Chapter 3
Functions

Introduction

The term *function* was first used by Leibniz in 1673 to denote the dependence of one quantity on another. In general, if a quantity y depends on a quantity x in such a way that each value of x determines exactly one value of y, then we say that y is a "function" of x.

Definition

A function is a rule which maps a number to another unique number.

We have numerical fonction is giving by : $f: I \to IR, I \text{ is a set of } IR.$

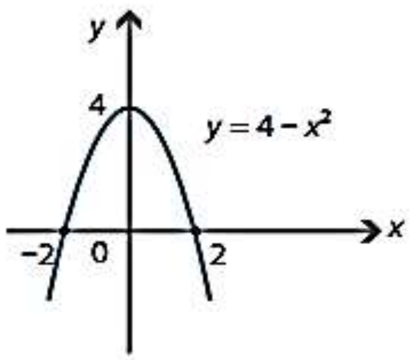
- **Range : the set of all images of points in the domain :** $R = \{f(x), x \in D\}.$
- **Domain :** $D(f) = \{ x \in IR, f(x) \text{ is defining}\}.$

Example

What is the domain and range of the function $f(x) = 4 - x^2$?

Solution

Here a graph of the function helps.



Since $f(x)$ is defined for all real numbers, we have domain$(f) = \mathbb{R}$.

We can see from the graph that range$(f) = \{y : y \leq 4\} = (-\infty, 4]$.

1.2.4 Classification of functions

- *Constant functions:* $f(x) = c$
- *Polynomial functions:* $f(x) = a_0 + a_1x_1 + \cdots + a_{n-1}x^{n-1} + a_nx_n$
- *Rational functions:* ratio of polynomials functions,
  $f(x) = \frac{a_0+a_1x_1+\cdots+a_{n-1}x^{n-1}+a_nx_n}{b_0+b_1x_1+\cdots+b_{n-1}x^{n-1}+b_nx_n}$
- *Irrational functions:* Root extractions,
  $f(x) = \sqrt[m]{\frac{a_0+a_1x_1+\cdots+a_{n-1}x^{n-1}+a_nx_n}{b_0+b_1x_1+\cdots+b_{n-1}x^{n-1}+b_nx_n}}$
- *Piece-wise functions.e.g.* $f(x) = | x - 1 |$
- *Transcendental:* trigonometric expressions, exponentials and logarithms[1].

For example:

- The function $y = \frac{1}{x}$ has domain $\{x \in \mathbb{R} \mid x \neq 0\}$, which is also written as $\mathbb{R} \setminus \{0\}$.
- The function $y = \log_2 x$ has domain $\{x \in \mathbb{R} \mid x > 0\}$, which is also written as $\mathbb{R}^+$.

1.2.3 Composition of functions

- *Composition* of f with g : $(f \circ g)(x) = f((g(x))$,the domain of $f \circ g$ consists

  of all x in the domain of g for which $g(x)$ is in the domain of f.

Example

Given $f(x) = 2x^2 + 1$ and $g(x) = 3x - 5$, find the following:

$$f(g(x)) = 2(3x-5)^2 + 1 = 2\left(9x^2 - 30x + 25\right) + 1 = 18x^2 - 60x + 51$$

$$g(f(x)) = 3(2x^2+1) - 5 = 6x^2 - 2$$

$$g(g(x)) = 3(3x-5) - 5 = 9x - 20$$

Lecture 5 : Continuous Functions

Definition 1 We say the function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a).$$

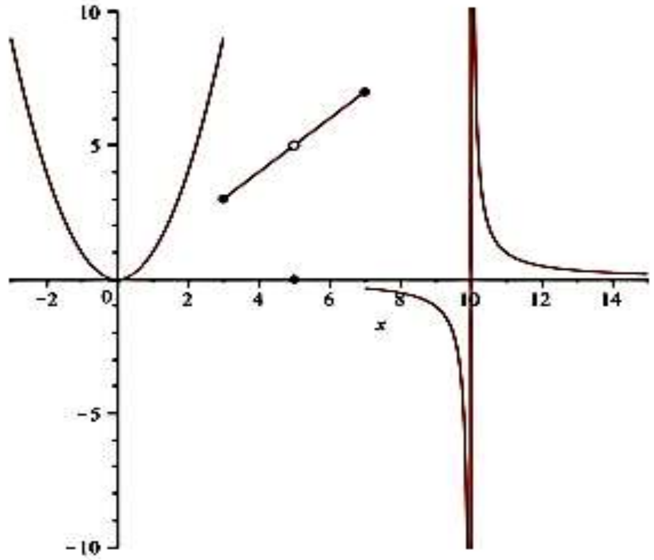
Note that this definition implies that the function f has the following three properties **if f is continuous at a:**

1. $f(a)$ is defined (a is in the domain of f).
2. $\lim_{x\to a} f(x)$ exists.
3. $\lim_{x\to a} f(x) = f(a)$. (Note that this implies that $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ both exist and are equal).

Example 2 **Consider the graph shown below of the function**

$$k(x) = \begin{cases} x^2 & -3 < x < 3 \\ x & 3 \le x < 5 \\ 0 & x = 5 \\ x & 5 < x \le 7 \\ \frac{1}{x-10} & x > 7 \end{cases}$$

Where is the function discontinuous and why?



Definition A function f is continuous from the right at a number a if $\lim_{x\to a^+} = f(a)$.
A function f is continuous from the left at a number a if $\lim_{x\to a^-} = f(a)$.

Definition A function f is continuous on an interval if it is continuous at every number in the interval. (If f is defined only on one side of an endpoint of the interval, we understand *continuous* at the endpoint to mean *continuous from the right or continuous from the left* at the endpoint as appropriate.)

Catalogue of functions continuous on their domains

From the last day we know:

Polynomials and Rational functions

- A polynomial function, $P(x) = c_0 + c_1x + c_2x^2 + \cdots + c_nx^n$, is continuous everywhere i.e. $\lim_{x\to a} P(x) = P(a)$ for all real numbers a.
- A rational function, $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials is continuous on its domain, i.e. $\lim_{x\to a} f(x) = \frac{P(a)}{Q(a)}$ for all values of a, where $Q(a) \neq 0$.

n th Root function

From #10 in last day's lecture, we also have that if $f(x) = \sqrt[n]{x}$, where n is a positive integer, then $f(x)$

is continuous on the interval $[0, \infty)$.

Example Find the domain of the following function and use the theorem above to show that it is continuous on its domain:

$$k(x) = \sqrt[3]{x}(x^2 + 2x + 1) + \frac{x+1}{x-10}.$$

$k(x)$ is continuous on its domain, since it is a combination of root functions, polynomials and rational functions using the operations $+, -, \cdot$ and $\div$. The domain of k is all values of x except $x = 10$ and this function is continuous on the intervals $(-\infty, 10)$ and $(10, \infty)$.

1.Exponential function

Definition

For any positive number $a > 0$, there is a function $f : \mathbb{R} \to (0, \infty)$ called an *exponential function* that is defined as $f(x) = a^x$.

For example, $f(x) = 3^x$ is an exponential function, and $g(x) = (\frac{4}{17})^x$ is an exponential function.

- **Natural exponential function is the function $f(x) = e^x, e \approx 2.7182\ldots$**

Properties of exponential functions

1) **The domain $D = IR$ and $\forall x \in IR : f(x) > 0$.**
2) **An exponential function is increasing when $a > 1$ and decreasing when $0 < a < 1$.**
3) **If $a^x = a^y$ then $x = y$.**
4) **We have**

$$e^0 = 1, lne^x = x, e^{lnx} = x; \forall x, y \in R, e^{x+y} = e^x.e^y, e^{x-y} = \frac{e^x}{e^y}$$

$$\forall x \in R, \left(e^{u(x)}\right)' = u'(x)e^{u(x)}$$

Some limits

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1, \quad \lim_{x\to +\infty} e^x = +\infty, \quad \lim_{x\to -\infty} e^x = 0, \lim_{x\to +\infty} \frac{e^x}{x^\alpha} = +\infty \ \alpha \in R$$

Example: Solve $4^{x^2} = 2^x$

(i) Rewrite the equation in the form $a^u = a^v$

Since $4 = 2^2$, we can rewrite the equation as

$$\left(2^2\right)^{x^2} = 2^x$$

Using properties of exponents we get $2^{2x^2} = 2^x$.

Since $2^{2x^2} = 2^x$ we have $2x^2 = x$.

Solve the equation u = v

$$2x^2 = x$$

$$2x^2 - x = 0$$

$$x(2x-1) = 0$$

$$x = 0 \qquad 2x - 1 = 0$$

$$x = 1/2$$

Solution set = {0, ½ }

2. Logarithmic function

A **logarithmic** function $f(x) = \log_a(x)$, $a > 0$, $a \neq 1$, $x > 0$ (logarithm to the base a of x) is the inverse of the exponential function $y = a^x$.

Therefore, we have the following properties for this function (as the inverse function)

(I) $y = \log_a(x)$ if and only if $a^y = x$

Natural logarithm is the logarithm with the base e (the inverse of $y = e^x$): $\ln(x) = \log_e(x)$

Example:

a) $\log_2(8)$ is an exponent to which 2 must be raised to obtain 8 (we can write this as $2^x = 8$) Clearly this exponent is 3, thus $\log_2(8) = 3$

b) $\log_{1/3}(9)$ is an exponent to which 1/3 must be raised to obtain 9: $(1/3)^x = 9$. Solving this equation for x, we get $3^{-x} = 3^2$ and $-x = 2$ or $x = -2$. Thus $\log_{1/3}(9) = -2$.

Properties of logarithm functions

1) **The domain** $D = (0, +\infty)$ and $R = IR$.
2) **We have :**

$$lnxy = lnx + lny\ ;\ ln\frac{1}{x} = -lnx\ ;\ ln\frac{x}{y} = lnx - lny\ ;\ lnx^2 = 2lnx$$

Some limits

$$\lim_{x\to 0} lnx = -\infty;\ \lim_{x\to+\infty} lnx = +\infty;\ \lim_{x\to 0}\frac{\ln(x+1)}{x} = 1;\ \lim_{x\to+\infty}\frac{\ln x}{x^{\alpha}} = 0\ ,\quad \alpha > 0$$

Example: Solve the following equations

a) $\log_5(x^2 + x + 4) = 2$

(i) Find the domain of the logarithm(s)

$x^2 + x + 4 > 0$

$x^2 + x + 4 = 0$

$x = \frac{-1 \pm \sqrt{1-4(1)(4)}}{2} = \frac{-1 \pm \sqrt{-15}}{2}$ not a real number

Since $y = x^2 + x + 4$ has no x-intercepts and the graph is a parabola that opens up, the graph must always stay above x-axis. Therefore, $x^2 + x + 4 > 0$ for all x

(ii) Change the equation to the exponential form and solve

$x^2 + x + 4 = 5^2$

$x^2 + x + 4 = 25$

$x^2 + x - 21 = 0$

$x = \frac{-1 \pm \sqrt{1-4(1)(-21)}}{2} = \frac{-1 \pm \sqrt{85}}{2}$

since there are no restrictions on x, above numbers are solutions of the equation.