Numercal Series

Definition: Given a sequence $(u_n)_{n\geq 0}$, we use the notation $\sum_{i=1}^{n} u_i$ to denote the finite sum $u_1 + u_2 + u_3 + \dots + u_n = S_n$.

We associate with the sequence $(u_n)_n$ a sequence $(S_n)_n$ of partial sums.

The Pair $((u_n), (S_n))$ is said to be a numercal series or a series, we denoted the series by the symbolic expression $\sum_{n=1}^{\infty} u_n$ or $\sum u_n$.

If $(S_n)_n$ Converges to a real number S, then the series is said to ~~be~~ converge and we write

$$\sum_{n=1}^{\infty} u_n = S.$$

We say that the number S is the sum of the series, where: $S = \lim_{n\to+\infty} S_n$.

If the sequence (S_n) diverges, then the series is said to diverge.

Theorem (Necessary condition)

If $\sum u_n$ Converges, then $\lim_{n\to+\infty} u_n = 0$.

Example ① Let the series: $\sum_{n \geq 0} e^n$

We have $\lim_{n \to +\infty} U_n = \lim_{n \to +\infty} e^n = +\infty \neq 0$

then the series $\sum_{n \geq 0} e^n$ diverges

② $\sum_{n \geq 1} \frac{1}{n}$, $\lim_{n \to +\infty} U_n = \lim_{n \to +\infty} \frac{1}{n} = 0$

but the series $\sum_{n \geq 1} \frac{1}{n}$ diverges (As we will see later)

Theorem: Let $\sum U_n$ and $\sum V_n$ be two convergent series.

a) For every $\alpha \in \mathbb{R}$: the series $\sum \alpha U_n$ and $\sum (U_n \pm V_n)$ are convergente, and we have

$\sum \alpha U_n = \alpha \sum U_n$ and $\sum (U_n \pm V_n) = \sum U_n \pm \sum V_n$

B) **Series of nonnegative terms**

We mean that $a_n \geq 0$, for every n.

Theorem (Comparaison test)

1) If $|U_n| \leq V_n$, $\forall n \geq n_0$ ($n_0 \in \mathbb{N}^*$), and if $\sum (V_n)$ converges, then $\sum U_n$ converges.

2) If $V_n \geq U_n \geq 0$, $\forall n \geq n_0$ ($n_0 \in \mathbb{N}^*$), and if $\sum (U_n)$ diverges, then $\sum V_n$ diverges.

(3) Suppose $U_n, V_n > 0$ for all n large enough and $\lim_{n \to +\infty} \frac{U_n}{V_n} = K \in [0, +\infty]$

then if $\sum V_n$ converges and $K < +\infty$, then $\sum U_n$ converges.

if $\sum V_n$ diverges and $K > 0$, then $\sum U_n$ diverges.

Theorem: (Geometric series)

- If $|x| < 1$, then $1 + \sum_{n=1}^{\infty} x^n = \frac{1}{1-x}$.
- If $|x| \geq 1$, the series diverges.

Tests of convergence

Theorem (~~Test~~ Criterion of Cauchy)

Given $\sum U_n$, if there exists the limit $\lim_{n \to +\infty} \sqrt[n]{|U_n|} = \ell$. Then:

a) If $\ell < 1$, $\sum U_n$ converges.

b) If $\ell > 1$, $\sum U_n$ diverges.

c) If $\ell = 1$, the test gives no information.

Example : Let the series (1) $\sum_{n \geq 0} e^{n}$

$$\lim_{n \to +\infty} \sqrt[n]{e^{n}} = \lim_{n \to +\infty} (e^{n})^{1/n} = \lim_{n \to +\infty} e = e > 1$$

Then $\sum_{n \geq 0} e^{n}$ diverges

(2) $\sum \frac{1}{2^{n}}$, $u_n = \frac{1}{2^{n}} \geq 0$

$$\lim_{n \to +\infty} \sqrt[n]{u_n} = \lim_{n \to +\infty} \sqrt[n]{\left(\frac{1}{2}\right)^{n}} = \frac{1}{2} < 1$$

Then $\sum \frac{1}{2^{n}}$ converges

Theorem (D'Alembert criterion)

If $u_n > 0$, and if the limit

$$\lim_{n \to +\infty} \frac{u_{n+1}}{u_n} = \ell$$

If $\ell < 1$, then $\sum u_n$ converges.

If $\ell > 1$, then $\sum u_n$ diverges.

If $\ell = 1$, There is no information.

Riemann Series

Theorem : The Riemann series $\sum_{n=1}^{\infty} \frac{1}{n^{p}}$ converges if and only if $p > 1$

Examples: (1) $\sum_{n\geq 1} \frac{1}{\sqrt{n}} = \sum_{n\geq 1} \frac{1}{n^{(1/2)}}$

The series ~~diverge~~ converges (Riemann series with $\alpha = 1/2 < 1$).

(2) $\sum_{n\geq 1} \frac{1}{n\sqrt{n}} = \sum_{n\geq 1} \frac{1}{n \cdot n^{1/2}} = \sum_{n\geq 1} \frac{1}{n^{(3/2)}}$

The series diverges (Riemann series with $\alpha = \frac{3}{2} > 1$)

I/ Multivariable functions

Functions with two variables

Let $D \subseteq \mathbb{R}^2$ and let $f : D \rightarrow \mathbb{R}$ be a function.

Fix $(x_0, y_0) \in D$.

We define the partial derivative of f with respect to x at (x_0, y_0) to be the limit

$$\lim_{h \to 0} \frac{f(x_0+h, y_0) - f(x_0, y_0)}{h}$$ provided

this limit exists. It is denoted by

$$\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0).$$

We define the partial derivative of f with respect to y at (x_0, y_0) to be the limit

$$\lim_{h \to 0} \frac{f(x_0, y_0+h) - f(x_0, y_0)}{h}$$

provided this limit exists, it is denoted by $f_y(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0)$.

Examples: Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x, y) \longmapsto f(x, y) = x^2 + y^2 + 10$

$\frac{\partial f}{\partial x}(x_0, y_0) = 2x_0$, $\frac{\partial f}{\partial y}(x_0, y_0) = 2y_0$, $\forall (x_0, y_0) \in \mathbb{R}^2$

Let $f(x, y) = x e^y$, then $\frac{\partial f}{\partial x}(x_0, y_0) = e^{y_0}$

$\frac{\partial f}{\partial y}(x_0, y_0) = x_0 e^{y_0}$.

The second partial derivatives

$$\frac{\partial^2 f}{\partial x^2}(x_0, y_0) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x} f_x(x_0, y_0) = f_{xx}(x_0, y_0)$$

$$\frac{\partial^2 f}{\partial y^2}(x_0, y_0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(f_y)(x_0, y_0) = f_{yy}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x_0, y_0) = \frac{\partial}{\partial x}(f_y)(x_0, y_0) = f_{yx}(x_0, y_0)$$

$$\frac{\partial^2 f}{\partial y \partial x}(x_0, y_0) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x_0, y_0) = \frac{\partial}{\partial y}(f_x)(x_0, y_0) = f_{xy}(x_0, y_0)$$

Multi-variable functions ($n \geq 2$).

Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, (x_0, y_0)

$(x_1, x_2, x_3, \ldots, x_n) \longmapsto f(x_1, x_2, x_3, \ldots, x_n)$.

$$\frac{\partial f}{\partial x_i}(x_{10}, x_{20}, \ldots, x_{n0}) = \lim_{h \to 0} \frac{f(x_{10}, x_{20}, \ldots, x_{i0}+h, x_{i+1,0}, \ldots, x_{n,0})}{h} - \frac{f(x_{1,0}, x_{2,0}, \ldots, x_{i,0}, \ldots, x_{n,0})}{h}.$$

Examples :

(1) $f(x_1, x_2, x_3) = x_1^2 + x_2^3 + x_3^4$

$\frac{\partial f}{\partial x_1}(x_0, y_0, z_0) = 2x_0 \Rightarrow \frac{\partial^2 f}{\partial x_1^2}(x_0, y_0, z_0) = 2$

$\frac{\partial f}{\partial x_2}(x_0, y_0, z_0) = 3y_0^2 \Rightarrow \frac{\partial^2 f}{\partial x_2^2}(x_0, y_0, z_0) = 6y_0$

$\frac{\partial f}{\partial x_3}(x_0, y_0, z_0) = 4z_0^3 \Rightarrow \frac{\partial^2 f}{\partial x_3^2}(x_0, y_0, z_0) = 12z_0^2$

(2) $f(x, y) = \cos x . \sin y$.

$\frac{\partial f}{\partial x}(x_0, y_0) = -\sin x_0 . \sin y_0 \Rightarrow \frac{\partial^2 f}{\partial x^2}(x_0, y_0) = -\cos x_0 \sin x_0$

$\frac{\partial f}{\partial y}(x_0, y_0) = \cos x_0 . \cos y_0 \Rightarrow \frac{\partial^2 f}{\partial y^2}(x_0, y_0) = -\cos x_0 . \sin y_0$