Chapter III: Kinematic of material point

Summary

1. Definitions

1.1. Kinematic

kinematics is the study of the movement of material point (position, velocity and acceleration), without taking into account their causes.

1.2. Material point

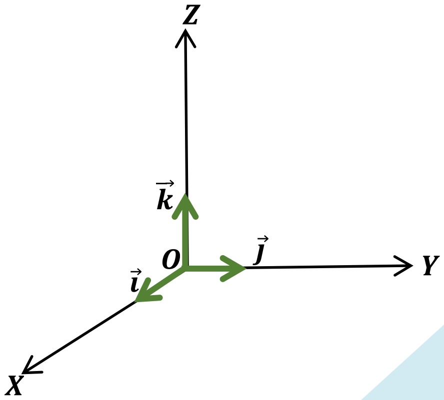
A **material point** is a mechanical system of one or more bodies that can be modeled by a geometric point **M** with which its mass m is associated (center of gravity).

1.3. Reference

Description of the movement is made according to a body, chosen as a reference, called an observation reference. For example: terrestrial reference, geocentric reference …etc.

1.4. Frame

To mathematically describe the characteristics of a movement, an observer (reference) uses a frame $\mathscr{R}$, A frame is determined by an origin O and by a base. Most often the base is orthonormal: the best-known frame is the Cartesian reference frame $(O, \vec{i}, \vec{j}, \vec{k})$.

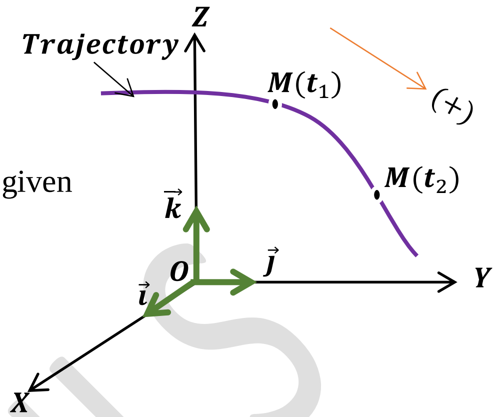


1.5. Trajectory

The trajectory is the set of successive positions occupied by the mobile M during its movement. It is represented by a curve in space. For example:

1. rectilinear trajectory
2. curvilinear trajectory
3. circular trajectory

Like any curve, the trajectory is determined, in a given reference frame, by its mathematical equation $f\ (x, y, z)$.



1.5. Movement

Movement or motion is a displacement of a body according to a reference at each instant "t". the movement is characterized by its trajectory and its velocity.

2. Kinematic quantities

The description of the movement of material point M need the definition of three vectors, namely: the position vector $\overrightarrow{OM}$, the velocity vector $\vec{V}$ and the acceleration vector $\vec{\gamma}$, which we call them kinematic quantities.

These vectors must be expressed in the base of a reference frame (R), most often orthonormal. We will see the kinematic quantities expressed in the Cartesian, Intrinsic, Polar, Cylindrical and Spherical coordinate systems, respectively.

The choice of system is arbitrary but, in practice, is guided by the trajectory and the forces acting on the mobile, and the choice will be made in such a way as to simplify the mathematical expressions.

2.1. The Cartesian coordinate system

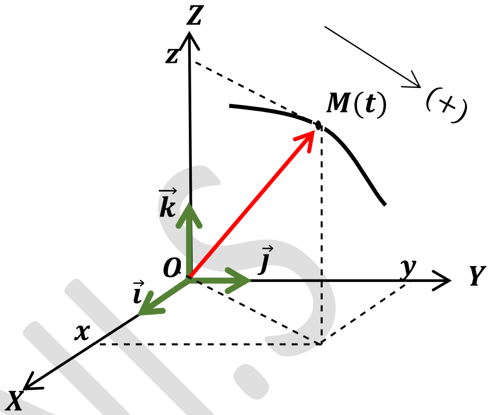
The cartesian coordinate system have orthonormal basis. It consists of an origin point O and a base of three-unit vectors $(\vec{\imath}, \vec{\jmath}, \vec{k})$ which determine the three usual directions of space $\{|Ox), |Oy), |Oz)\}$. Thus, the Cartesian reference frame will be denoted $R\ (O, \vec{\imath}, \vec{\jmath}, \vec{k})$.

$$\vec{\imath} \perp \vec{\jmath}, \qquad \vec{\jmath} \perp \vec{k}, \qquad \vec{k} \perp \vec{\imath}; \quad \|\vec{\imath}\| = \|\vec{\jmath}\| = \|\vec{k}\| = 1; \quad \vec{\imath} \wedge \vec{\jmath} = \vec{k}$$

1/ Position vector $\vec{r}(t)$

Let $\mathcal{R}$ be a direct orthonormal Cartesian reference frame of origin O and base $(\vec{\imath}, \vec{\jmath}, \vec{k})$. The position of material point M at time t is defined by the position vector. The position vector $\overrightarrow{OM}(t)$ is identified by its Cartesian coordinates (x, y, z) as follows:

$$\vec{r}(t) = \overrightarrow{OM}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$$



- Its magnitude is: $\|\overrightarrow{OM}\| = \sqrt{x^2 + y^2 + z^2}$
- Its unit in International System (S.I.) is the meter (m).
- The expressions of x(t), y(t) et z(t) are called the hourly equations of movement.

2/ Velocity vector $\vec{V}(t)$

The velocity vector is defined as the instantaneous variation of the position vector with respect to time, in other words:

$$\vec{V}(t) = \frac{d\vec{r}(t)}{dt} = \frac{d\overrightarrow{OM}(t)}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}$$

The unit vectors $(\vec{\imath}, \vec{\jmath}, \vec{k})$ are fixed in the Cartesian frame, therefore:

$$\frac{d\vec{\imath}}{dt} = \frac{d\vec{\jmath}}{dt} = \frac{d\vec{k}}{dt} = 0$$

We also note (from Newton): $\vec{V}(t) = \dot{x}(t)\vec{\imath} + \dot{y}(t)\vec{\jmath} + \dot{z}(t)\vec{k}$

Finally, we write: $\vec{V}(t) = V_x(t)\vec{\imath} + V_y(t)\vec{\jmath} + V_z(t)\vec{k}$

V_x, V_y *and* V_z are the components of velocity vector, where:

$$V_x = \frac{dx}{dt}; \qquad V_y = \frac{dy}{dt}; \qquad V_z = \frac{dz}{dt}$$

The magnitude of $\vec{V}(t)$ is : $\|\vec{V}\| = \sqrt{{V_x}^2 + {V_y}^2 + {V_z}^2}$

The velocity vector $V(t)$ is tangent to the trajectory at the point $M(t)$ and it is directed in the direction of movement.

3/ Acceleration vector $\vec{\gamma}(t)$

It is defined by the instantaneous variation of the velocity vector with respect to time, in other words:

$$\vec{\gamma}(t) = \frac{d\vec{V}(t)}{dt} = \frac{d^2\vec{r}(t)}{dt^2} = \frac{d^2\overrightarrow{OM}(t)}{dt^2}$$

$$\vec{\gamma}(t) = \frac{dV_x}{dt}\vec{i} + \frac{dV_y}{dt}\vec{j} + \frac{dV_z}{dt}\vec{k}$$

$$\vec{\gamma}(t) = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k}$$

We also note: $\vec{\gamma}(t) = \ddot{x}(t)\vec{i} + \ddot{y}(t)\vec{j} + \ddot{z}(t)\vec{k}$

We finally write: $\boldsymbol{\vec{\gamma}(t) = \gamma_x(t)\vec{i} + \gamma_y(t)\vec{j} + \gamma_z(t)\vec{k}}$

γ_x, γ_y *and* γ_z : are the components of acceleration vector, thus:

$$\gamma_x = \frac{dV_x}{dt}; \quad \gamma_y = \frac{dV_y}{dt}; \quad \gamma_z = \frac{dV_z}{dt}$$

The magnitude of $\vec{\gamma}(t)$ is: $\|\vec{\gamma}\| = \sqrt{\gamma_x^2 + \gamma_y^2 + \gamma_z^2}$

The acceleration vector $\vec{\gamma}$ is directed towards the concavity of the trajectory of the material point M.

2.2. The Intrinsic Coordinate System

The Intrinsic coordinate system (or Frenet) consists of two-unit vectors: $(\vec{u_T}, \vec{u_N})$ mobile in the observation frame of Cartesian system $R\ (O, \vec{i}, \vec{j}, \vec{k})$. This system is very suitable for the analysis of movement which admits a curvilinear trajectory, to determine the radius of curvature R_c and the center of curvature.

- $\vec{u_T}$: Tangential unit vector, it is tangent to the trajectory at the point of curvilinear abscissa $S(t)$ and oriented according to the positive time.
- $\vec{u_N}$: Normal unit vector, it is perpendicular to $\vec{u_T}$ and oriented according to the concavity of the trajectory.

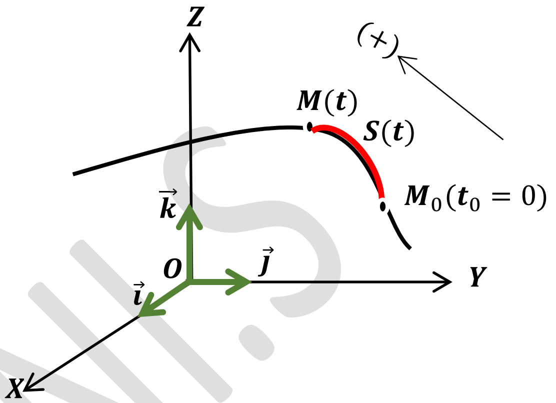
The projection of these vectors in the Cartesian frame gives:

$$\begin{cases} \overrightarrow{u_T} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j} \\ \overrightarrow{u_N} = -\cos\theta\,\vec{i} - \sin\theta\,\vec{j} \end{cases}$$

1/ Curvilinear abscissa

We call the curvilinear abscissa at time t, denoted $S(t)$, of material point M, the length of the arc of the trajectory of M counted from an origin M_0 at $t_0 = 0$:

$$S(t) = S(M) = \overline{M_0M}(t)$$



2/ Position vector $\vec{r}\ (t)$

In the Intrinsic frame, we cannot explicitly define the vector $\vec{r}(t)$ *or* $\overrightarrow{OM}(t)$. We will use the curvilinear abscissa $S(t)$ to locate the position of point M in time knowing that the trajectory is known.

3/ Velocity vector $\vec{V}(t)$

By definition, the velocity vector is the temporal derivative of the position vector $\vec{r}(t)$:

$$\vec{V}(t) = \frac{d\vec{r}(t)}{dt} = \frac{d\vec{r}}{dS}.\frac{dS}{dt}$$

The velocity in magnitude is expressed by:

$$V(t) = \lim_{\Delta t \to 0} \frac{\overline{M_0M}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t} = \frac{dS}{dt}$$

We know that the velocity vector is tangential to the trajectory of the mobile which means that

$$\overrightarrow{u_T} = \frac{d\vec{r}(t)}{dS}$$

$$\vec{V}(t) = \frac{d\vec{r}}{ds}.\frac{ds}{dt} = V.\overrightarrow{u_T}$$

So, the expression of the velocity takes the form: $\boldsymbol{\vec{V}(t) = V.\overrightarrow{u_T}}$

4/ Acceleration vector $\vec{\gamma}(t)$

By definition, the acceleration vector is the temporal derivative of the velocity vector $\vec{V}(t)$:

$$\vec{\gamma} = \frac{d\vec{V}}{dt} = \frac{d}{dt}(V.\overrightarrow{u_T}) = \frac{dV}{dt}.\overrightarrow{u_T} + V.\frac{d\overrightarrow{u_T}}{dt}$$

The intervention of the curvilinear abscissa S and the angle θ formed between the Ox axis and the position vector $\overrightarrow{OM}$ makes it possible to **change the variable** for determining the expression of $\frac{d\overrightarrow{u_T}}{dt}$:

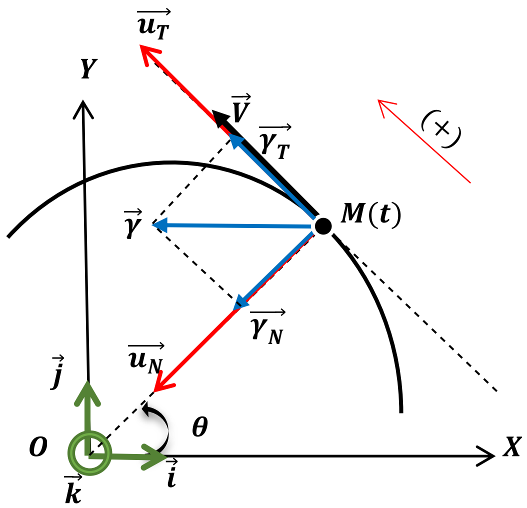
$$\frac{d\overrightarrow{u_T}}{dt} = \frac{d\overrightarrow{u_T}}{d\theta}.\frac{d\theta}{dt} = \frac{d\overrightarrow{u_T}}{d\theta}.\frac{d\theta}{dS}.\frac{dS}{dt}$$

1. We know that the length of the arc $S(t)$ (curvilinear abscissa) is written as follow:

$$S = R.\theta \quad \Longrightarrow \quad \theta = \frac{S}{R} \quad \Longrightarrow \quad \frac{d\theta}{dS} = \frac{1}{R}$$

2. In turn the magnitude of the velocity is given by: $V = \frac{dS}{dt}$
3. In addition, the derivative of the tangential unit vector with respect to θ gives the normal unit vector: $\frac{d\overrightarrow{u_T}}{d\theta} = \overrightarrow{u_N}$

we summarize: $$\begin{cases} \frac{d\overrightarrow{u_T}}{d\theta} = \overrightarrow{u_N} \\ \frac{d\theta}{dS} = \frac{1}{R} \\ \frac{dS}{dt} = V \end{cases} \Longrightarrow \frac{d\overrightarrow{u_T}}{dt} = \frac{V}{R}.\overrightarrow{u_N}$$



We finally obtain the acceleration vector in the form:

$$\vec{\gamma}(t) = \frac{dV}{dt}.\overrightarrow{u_T} + \frac{V^2}{R}.\overrightarrow{u_N}$$

Notes:

- The acceleration can be written as: $\vec{\gamma}(t) = \gamma_T\,\overrightarrow{u_T} + \gamma_N.\overrightarrow{u_N}$
- With: γ_T and γ_N are the components of the acceleration in the base $(\overrightarrow{u_T},\ \overrightarrow{u_N})$, where:

$\gamma_T = \frac{dV}{dt}$: is the tangential acceleration which results from the variation of the magnitude of $\vec{V}$

$\gamma_N = \frac{V^2}{R}$: is the normal acceleration which results from the variation in the direction of $\vec{V}$

- The magnitude of acceleration is then written: $\|\vec{\gamma}\| = \sqrt{\gamma_T^2 + \gamma_N^2}$
- Knowing $\|\vec{V}\|$ and $\|\vec{\gamma}\|$, we can determine the radius of curvature R_c :

$$R_c = \frac{V^2}{\sqrt{\gamma^2 - \gamma_T^2}}$$

2.3. The Polar coordinate system

The Polar coordinate system is a plane reference frame with rotational symmetry. It consists of two-unit vectors: $(\vec{u_r}, \vec{u_\theta})$ moving in the observation frame of Cartesian system $\mathcal{R}\ (O, \vec{i}, \vec{j}, \vec{k})$. This system is very suitable for the analysis of circular movements.

In this frame, each point is determined by the polar coordinates, which are the radial coordinate (polar radius r) and the angular coordinate (the polar angle θ). More precisely:

- $r\ (t)$: polar radius, represents the magnitude of the position vector $\overrightarrow{OM}$.
- $\theta\ (t)$: polar angle, represents the angle between the Ox axis and the position vector $\overrightarrow{OM}$.
- $\vec{u_r}$: radial unit vector having the same direction and sense of the position vector $\vec{r}$.
- $\vec{u_\theta}$: angular unit vector is perpendicular to $\vec{u_r}$ and oriented towards the direction of increase of the angle θ.

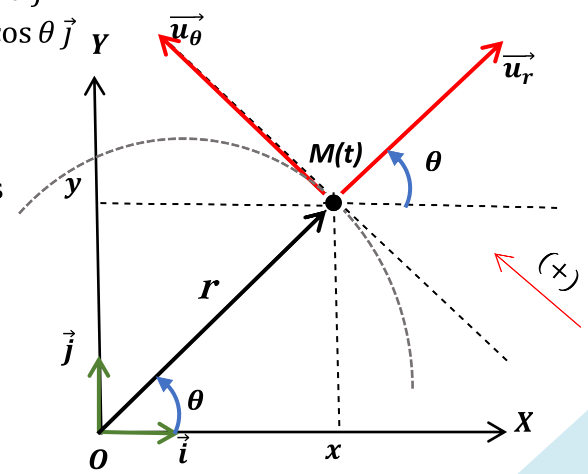
The projection of these vectors in the Cartesian frame gives:

$$\begin{cases} \vec{u_r} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} \\ \vec{u_\theta} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j} \end{cases}$$

1/ Position vector $\vec{r}(t)$

At time t, the position of material point $M\ (r, \theta)$ is identified by the instantaneous position vector:

$$\vec{r}(t) = \overrightarrow{OM}(t) = r(t)\,\vec{u_r}$$



2/ Velocity vector $\vec{V}(t)$

Similarly, the same definition as in Cartesian and Frenet coordinates, the expression of the velocity vector is as follows:

$$\vec{V}(t) = \frac{d\vec{r}}{dt} = \frac{d\overrightarrow{OM}}{dt} = \frac{d}{dt}\left(r\,.\overrightarrow{u_r}\right) = \frac{dr}{dt}.\overrightarrow{u_r} + r.\frac{d\overrightarrow{u_r}}{dt}$$

The derivative of the unit vector $\overrightarrow{\mathbf{u_r}}$ can be determined via the variable θ:

$$\frac{d\overrightarrow{u_r}}{dt} = \frac{d\overrightarrow{u_r}}{d\theta}.\frac{d\theta}{dt} = \frac{d\theta}{dt}.\overrightarrow{u_\theta}$$

The expression of the velocity will become:

$$\vec{V}(\boldsymbol{t}) = \frac{\boldsymbol{dr}}{\boldsymbol{dt}}.\overrightarrow{\mathbf{u_r}} + \boldsymbol{r}.\frac{\boldsymbol{d\theta}}{\boldsymbol{dt}}\overrightarrow{\mathbf{u_\theta}}$$

We can also write the form: $\vec{\boldsymbol{V}}(\boldsymbol{t}) = \boldsymbol{V_r}.\overrightarrow{\mathbf{u_r}} + \boldsymbol{V_\theta}.\overrightarrow{\mathbf{u_\theta}}$

- $V_r = \frac{dr}{dt}$: is the radial component of the velocity vector,
- $V_\theta = r.\frac{d\theta}{dt}$: is the orthoradial component of the velocity vector,
- $\omega = \frac{d\theta}{dt}$: is the angular velocity of point M.

3/ Acceleration vector $\vec{\gamma}(t)$

By definition:

$$\vec{\gamma} = \frac{d\vec{V}}{dt} = \frac{d}{dt}\left(\frac{dr}{dt}\overrightarrow{u_r} + r.\frac{d\theta}{dt}\overrightarrow{u_\theta}\right) = \frac{d^2r}{dt^2}\overrightarrow{u_r} + \frac{dr}{dt}\frac{d\overrightarrow{u_r}}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\overrightarrow{u_\theta} + r.\frac{d^2\theta}{dt^2}\overrightarrow{u_\theta} + r.\frac{d\theta}{dt}\frac{d\overrightarrow{u_\theta}}{dt}$$

The derivative of the unit vector $\overrightarrow{u_\theta}$ can be determined via the variable θ:

$$\frac{d\overrightarrow{u_\theta}}{dt} = \frac{d\overrightarrow{u_\theta}}{d\theta}.\frac{d\theta}{dt} = -\frac{d\theta}{dt}.\overrightarrow{u_r}$$

So, the expression of the acceleration takes the form:

$$\vec{\gamma} = \left[\frac{\boldsymbol{d^2r}}{\boldsymbol{dt^2}} - \boldsymbol{r}\left(\frac{\mathbf{d\theta}}{\boldsymbol{dt}}\right)^2\right]\overrightarrow{\mathbf{u_r}} + \left[2\frac{\boldsymbol{dr}}{\boldsymbol{dt}}\frac{\boldsymbol{d\theta}}{\boldsymbol{dt}} + \boldsymbol{r}\frac{\boldsymbol{d^2\theta}}{\boldsymbol{dt^2}}\right]\overrightarrow{\mathbf{u_\theta}}$$

Notes:

- The expression of acceleration can be written as: $\vec{\boldsymbol{\gamma}} = \boldsymbol{\gamma_r}\overrightarrow{\mathbf{u_r}} + \boldsymbol{\gamma_\theta}\overrightarrow{\mathbf{u_\theta}}$

- γ_r and γ_θ are the components of acceleration in the base $(\overrightarrow{u_r}, \overrightarrow{u_\theta})$, where:

- $$\begin{cases} \gamma_r = \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2 & ;\ \textit{the radial component of the acceleration vector.} \\ \gamma_\theta = 2\dfrac{dr}{dt}\dfrac{d\theta}{dt} + r\dfrac{d^2\theta}{dt^2} & ;\ \textit{the ortho} - \textit{radial component of the acceleration vector.} \end{cases}$$

- $\dot{\omega} = \dfrac{d^2\theta}{dt^2}$: the angular acceleration of the point M.

- The magnitude of acceleration is then written: $\|\vec{\gamma}\| = \sqrt{\gamma_r^2 + \gamma_\theta^2}$

- The relations between the elements of the Cartesian system (x,y) and those of the polar system (r, θ) are given by:

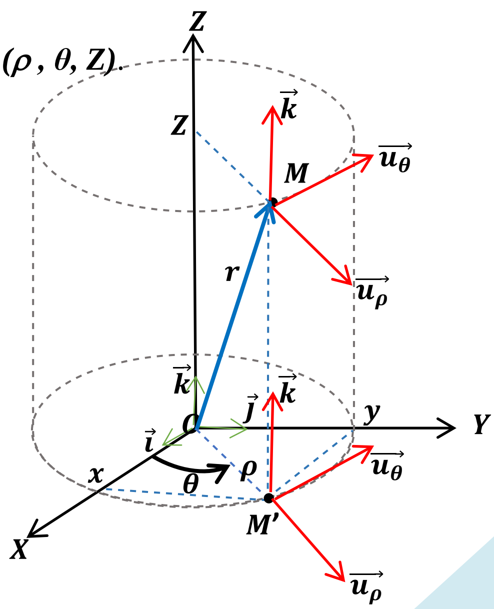
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}, \quad \textbf{or} \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = \arctan\left(\dfrac{y}{x}\right) \end{cases}$$

2.4. The cylindrical coordinate system

The cylindrical coordinate system is a spatial and orthonormal reference frame. It consists of three-unit vectors $(\overrightarrow{u_\rho}, \overrightarrow{u_\theta}, \vec{k})$. $\vec{k}$ is immobile in the chosen reference frame, and it is identical to $\vec{k}$ of the Cartesian system, while the other unit vectors $\overrightarrow{u_\rho}$ and $\overrightarrow{u_\theta}$ are mobile. This system is very suitable for the analysis of circular and translational movements according to the Oz axis: (circular, elliptical, helical, etc.)

Thus, the movement of a mobile M in the cylindrical coordinate system is decomposed into two movement: a rotational movement in the polar frame (projection of M on the plane (Oxy)) and a translational movement along the Oz axis.

We identify the position M by the cylindrical coordinates (ρ, θ, Z).



- $\rho\ (t)$: the magnitude of the vector $\overrightarrow{OM'}$ which represents the projection of the vector $\overrightarrow{OM}$ on the (Oxy) plane.

- $\theta(t)$: the angle formed between the Ox axis and the vector $\overrightarrow{OM'}$.

- $Z\ (t)$: represents the height of point M from the plane (Oxy).

1/ Position vector $\vec{r}(t)$

According to the figure above, the position vector is therefore written:

$$\vec{r} = \overrightarrow{OM} = \rho\,\overrightarrow{u_\rho} + z\,\vec{k}$$

2/ Velocity vector $\vec{V}(t)$

As previously seen:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\rho\,\overrightarrow{u_\rho} + z\,\vec{k}\right) = \frac{d\rho}{dt}\overrightarrow{u_\rho} + \rho\,\frac{d\overrightarrow{u_\rho}}{dt} + \frac{dz}{dt}\,\vec{k} + z\,\frac{d\vec{k}}{dt}$$

Thus, $$\vec{v} = \frac{d\rho}{dt}\overrightarrow{u_\rho} + \rho\,\frac{d\theta}{dt}\overrightarrow{u_\theta} + \frac{dz}{dt}\,\vec{k}$$

The expression of velocity can take the form: $\vec{V} = V_\rho\,\overrightarrow{u_\rho} + V_\theta\,\overrightarrow{u_\theta} + V_z\,\vec{k}$, where:

$$V_\rho = \frac{d\rho}{dt}; \qquad V_\theta = \rho\,\frac{d\theta}{dt}; \qquad V_z = \frac{dz}{dt}$$

3/ Acceleration vector $\vec{\gamma}(t)$

As previously seen: $\vec{\gamma} = \dfrac{d\vec{V}}{dt} = \dfrac{d}{dt}\left[\dfrac{d\rho}{dt}\overrightarrow{u_\rho} + \rho\,\dfrac{d\theta}{dt}\overrightarrow{u_\theta} + \dfrac{dz}{dt}\,\vec{k}\right]$

$$\vec{\gamma} = \frac{d^2\rho}{dt^2}\overrightarrow{u_\rho} + \frac{d\rho}{dt}\frac{d\overrightarrow{u_\rho}}{dt} + \frac{d\rho}{dt}\frac{d\theta}{dt}\overrightarrow{u_\theta} + \rho\,\frac{d^2\theta}{dt^2}\overrightarrow{u_\theta} + \rho\,\frac{d\theta}{dt}\frac{d\overrightarrow{u_\theta}}{dt} + \frac{d^2z}{dt^2}\,\vec{k}$$

The final expression of acceleration in the cylindrical coordinate system is:

$$\vec{\gamma} = \left[\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\theta}{dt}\right)^2\right]\overrightarrow{u_\rho} + \left[2\,\frac{d\rho}{dt}\,\frac{d\theta}{dt} + \rho\,\frac{d^2\theta}{dt^2}\right]\overrightarrow{u_\theta} + \frac{d^2z}{dt^2}\,\vec{k}$$

Notes:

- The expression for acceleration can be written as: $\vec{\gamma} = \gamma_\rho\overrightarrow{u_\rho} + \gamma_\theta\overrightarrow{u_\theta} + \gamma_z\,\vec{k}$
- γ_ρ , γ_θ and γ_z are the components of the acceleration in the base $(\overrightarrow{u_\rho},\ \overrightarrow{u_\theta},\ \vec{k})$, where:

$$\begin{cases} \gamma_\rho = \dfrac{d^2\rho}{dt^2} - \rho\left(\dfrac{d\theta}{dt}\right)^2 \\ \gamma_\theta = 2\dfrac{d\rho}{dt}\dfrac{d\theta}{dt} + \rho\dfrac{d^2\theta}{dt^2} \\ \gamma_z = \dfrac{d^2z}{dt^2} \end{cases}$$

- The module of acceleration is then written: $\|\vec{\gamma}\| = \sqrt{\gamma_\rho^2 + \gamma_\theta^2 + \gamma_z^2}$

- The relationships between the elements of the Cartesian frame (x, y, z) and those of the cylindrical frame (ρ, θ, z) are given by:

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}, \quad \textbf{or} \quad \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \theta = \arctan\left(\frac{y}{x}\right) \\ z = z \end{cases}$$

2.5. The spherical coordinate system

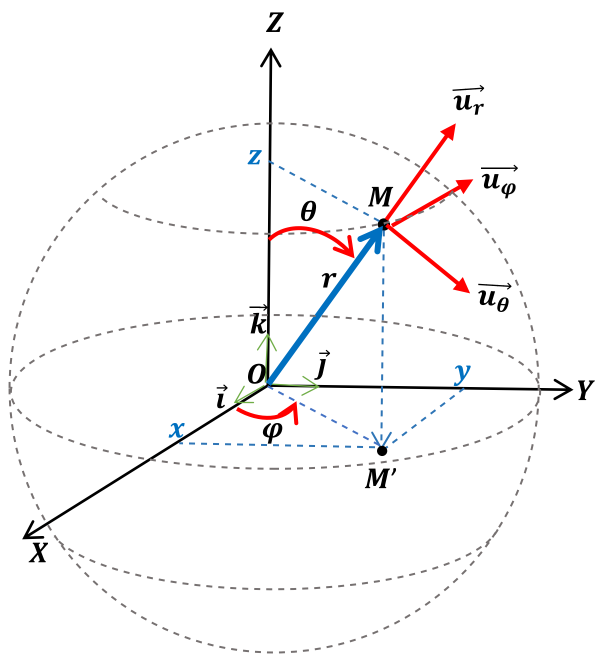
The spherical coordinate system is a spatial and orthonormal reference frame. It consists of three-unit vectors $(\vec{u_r}, \vec{u_\theta}, \vec{u_\varphi})$ moving in the observation frame of Cartesian system $\mathcal{R}\,(O, \vec{i}, \vec{j}, \vec{k})$. This system is very suitable for astrometric and geographic location (altitude, latitude, and longitude, etc.).

The spherical coordinates of point M are: ***r, θ and φ***, where:

- $r\,(t)$: spherical radius, represents the magnitude of the position vector $\overrightarrow{OM}$.

- $\theta\,(t)$: latitude, represents the angle between the Oz axis and the position vector $\overrightarrow{OM}$. This angle is between 0 and π.

- $\varphi\,(t)$: longitude, represents the angle between the axis Ox and the position vector $\overrightarrow{OM'}$. This angle is between 0 and 2π.



- $\vec{u_r}$: radial unit vector having the same direction and sense of the position vector $\overrightarrow{OM}$.

- $\vec{u_\theta}$: angular unit vector is perpendicular to $\vec{u_r}$ and oriented towards the direction of the angle θ.

- $\vec{u_\varphi}$: angular unit vector oriented towards the direction of angle φ, where: $\vec{u_\varphi} = \vec{u_r} \wedge \vec{u_\theta}$.

- The projection of these vectors in the Cartesian frame gives:

$$\begin{cases} \overrightarrow{u_r} = \sin\theta\cos\varphi\,\vec{i} + \sin\theta\sin\varphi\,\vec{j} + \cos\theta\,\vec{k} \\ \overrightarrow{u_\theta} = \cos\theta\cos\varphi\,\vec{i} + \cos\theta\sin\varphi\,\vec{j} - \sin\theta\,\vec{k} \\ \overrightarrow{u_\varphi} = -\sin\varphi\,\vec{i} + \cos\varphi\,\vec{j} \end{cases}$$

1/ Position vector $\vec{r}(t)$

According to the figure above, the position vector is therefore written:

$$\boldsymbol{\vec{r} = \overrightarrow{OM} = r\,\overrightarrow{u_r}}$$

2/ Velocity vector $\vec{V}(t)$

As seen previously:

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(r\,\overrightarrow{u_r}) = \frac{dr}{dt}\overrightarrow{u_r} + r\frac{d\overrightarrow{u_r}}{dt}$$

The radial unit vector $\overrightarrow{u_r}$ depend on two variables θ and φ. So, its temporal derivative is given by:

$$\frac{d\overrightarrow{u_r}}{dt} = \frac{d\overrightarrow{u_r}}{d\theta}\frac{d\theta}{dt} + \frac{d\overrightarrow{u_r}}{d\varphi}\frac{d\varphi}{dt} \quad where, \begin{cases} \frac{d\overrightarrow{u_r}}{d\theta} = \overrightarrow{u_\theta} \\ \frac{d\overrightarrow{u_r}}{d\varphi} = \sin\theta\,\overrightarrow{u_\varphi} \end{cases}$$

$$\frac{d\overrightarrow{u_r}}{dt} = \frac{d\theta}{dt}\overrightarrow{u_\theta} + \sin\theta\,\frac{d\varphi}{dt}\,\overrightarrow{u_\varphi}$$

Finally, the expression of the velocity in a spherical frame is given by:

$$\boldsymbol{\vec{V} = \frac{dr}{dt}\overrightarrow{u_r} + r\frac{d\theta}{dt}\overrightarrow{u_\theta} + r\sin\theta\,\frac{d\varphi}{dt}\,\overrightarrow{u_\varphi}}$$

It can also be noted that: $\boldsymbol{\vec{V} = V_r\,\overrightarrow{u_r} + V_\theta\,\overrightarrow{u_\theta} + V_\varphi\,\overrightarrow{u_\varphi}}$, where:

$$V_r = \frac{dr}{dt}; \qquad V_\theta = r\frac{d\theta}{dt}; \qquad V_\varphi = r\sin\theta\,\frac{d\varphi}{dt}$$

3/ Acceleration vector $\vec{\gamma}(t)$

As seen previously:

$$\vec{\gamma} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left[\frac{dr}{dt}\overrightarrow{u_r} + r\frac{d\theta}{dt}\overrightarrow{u_\theta} + r\sin\theta\,\frac{d\varphi}{dt}\,\overrightarrow{u_\varphi}\right]$$

Unit vectors $\overrightarrow{u_r}$, $\overrightarrow{u_\theta}$, $\overrightarrow{u_\varphi}$ depend on two variables θ and φ, therefore its temporal derivatives are given respectively by:

$$\frac{d\overrightarrow{u_\theta}}{dt} = -\frac{d\theta}{dt}\overrightarrow{u_r} + \cos\theta\,\frac{d\varphi}{dt}\,\overrightarrow{u_\varphi}$$

By definition $\overrightarrow{u_\varphi} = \overrightarrow{u_r} \wedge \overrightarrow{u_\theta}$: $d\overrightarrow{u_\varphi}$ is given by:

$$\frac{d\overrightarrow{u_\varphi}}{dt} = \frac{d}{dt}\left(\overrightarrow{u_r} \wedge \overrightarrow{u_\theta}\right) = -\sin\theta\,\frac{d\varphi}{dt}\,\overrightarrow{u_r} - \cos\theta\,\frac{d\varphi}{dt}\,\overrightarrow{u_\theta}$$

By replacing these temporal derivatives of the unit vector in the relation of $\vec{\gamma}$

The expression of $\vec{\gamma}(t)$ is therefore:

$$\boldsymbol{\vec{\gamma} = \gamma_r \overrightarrow{u_r} + \gamma_\theta \overrightarrow{u_\theta} + \gamma_\varphi \overrightarrow{u_\varphi}}$$

Notes:

- γ_e , γ_θ and γ_φ re the components of the acceleration in the base $(\overrightarrow{u_r}, \overrightarrow{u_\theta}, \overrightarrow{u_\varphi})$, where:

$$\begin{cases} \boldsymbol{\gamma_r} = \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2 - r\sin^2\theta\left(\dfrac{d\varphi}{dt}\right)^2 \\ \boldsymbol{\gamma_\theta} = 2\,\dfrac{dr}{dt}\dfrac{d\theta}{dt} + r\,\dfrac{d^2\theta}{dt^2} - r\cos\theta\sin\theta\left(\dfrac{d\varphi}{dt}\right)^2 \\ \boldsymbol{\gamma_\varphi} = 2r\cos\theta\,\dfrac{d\theta}{dt}\dfrac{d\varphi}{dt} + 2\,\dfrac{dr}{dt}\dfrac{d\varphi}{dt}\sin\theta + r\sin\theta\,\dfrac{d^2\varphi}{dt^2} \end{cases}$$

- The magnitude of acceleration is therefore written: $\|\vec{\gamma}\| = \sqrt{\gamma_r^2 + \gamma_\theta^2 + \gamma_\varphi^2}$

- The relations between the elements of the Cartesian frame (x, y, z) and those of the spherical frame (*r*, *θ*, *φ*) are given by:

$$\begin{cases} x = r\sin\theta\,\cos\varphi \\ y = r\sin\theta\,\sin\varphi \\ z = r\cos\theta \end{cases}, \quad \textbf{or} \quad \begin{cases} r = \sqrt{x^2+y^2+z^2} \\ \theta = \arctan\left(\sqrt{x^2+y^2}/z\right) \\ \varphi = \arctan\left(y/x\right) \end{cases}$$

3. Particular movements

A movement is a displacement of a body according to a frame of reference with respect to time t. The movement is characterized by its trajectory and velocity, for example: rectilinear, curvilinear, vibratory movements, etc. Or the combination between them.

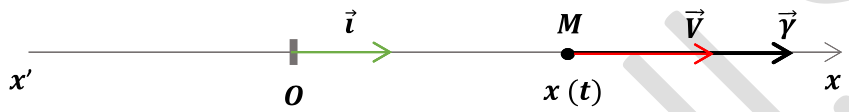
3.1. Rectilinear movement

A movement is called rectilinear if the trajectory followed by the mobile, is a straight line.

a) Position

Consider M a material point moving on a straight line $x'Ox$ with a unit vector $\vec{\boldsymbol{i}}$, the position M of the mobile is identified by the position vector $\overrightarrow{\boldsymbol{OM}}$:

$$\overrightarrow{\boldsymbol{OM}} = \boldsymbol{x(t)\vec{i}}$$



b) Velocity

- **Average velocity:** let x_1 and x_2 be two positions of the mobile at two times t_1 and t_2, the average velocity of the mobile between t_1 and t_2 is given by:

$$V_{avg} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t}$$

- **Instantaneous velocity:** we define the instantaneous velocity at time t as the instantaneous variation of displacement with respect to time:

$$V = \lim_{\Delta t \mapsto 0} \frac{\Delta x}{\Delta t} = \frac{dx}{dt}$$

c) Acceleration

The velocity is changing over the course of time, we characterize this change by acceleration.

- **Average acceleration**: the average acceleration is defined by:

$$\gamma_{moy} = \frac{V_2 - V_1}{t_2 - t_1} = \frac{\Delta V}{\Delta t}$$

- **Instantaneous acceleration:** the instantaneous acceleration of a mobile is given by:

$$\gamma = \lim_{\Delta t \mapsto 0} \frac{\Delta V}{\Delta t} = \frac{dV}{dt}$$

3.1.1. Uniform rectilinear movement

A rectilinear movement is called uniform when the velocity is constant: $V = C^{te}$, and consequently the acceleration is zero: $\gamma = 0$, hence:

$$\gamma = \frac{dV}{dt} = 0$$

$$V = \frac{dx}{dt} \Rightarrow \int_{x_0}^{x} x = \int_{t_0}^{t} V\, dt$$

It is a first order differential equation that we will integrate to find the hourly equation of movement x(t):

$$\Rightarrow \quad \boldsymbol{x(t) = V(t - t_0) + x_0}$$

3.1.2. Uniformly varied rectilinear movement

Rectilinear movement is called uniformly varied when the acceleration is constant $= C^{te}$. Furthermore, if the velocity increases, the movement is accelerated and if the velocity decreases the movement is decelerated, where:

$$\gamma = \frac{dV}{dt} = C^{te}$$

$$\int_{V_0}^{V} V = \int_{t_0}^{t} \gamma\, dt$$

It is a first order differential equation that we will integrate to find the hourly equation of the velocity v(t) :

$$\Rightarrow \boldsymbol{V(t) = \gamma(t - t_0) + V_0}$$

Searching now $x(t)$, we know that: $V = \frac{dx}{dt}$, where:

$$\int_{x_0}^{x} x = \int_{t_0}^{t} V\, dt = \int_{t_0}^{t} (\gamma (t - t_0) + V_0)\, dt = \int_{t_0}^{t} \gamma (t - t_0)\, dt + \int_{t_0}^{t} V_0\, dt$$

So, the hourly equation of movement x(t) is:$\Rightarrow$ $\boldsymbol{x(t) = \frac{1}{2}\gamma (t - t_0)^2 + V_0 (t - t_0) + x_0}$

3.2. Curvilinear movement

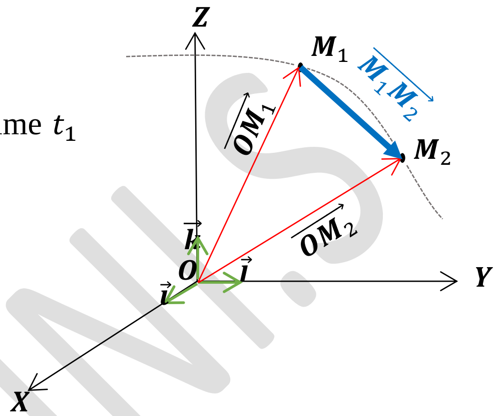
The movement of an object is called curvilinear movement if its trajectory is a curve. To describe the curvilinear movement of a mobile, we must choose an origin O and its position is identified at each instant t by the following position vector:

$$\overrightarrow{OM}\,(t) = \vec{r}\,(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$$

1/ Displacement vector

The displacement of a mobile initially in M_1 at time t_1 and arriving in M_2 at time t_2 is obviously:

$$\overrightarrow{M_1M_2} = \overrightarrow{OM_2} - \overrightarrow{OM_1} = \Delta\overrightarrow{OM} = \Delta\vec{r}$$



The curvilinear abscissa at time t, denoted $S\,(t)$, is the length of the trajectory (arc) from M_1 to M_2 hence: $S\,(t) = \overline{M_1M_2}$.

2/ Velocity vector

- **Average velocity vector:**

The average velocity during the displacement from M_1 to M_2 is written:

$$\vec{V}_{avg} = \frac{\overrightarrow{M_1M_2}}{t_2 - t_1} = \frac{\overrightarrow{M_1M_2}}{\Delta t} = \frac{\Delta\overrightarrow{OM}}{\Delta t} = \frac{\Delta\vec{r}}{\Delta t}$$

- **Instantaneous velocity vector:**

The instantaneous velocity is the velocity of the mobile at a time t, where $\vec{V}$ is expressed by the relation:

$$\vec{V} = \lim_{\Delta t \mapsto 0} \frac{\overrightarrow{M_1M_2}}{\Delta t} = \frac{d\overrightarrow{OM}}{dt} = \frac{d\vec{r}}{dt}$$

The instantaneous velocity vector $\vec{V}$ at each instant t is tangent to the trajectory. Its direction is that of movement.

3/ Acceleration vector

- **Average acceleration vector:**

Between times t_1 and t_2 the average acceleration vector is defined by:

$$\gamma_{moy} = \frac{V_2 - V_1}{t_2 - t_1} = \frac{\Delta V}{\Delta t}$$

- **instantaneous acceleration vector:**

The instantaneous acceleration vector can be obtained by the temporal derivative of the instantaneous velocity vector:

$$\vec{\gamma} = \lim_{\Delta t \mapsto 0} \frac{\Delta \vec{V}}{\Delta t} = \frac{d\vec{V}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

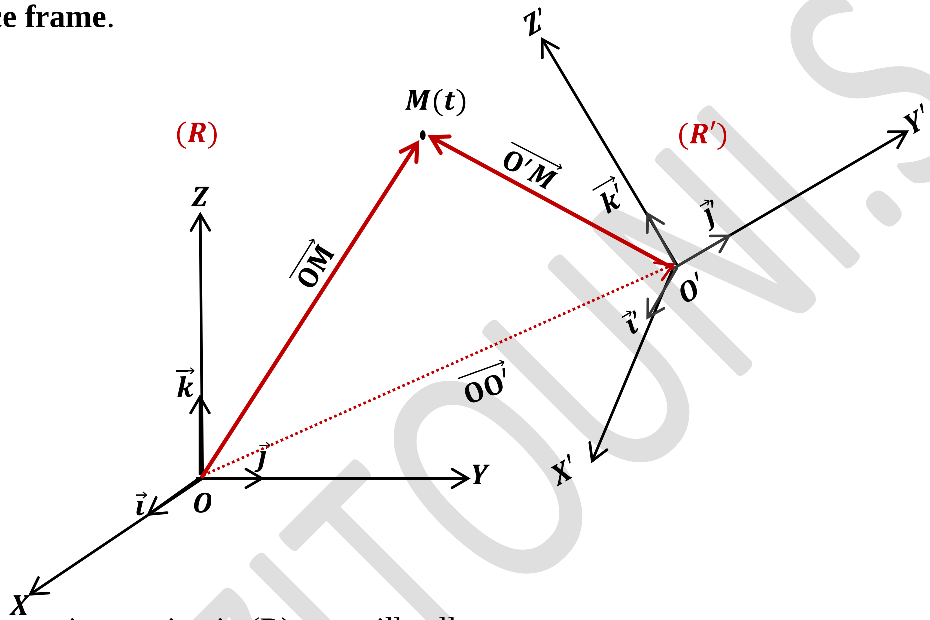
3.3. Other movements

Some movements are a little more complex than rectilinear, curvilinear and circular movements. Let us cite, as an example...the helical movement, the spiral movement, the cycloidal movement, the vibratory movements, the elliptical movement, the hyperbolic movement and the parabolic movement etc….

4. Relative movement

4.1. Definition

We consider a reference frame (R) provided with a fixed orthonormal basis $R\ (O,\vec{\imath},\vec{\jmath},\vec{k})$. This fixed reference frame will be called an **absolute reference frame**. On the other hand, we consider another reference frame (R') provided with an orthonormal basis $R'(O',\vec{\imath'},\vec{\jmath'},\vec{k'})$ in motion with respect to (R), this moving reference frame will be called a **relative reference frame**.



Let M be a point moving in (R), we will call:

- **Absolute movement**, the movement of M with respect to (R)
- **Relative movement**, the movement of M with respect to (R')
- **Training movement**, the movement of (R') with respect to (R)

4.2. Calculation of velocity and acceleration

The calculation of the absolute velocity as well as the absolute acceleration of the mobile M with respect to the fixed reference frame (R) is carried out by two methods: the direct method (or direct derivative) and the compositios of velocities and accelerations method.

4.2.1. Direct method

The velocity and acceleration of the mobile M with respect to the absolute reference (R) and the relative reference (R') are given by:

a) Absolute quantities:

The movement of M with respect to the absolute reference frame (R) is characterized by:

Position vector: $\overrightarrow{OM} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

Absolute velocity : $\vec{V_a}(M) = \left.\frac{d\overrightarrow{OM}}{dt}\right|_R = \frac{dx}{dt}\,\vec{i} + \frac{dy}{dt}\,\vec{j} + \frac{dz}{dt}\,\vec{k}$

Absolute acceleration : $\vec{\gamma_a}(M) = \left.\frac{d\vec{V_a}}{dt}\right|_R = \left.\frac{d^2\overrightarrow{OM}}{dt^2}\right|_R = \frac{d^2x}{dt^2}\,\vec{i} + \frac{d^2y}{dt^2}\,\vec{j} + \frac{d^2z}{dt^2}\,\vec{k}$

b) Relative quantities:

The movement of M with respect to the relative reference frame (R') is characterized by:

Position vectory: $\overrightarrow{OM'} = x'\,\vec{i'} + y'\,\vec{j'} + z'\,\vec{k'}$

Relative velocity : $\vec{V_r}(M) = \left.\frac{d\overrightarrow{O'M}}{dt}\right|_{R'} = \frac{dx'}{dt}\,\vec{i'} + \frac{dy'}{dt}\,\vec{j'} + \frac{dz'}{dt}\,\vec{k'}$

Relative *acceleration* : $\vec{\gamma_r}(M) = \left.\frac{d\vec{V_r}}{dt}\right|_{R'} = \left.\frac{d^2\overrightarrow{O'M}}{dt^2}\right|_{R'} = \frac{d^2x'}{dt^2}\vec{i'} + \frac{d^2y'}{dt^2}\,\vec{j'} + \frac{d^2z'}{dt^2}\,\vec{k'}$

4.2.2. Composition of velocities and accelerations

1/ Composition of velocities

The composition of the absolute velocity vector $\vec{V_a}$ is expressed as a function of the relative velocity vector $\vec{V_r}$, as follows.

We have:

$$\vec{V_a} = \left.\frac{d\overrightarrow{OM}}{dt}\right|_R = \frac{d}{dt}\left(\overrightarrow{OO'} + \overrightarrow{O'M}\right) = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \left.\frac{d\overrightarrow{O'M}}{dt}\right|_R$$

$$\vec{V_a} = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \left[\frac{dx'}{dt}\,\vec{i'} + \frac{dy'}{dt}\,\vec{j'} + \frac{dz'}{dt}\,\vec{k'}\right] + \left[\frac{d\vec{i'}}{dt}\,x' + \frac{d\vec{j'}}{dt}\,y' + \frac{d\vec{k'}}{dt}\,z'\right]$$

The reference frame (R') is mobile, the unit vectors $(\vec{i'}, \vec{j'}, \vec{k'})$ are therefore not constant over time.

The composition of the absolute velocity is therefore written:

$$\vec{V_a} = \vec{V_r} + \left[\left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \left[\frac{d\vec{\imath'}}{dt}x' + \frac{d\vec{\jmath'}}{dt}y' + \frac{d\vec{k'}}{dt}z'\right]\right]$$

The term in the square brackets describes the movement of the reference $R'(O', \vec{\imath'}, \vec{\jmath'}, \vec{k'})$ with respect to the reference (R) and we call it the training velocity of (R') with respect to (R), denoted $\vec{V_e}$, and we write:

$$\vec{v_e} = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \left[\frac{d\vec{\imath'}}{dt}x' + \frac{d\vec{\jmath'}}{dt}y' + \frac{d\vec{k'}}{dt}z'\right]$$

Then, the absolute velocity is written by the law of composition of velocities, as follows:

$$\vec{\boldsymbol{V_a}} = \vec{\boldsymbol{V_r}} + \vec{\boldsymbol{V_e}}$$

Discussion on training velocity $\vec{V_e}$:

The term between the square brackets of the training velocity $\vec{V_e}$ contains the derivative of the unit vectors $(\vec{\imath'}, \vec{\jmath'}, \vec{k'})$ of the moving frame (R'). So, to identify this term we distinguish two following cases:

- **(R') in translation with respect to (R):**

  (R') in translation with respect to (R), i.e.: $\vec{\imath} = \vec{\imath'}, \vec{\jmath} = \vec{\jmath'}, \vec{k} = \vec{k'}$

$$\left[\frac{d\vec{\imath'}}{dt}x' + \frac{d\vec{\jmath'}}{dt}y' + \frac{d\vec{k'}}{dt}z'\right] = \vec{0}$$

  The training velocity will then write:

$$\vec{V_e} = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R$$

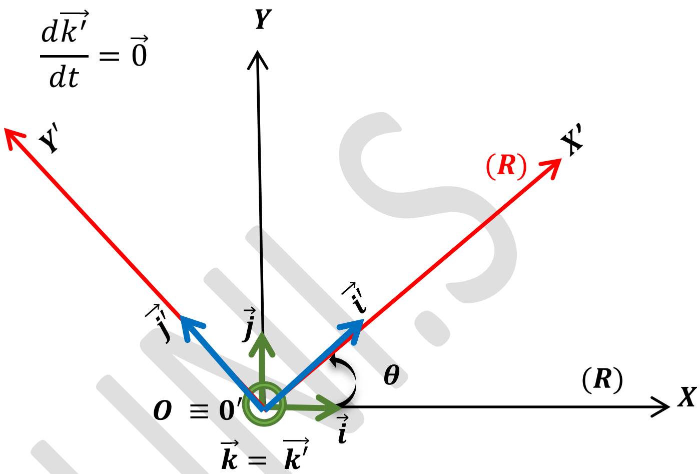
- **(R') in rotation with respect to (R):**

We consider that the axis of rotation of (R') with respect to (R) is Oz $(k = k')$. The angular velocity vector (rotation) is therefore: $\vec{\omega} = \frac{d\theta}{dt}\vec{k}$, or : $\vec{\omega} = \omega\,\vec{k}$. We know that:

$$\vec{V_e} = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \left[\frac{d\vec{\imath'}}{dt}x' + \frac{d\vec{\jmath'}}{dt}y' + \frac{d\vec{k'}}{dt}z'\right]$$

The unit vectors of the moving reference frame (R') can be given as a function of that of the absolute reference frame according to the following relations:

$$\begin{cases} \vec{\imath'} = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath} \\ \vec{\jmath'} = -\sin\theta\,\vec{\imath} + \cos\theta\,\vec{\jmath} \end{cases}, \quad \text{where :} \quad \begin{cases} \dfrac{d\vec{\imath'}}{dt} = \dfrac{d\vec{\imath'}}{d\theta}\dfrac{d\theta}{dt} = \omega\vec{\jmath'} \\ \dfrac{d\vec{\jmath'}}{dt} = \dfrac{d\vec{\jmath'}}{d\theta}\dfrac{d\theta}{dt} = -\omega\vec{\imath'} \\ \dfrac{d\vec{k'}}{dt} = \vec{0} \end{cases}$$



We replace them in the expression of the training velocity:

$$\vec{V_e} = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \left[\omega\vec{\jmath'}\,x' - \omega\vec{\imath'}\,y' + \vec{0}.z'\right]$$

$$\Rightarrow \quad \vec{V_e} = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \omega\left[x'\vec{\jmath'} - y'\vec{\imath'}\right]$$

The expression in parentheses can be determined by the vector product between the angular velocity vector $\vec{\omega}$ and the position vector $\overrightarrow{O'M}$:

$$\Rightarrow \boldsymbol{\vec{V_e}} = \left.\frac{\boldsymbol{d\overrightarrow{OO'}}}{\boldsymbol{dt}}\right|_{\boldsymbol{R}} + \boldsymbol{\vec{\omega}} \wedge \boldsymbol{\overrightarrow{O'M}}$$

We summarize:

$$\boldsymbol{\vec{V_a}} = \boldsymbol{\vec{V_r}} + \boldsymbol{\vec{V_e}}$$

$$\vec{V_r} = \left.\frac{d\overrightarrow{O'M}}{dt}\right|_{R'} = \frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'}$$

$$\vec{V_e} = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \vec{\omega} \wedge \overrightarrow{O'M}$$

To determine the derivative of a vector which belongs to the moving reference frame (R') with respect to the fixed reference frame (R), we use the following procedure:

$$\left.\frac{d\overrightarrow{O'M}}{dt}\right|_R = \left[\frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'}\right] + \left[\frac{d\vec{\imath'}}{dt}x' + \frac{d\vec{\jmath'}}{dt}y' + \frac{d\vec{k'}}{dt}z'\right]$$

We finally write:

$$\left.\frac{d\overrightarrow{O'M}}{dt}\right|_R = \left.\frac{d\overrightarrow{O'M}}{dt}\right|_{R'} + \vec{\omega}\wedge\overrightarrow{O'M}$$

2/ Composition of accelerations

The composition of the absolute acceleration vector $\overrightarrow{\gamma_a}$ is expressed as a function of the relative acceleration vector $\overrightarrow{\gamma_r}$:

we have : $$\overrightarrow{\gamma_a} = \left.\frac{d\overrightarrow{V_a}}{dt}\right|_R = \left.\frac{d(\overrightarrow{V_r}+\overrightarrow{V_e})}{dt}\right|_R = \left.\frac{d\overrightarrow{V_r}}{dt}\right|_R + \left.\frac{d\overrightarrow{V_e}}{dt}\right|_R$$

We will now search to $\left.\frac{d\overrightarrow{V_r}}{dt}\right|_R$ and $\left.\frac{d\overrightarrow{V_e}}{dt}\right|_R$:

$\overrightarrow{V_r}$: is a vector defined in the frame (R'), therefore its derivative with respect to (R) takes the form:

$$\left.\frac{d\overrightarrow{V_r}}{dt}\right|_R = \left.\frac{d\overrightarrow{V_r}}{dt}\right|_{R'} + \vec{\omega}\wedge\overrightarrow{V_r};$$

$$\left.\frac{d\overrightarrow{V_e}}{dt}\right|_R = \frac{d}{dt}\left[\left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \omega\wedge\overrightarrow{O'M}\right]_R = \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R + \left.\frac{d}{dt}(\vec{\omega}\wedge\overrightarrow{O'M})\right|_R$$

$$\left.\frac{d\overrightarrow{V_e}}{dt}\right|_R = \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R + \left.\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M}\right|_R + \vec{\omega}\wedge\left.\frac{d\overrightarrow{O'M}}{dt}\right|_R$$

$$\left.\frac{d\overrightarrow{V_e}}{dt}\right|_R = \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R + \left.\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M}\right|_R + \vec{\omega}\wedge\left[\left.\frac{d\overrightarrow{O'M}}{dt}\right|_{R'} + \vec{\omega}\wedge\overrightarrow{O'M}\right]$$

$$\left.\frac{d\overrightarrow{V_e}}{dt}\right|_R = \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R + \left.\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M}\right|_R + \vec{\omega}\wedge\left[\overrightarrow{V_r} + \vec{\omega}\wedge\overrightarrow{O'M}\right]$$

$$\left.\frac{d\overrightarrow{V_e}}{dt}\right|_R = \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R + \left.\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M}\right|_R + \vec{\omega}\wedge\overrightarrow{v_r} + \vec{\omega}\wedge\left(\vec{\omega}\wedge\overrightarrow{O'M}\right)$$

We do now the sum between $\left.\frac{d\overrightarrow{V_r}}{dt}\right|_R$ and $\left.\frac{d\overrightarrow{V_e}}{dt}\right|_R$:

$$\overrightarrow{\gamma_a} = \left.\frac{d\overrightarrow{V_r}}{dt}\right|_R + \left.\frac{d\overrightarrow{V_e}}{dt}\right|_R$$

$$\overrightarrow{\gamma_a} = \left.\frac{d\overrightarrow{V_r}}{dt}\right|_{R'} + \vec{\omega} \wedge \overrightarrow{V_r} + \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_{R} + \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O'M} + \vec{\omega} \wedge \overrightarrow{V_r} + \vec{\omega} \wedge \left(\vec{\omega} \wedge \overrightarrow{O'M}\right)$$

$$\boldsymbol{\overrightarrow{\gamma_a} = \left.\frac{d\overrightarrow{V_r}}{dt}\right|_{R'} + 2\vec{\omega} \wedge \overrightarrow{V_r} + \left[\left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_{R} + \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O'M} + \vec{\omega} \wedge \left(\vec{\omega} \wedge \overrightarrow{O'M}\right)\right]}$$

We define:

- Relative acceleration: $\overrightarrow{\gamma_r} = \left.\frac{d\overrightarrow{V_r}}{dt}\right|_{R'}$
- Coriolis acceleration: $\overrightarrow{\gamma_c} = 2\vec{\omega} \wedge \overrightarrow{V_r}$
- Training acceleration: $\overrightarrow{\gamma_e} = \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_{R} + \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O'M} + \vec{\omega} \wedge \left(\vec{\omega} \wedge \overrightarrow{O'M}\right)$.

Then, the absolute acceleration is written by the law of composition of accelerations, as follows:

$$\boldsymbol{\overrightarrow{\gamma_a} = \overrightarrow{\gamma_r} + \overrightarrow{\gamma_c} + \overrightarrow{\gamma_e}}$$

5. Applications

☒ Exercise -01

The position of a particle moving in the (Oxy) plane at any time t is given by the following equations:

$$x = 2t \; ; \quad y = 4t^2 - 4t$$

1/ Find the trajectory equation of the movement $y = f(x)$ and determine its nature?

2/ Calculate the velocity of the particle,

3/ Show that its acceleration is constant,

4/ Determine the normal and tangential components of the acceleration in the intrinsic coordinate system.

5/ Deduce the radius of curvature.

☒ Solution

1/ Trajectory equation: We eliminate the time between the time equations to obtain $y = f(x) : \rightarrow \; t = \frac{1}{2}x$, $\boldsymbol{y = x^2 - 2x}$, The trajectory is a parabolic.

2/ The velocity of the mobile: We derive the position vector with respect to time:

$$\begin{cases} V_x = \dfrac{dx}{dt} = 2 \\ V_y = \dfrac{dy}{dt} = 8t - 4 \end{cases} \Rightarrow V = \sqrt{V_x^2 + V_y^2} \Rightarrow V = \sqrt{(8t-4)^2 + 4} \ \ (m.s^{-1})$$

3/ The acceleration of the mobile: By deriving the velocity vector with respect to time we arrive at:

$$\begin{cases} \gamma_x = \dfrac{dV_x}{dt} = 0 \\ \gamma_y = \dfrac{dV_y}{dt} = 8 \end{cases} \Rightarrow \gamma = \sqrt{\gamma_x^2 + \gamma_y^2} \Rightarrow \gamma = 8\,(m.s^{-2}) \Rightarrow \gamma = C^t$$

4/ The tangential acceleration is the derivative of the magnitude of the velocity with respect to time, therefore:

$$\gamma_T = \frac{dV}{dt} = \frac{8\,(8t-4)}{\sqrt{(8t-4)^2+4}} \ (m.s^{-2})$$

The normal acceleration is:

$$\gamma_N = \sqrt{\gamma^2 - \gamma_T^2} = \frac{16}{\sqrt{(8t-4)^2+4}} \ (m.s^{-2})$$

5/ The radius of curvature is:

$$\gamma_N = \frac{V^2}{R} \Rightarrow R = \frac{V^2}{\gamma_N} = \frac{\sqrt{(8t-4)^2+4}^{\,3}}{16} \ (m)$$