the action exerted by*** (A) ***on*** (B)***:*** $\vec{F}_{A/B}$ ***is equal and opposite to that exerted by*** (B) ***on*** (A)***:*** $\vec{F}_{B/A}$ ».

$$\vec{F}_{A/B} = -\vec{F}_{B/A}$$

$$\left\|\vec{F}_{A/B}\right\| = \left\|\vec{F}_{B/A}\right\|$$

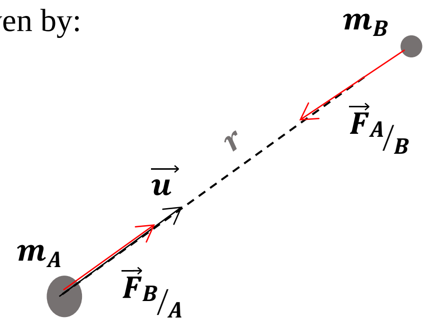
These two forces have the same nature

3. Some usual forces

3.1. Forces at a distance

a. The gravitational force

The gravitational force is an attractive force that acts at a distance between two celestial bodies. The expression of the force exerted between a body A of mass m_A (unit kg) and a body B of mass m_B (unit kg) separated by a distance r is given by:



$$\vec{F}_{A/B} = -\vec{F}_{B/A}$$

$$\vec{F}_{B/A} = G\,\frac{m_A m_B}{r^2}\,\vec{u}$$

Where:

- **G**: is the universal gravitational constant, $G = 6{,}67384 \times 10^{-11}\ N.m^2.kg^{-2}$ in the international system (IS).
- $\vec{u}$: unit vector of $\vec{r} = \overrightarrow{AB}$, $\vec{u} = \frac{\vec{r}}{r}$

- We can write: $\vec{F} = m\vec{g}$, where: $\vec{g} = G\,\frac{m_A}{r^2}\,\vec{u}$, r is the radius of the earth and m_A, is the mass of the earth, the norm of this vector is the gravitational field on the earth's surface: $g_0 \sim 9,81\ m.s^{-2}$

b. The electrostatic force

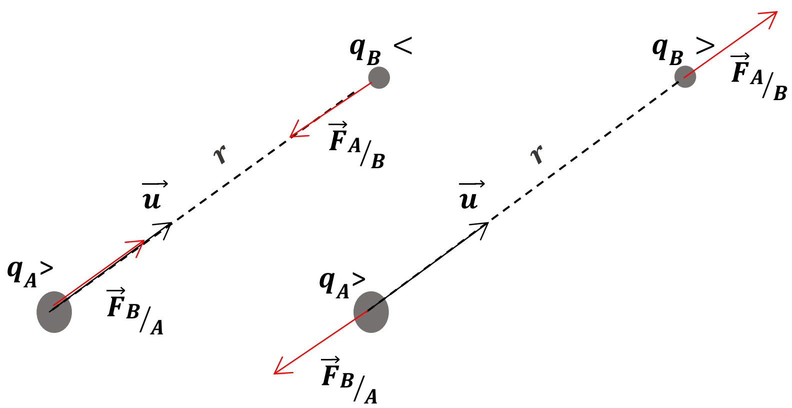
The electric or electrostatic force presents the interaction between two electric charges q_A and q_B at rest, its intensity is proportional to the product of these two charges and inversely proportional to the square of the distance r separating between them.

This force is attractive if the charges have the opposite signs and repulsive if the charges have the same signs. The expression of the electrostatic force is given by Coulomb's law:

$$\vec{F}_{A/B} = -\vec{F}_{B/A}$$

$$\vec{F}_{B/A} = k\,\frac{q_A q_B}{r^2}\,\vec{u}$$

$$k = \frac{1}{4\,\pi\,\varepsilon_0}$$



where:

- **k** : is Coulomb's constant, $K = 9 \times 10^9\ N.m^2.C^{-2}$ in international system (SI.).
- ε_0 **:** vacuum permittivity, $\varepsilon_0 = 8{,}854 \times 10^{-12}\ F\ m^{-1}$
- $\vec{u}$: unit vector of $\vec{r} = \overrightarrow{AB}$, $\vec{u} = \frac{\vec{r}}{r}$

c. The electromagnetic force or Lorentz force $\vec{F_l}$

The force exerted on an electric charge moved by a velocity $\vec{v}$ in an electric field $\vec{E}$ (subject to an electric force $\vec{F_e} = q\vec{E}$) and a magnetic field $\vec{B}$ (subject to a magnetic force ($\vec{F_m} = q(\vec{v} \wedge \vec{B}$)) is called the electromagnetic force or Lorentz force:

$$\vec{F_l} = q\left(\vec{E} + \vec{v} \wedge \vec{B}\right)$$

d. The strong and weak nuclear interaction

The **strong nuclear interaction** is responsible for the cohesion of protons and neutrons. Its intensity is 100 times stronger than the electromagnetic interaction but only occurs on a scale of $10^{-15}m$.

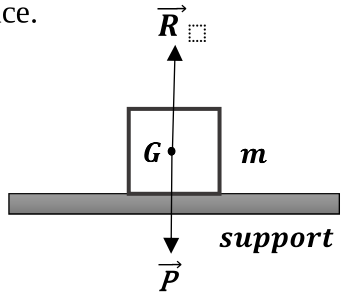
The **weak nuclear interaction** manifests in the β decay of the neutrino. Its range is even weaker than the strong nuclear interaction and its intensity is weaker than the electromagnetic interaction.

3.2. Contact forces

These are another forces that take place once contact is established. We can cite a few of them:

a. Reaction force of the support

The force exerted on a body of mass m and weight $\vec{P} = m\vec{g}$, placed on a horizontal support, is called the reaction force of the support $\vec{R}$. The reaction of the support on the body is distributed over the entire support-body contact surface.



The object being in equilibrium: $\vec{R} = \vec{P}$

b. Friction force

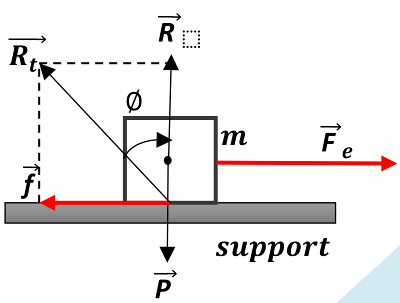
The movement of a body of mass m on a rigid support (solid) or in a fluid medium (gas or liquid) is always associated with a variation in its quantity of movement $\vec{p}$ due to the interaction between the atoms of the two body. This interaction which opposes the direction of movement is called **friction force** $\vec{f}$. There are two types of friction:

- Solid friction (solid-solid contact)
- Viscous friction (solid-fluid contact)

1) Sliding Friction Force

The sliding friction force appears between two solid bodies in contact. According to this figure, the body of mass m moves under the action of a driving force $\vec{F}_e$ on a non-smooth support, of which the reaction of the support on the body is decompose into:

1. **Normal component: Reaction force** of the support $\vec{R}$ or $\vec{N}$.
2. **Parallel component**: **Friction force** $\vec{f}$. This force is proportional to the normal component $\vec{R}$, and the proportionality constant is called the friction coefficient μ. From where :



$$\vec{f} = \mu\vec{R} \quad ; \quad \mu = tg\emptyset$$

The friction coefficient μ depends on the nature of the surfaces in contact. And ∅ is called an angle of friction. There is two type of friction coefficient μ which are: static friction coefficient μ_s and kinetic friction coefficient μ_c.

2) Fluid Friction force

Fluid friction force is also called viscous friction force which describes the friction force exerted by fluids. The viscous friction force $\vec{f_v}$ appears when a solid body moves within a fluid medium such as gas or liquid. It is due to the shocks generated by the free molecules of the fluid on the solid body. This force is proportional to the velocity $\vec{v}$. From where:

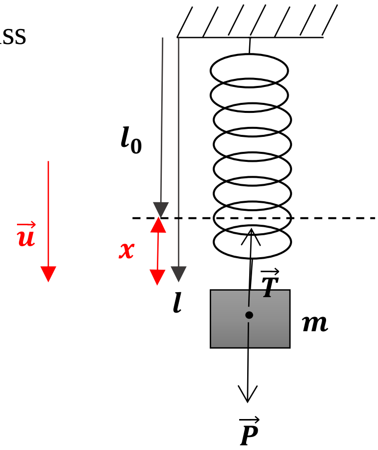
$$\vec{f}_v = -k.\eta.\vec{v},$$

(Valid only for low velocities)

- **k** : is a coefficient related to the shape of the solid body;
- η : is the viscosity coefficient ($kg.\ m^{-1}.\ s^{-1}$) of the considered fluid.
- We also find the following form: $\vec{f_v} = -\alpha.\vec{v}$

c. Tension force

We consider a spring fixed at one of its extremities and at the end of which the body of mass m is attached. When the spring extends, an elastic restoring force, proportional to this elongation and called tension force $\vec{T}$, is exerted on the mass



$$\vec{T} = -k(l - l_0).\vec{u}$$

$$T = kx$$

- k : spring stiffness constant
- l : length at time t of the spring
- l_0 : empty length of the spring
- $\vec{u}$: unit vector

4. kinetic moment or angular momentum

4.1. Definition

The kinetic moment $\vec{L}_O$ of a material point M of mass (m) with respect to the fixed point O, moving by velocity $\vec{V}$ and having a quantity of movement $\vec{p}$, is defined by the vector product:

$$\vec{L}_O = \overrightarrow{OM} \wedge \vec{p}$$

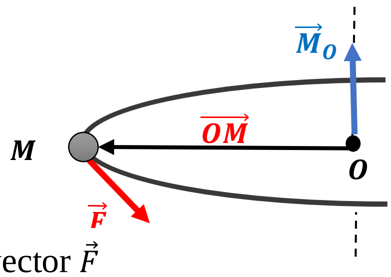
$$\vec{L}_O = \overrightarrow{OM} \wedge m\vec{V}$$

$\rightarrow M$: a point belongs to the carrier axis of the vector $\vec{p}$

Likewise, we give the moment of a force $\vec{M}_O(\vec{F})$ with respect to a fixed-point O, applied to a material point M, as follows:

$$\vec{M}_O(\vec{F}) = \overrightarrow{OM} \wedge \vec{F}$$

$$\vec{M}_O(\vec{F}) = \overrightarrow{OM} \wedge m\vec{\gamma}$$



$\rightarrow M$: a point belongs to the carrier axis of the vector $\vec{F}$

4.2. Kinetic moment theorem or Angular Momentum Theorem

In a Galilean reference frame (R), the temporal derivative of the angular momentum of a material point M, with respect to a fixed point O, is equal to the moment of the resultant of the external forces applied to the material point M with respect to the same fixed point O. Hence:

$$\vec{L}_O = \overrightarrow{OM} \wedge \vec{p}$$

$$\frac{d\vec{L}_O}{dt} = \overrightarrow{OM} \wedge \frac{d\vec{p}}{dt} = \overrightarrow{OM} \wedge \frac{d(m\vec{v})}{dt} = \overrightarrow{OM} \wedge m\frac{d\vec{v}}{dt} = \overrightarrow{OM} \wedge m\,\vec{\gamma} = \overrightarrow{OM} \wedge \vec{F}$$

$$\boldsymbol{\frac{d\vec{L}_O}{dt} = \vec{M}_O(\vec{F})}$$

4.3. The Kinetic moment in the presence of a central force

We say that a movement is central (under the effect of a central force $\vec{F}$), if the result of the forces applied to a body is directed towards a fixed point, which called the center of accelerations. So, the support of this force is at each instant t is directed towards a fixed point O, and this means that: $\overrightarrow{OM} \parallel \vec{F}$

$$\overrightarrow{OM} \parallel \vec{F} \implies \overrightarrow{OM} \wedge \vec{F} = \vec{0}$$

So, the moment of force applied to a material point M is zero $\vec{M}_O(\vec{F}) = \vec{0}$:

$$\frac{d\vec{L}_O}{dt} = \vec{M}_O(\vec{F}) = \vec{0} \implies \vec{L}_O = C^t$$

Therefore, **“the moment of central force $\vec{M}_O(\vec{F})$ is always zero, while the Kinetic moment $\vec{L}_O$ of a material point is constant”.**

5. Differential equations

5.1. Definition

We call a differential equation (d.eq) of order n an equation of the form:

$$F(y^{(n)}(t),\ y^{(n-1)}(t), \dots, y'(t), y(t), t) = 0$$

Where $y(t)$ is the unknown scalar function, t: is the variable and n is the number of times differentiable.

We call solution of the differential equation any function $y(t)$ of n times differentiable, verifying the differential equation.

5.2. Differential equation of the first order

a) Differential equation of the first order without second member

A linear differential equation of the **1st order** without a **second member** is given by:

$$ay' + b\,y = 0$$

Where: a and b are real constants.

The set of solutions of the homogeneous linear differential equation of the **1st order** is:

$$y(t) = C\,e^{-\frac{b}{a}t}$$

Where C : is a real constant.

b) Differential equation of the first order with second member

A linear differential equation of the **1st order** with second member is given by:

$$ay' + b\,y = f(t)$$

Where: **a** and **b** are real constants and $f(t)$ is a continuous function.

The solution to this equation consists of two $(\mathbf{02})$ general and particular solutions:

$$\boldsymbol{y(t) = y_g(t) + y_p(t)}$$

$\boldsymbol{y_g}$: is the general solution consists of putting the second member: $f(t) = 0$, therefore:

$$y_g(t) = C\, e^{-\frac{b}{a}t}$$

$\boldsymbol{y_p}$: is the particular solution takes the same form as that of the second member ***f (t)***.

$$y_p(t) = \int f(t)dt \text{ , So}$$

$$\boldsymbol{y(t) = C\, e^{-\frac{b}{a}t} + \int f(t)dt}$$

5.3. Differential equation of the second order

A linear differential equation of the **2nd order** is given by:

$$\boldsymbol{ay'' + b\,y' + cy = f(t)}$$

Where: **a, b** and **c** are real constant and $\boldsymbol{f(t)}$ is a continuous function.

The solution of this equation is also made up of two $(\mathbf{02})$ general and particular solutions $\boldsymbol{y = y_g + y_p}$, $\boldsymbol{y_p}$: takes the same form as previously:

$$y_p(t) = \int f(t)dt$$

$\boldsymbol{y_g}$: is the general solution consists of putting the second member: $f(t) = 0$, therefore:

$$\boldsymbol{ay'' + b\,y' + cy = 0}$$

We must therefore calculate the determinant $\Delta = b^2 - 4ac$ of equation: $\boldsymbol{ay^2 + b\,y + c = 0}$

If: $\Delta > 0 \Rightarrow s_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2a} \Rightarrow \boldsymbol{y_g = C_1\, e^{s_1 t} + C_2\, e^{s_2 t}}$

If: $\Delta = 0 \Rightarrow s_1 = s_2 = \frac{-b}{2a} \Rightarrow \boldsymbol{y_g = C_1\, e^{s_1 t} + C_2\, e^{s_1 t}}$

If: $\Delta < 0 \Rightarrow s_{1,2} = \lambda \pm i\omega,\ \left[\lambda = \frac{-b}{2a}\ ;\ \omega = \frac{\sqrt{\Delta}}{2a}\right] \Rightarrow \boldsymbol{y_g = [C_1 \cos \omega t + C_2 \sin \omega t\,]e^{\lambda t}}$

a) <u>Differential equation of the form $y'' + \omega^2 y = 0$</u>

The solution takes the form:

$$\boldsymbol{y = C_1 \cos\omega t + C_2 \sin\omega t.}$$

b) <u>Differential equation of the form $y'' - \omega^2 y = 0$</u>

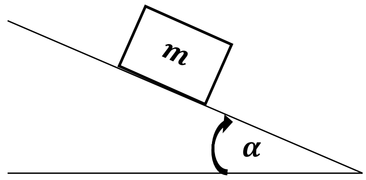
The solution takes the form:

$$y = C_1 \frac{e^{\omega t} + e^{-\omega t}}{2} + C_2 \frac{e^{\omega t} - e^{-\omega t}}{2}$$

6. Applications

The **figure below** represents a body with a mass of $800\ g$, moving over a rough plane inclined by α. The kinetic friction coefficient μ is 0.40. We take $g = 10\ ms^{-2}$.

1. What should be the angle of inclination so that the body slides with a constant velocity?
2. What is the normal reaction force of the plane N?
3. What is the friction force f?
4. What is the acceleration for an inclination of $\alpha = 35°$?

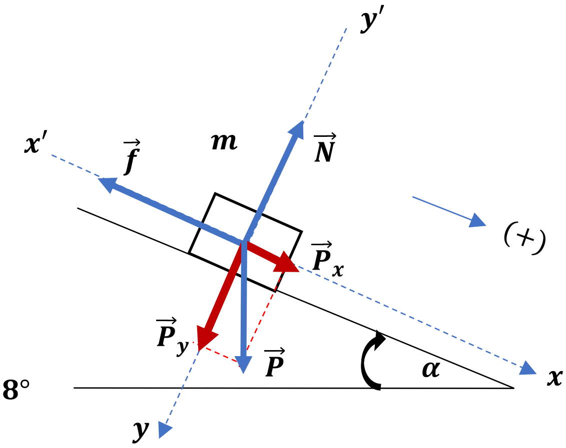


☒ Solution

1/ Angle of inclination α necessary for the body to move at constant velocity, this means that the sum of the forces must be zero (principle of inertia, Newton's 1st law): :

$$V = C^{te} \Rightarrow \sum \vec{F}_{ext} = \vec{0}$$

$$\vec{p} + \vec{f} + \vec{N} = \vec{0}$$



By projection on the two axes, it comes:

$$\begin{cases} P_x - f = 0 \\ p_y - N = 0 \end{cases} \Rightarrow \begin{cases} P_x = f \\ p_y = N \end{cases} \Rightarrow \begin{cases} p \sin\alpha = \mu N \\ p \cos\alpha = N \end{cases}$$

$$\Rightarrow \frac{p \sin\alpha}{p \cos\alpha} = \frac{\mu N}{N} \Rightarrow \tan\alpha = \mu = 0.4 \Rightarrow \boldsymbol{\alpha = 21.8°}$$

2/ The normal reaction force of the plane N for $\alpha = 35°$:

$$p \cos\alpha = N \Rightarrow N = mg\cos\alpha = 8\cos 35 = 6.55\ N \Rightarrow \boldsymbol{N = 6.55\ N}$$

3/ the friction force f for $\alpha = 35°$: $f = \mu N \Rightarrow \boldsymbol{f = 2.62\ N}$

4/ Acceleration for α= 35°: Angle of inclination 35° >21.8° (case of constant Velocity), this means that the Velocity is variable (fundamental principle of dynamics, Newton's 2nd law):

$$PFD \Rightarrow \sum \vec{F}_{ext} = m\vec{\gamma}$$

By projection on the two axes, it comes:

$$\begin{cases} P_x - f = m\gamma \\ p_y - N = 0 \end{cases} \Rightarrow p \sin\alpha - f = m\gamma \Rightarrow \gamma = \frac{mg \sin\alpha - f}{m} \Rightarrow \boldsymbol{\gamma = 2.46}$$