Chapter II: Vector Calculus

Summary

1. Definitions

Physics is a science based on physical phenomena, and for studying these phenomena requires defining physical quantities. A physical quantity is a measurable physical property, such as: pressure (P), volume (V), mass (M), temperature (T)...etc.

In physics, physical quantities are in nature: scalar and vector.

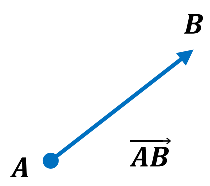
A scalar is a number: positive, negative or zero, used to represent various quantities: time, temperature, mass, energy, etc.

A vector has a magnitude and a direction, used to represent various quantities: velocity, acceleration, force, electric and magnetic fields, etc.

2. Notion of a vector "V"

Mechanics is the study of the motion (movement) of material point, which means the study of the parameters of its motion (such as its position, velocity, and acceleration) as well as the forces applied to it. Therefore, the notion of a vector "V" must be introduced.

In a general way, vectors $\vec{V}$ are mathematical objects represented graphically by oriented line segments, for example: $\overrightarrow{AB}$. This representation makes it possible to define the four characteristics of the vector:



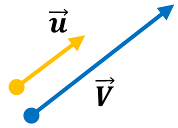
- The **origin** A.
- The **support** which is the straight line (AB).
- The **direction** (from A to B).

- The **norm** or **magnitude** (related to segment size).

3. Unit Vector

Each vector $\vec{V}$ can be expressed in terms of its **magnitude** $\|\vec{V}\|$ and its **unit vector** $\vec{u}$. we write:

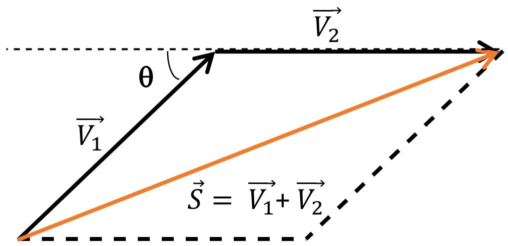
- $\vec{V} = \|\vec{V}\|\,\vec{u}$; $\vec{V} \parallel \vec{u}$
- $|\vec{u}| = u = 1$
- $\|\vec{V}\| = V$



4. Addition of vectors

Vectors can be added together to form another vector called a sum vector or result (S).

$$\vec{S} = \vec{V_1} + \vec{V_2}$$



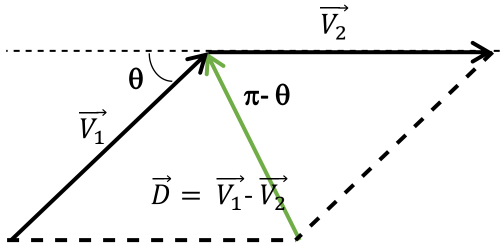
Geometrically, the addition of two vectors produces a vector corresponding to the long diagonal of the parallelogram formed by the two vectors to be added (when the latter have the same "hanging point").

- **This operation has the following properties:**
  - Commutativity: $\vec{V_1} + \vec{V_2} = \vec{V_2} + \vec{V_1}$
  - Associativity: $\vec{V_1} + (\vec{V_2} + \vec{V_3}) = (\vec{V_1} + \vec{V_2}) + \vec{V_3}$
  - There is a neutral element 0, such as $\vec{V} + \vec{0} = \vec{V}$
  - Each vector $\vec{V}$ has an opposite element, denoted $-\vec{V}$
  - The parallel condition of two vectors: $\vec{V_1} = K\,\vec{V_2},\ K = C^{te}$
  - Magnitude of $\vec{S}$: $S = \sqrt{V_1^2 + V_2^2 + 2V_1V_2cos(\theta)}$

5. Subtraction of vector

Similarly, the vectors can also be subtracted to form another vector called the difference vector (D).

$$\vec{D} = \vec{V_1} - \vec{V_2}$$



Geometrically, the subtraction of two vectors forms the small diagonal of the parallelogram.

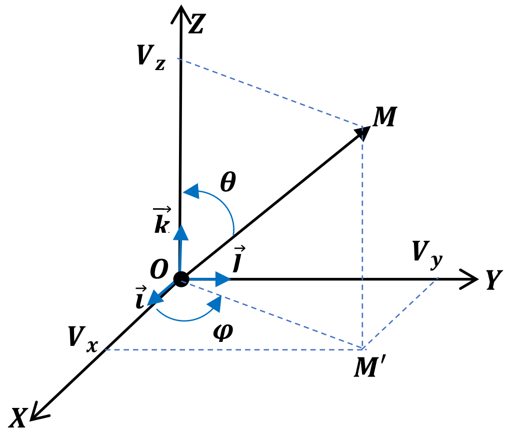
- **This operation has the following properties:**
  - Non-commutativity: $\vec{V_1} - \vec{V_2} \neq \vec{V_2} - \vec{V_1}$
  - Non-associativity: $\vec{V_1} + (\vec{V_2} - \vec{V_3}) \neq (\vec{V_1} + \vec{V_2}) - \vec{V_3}$
  - There is a neutral element 0, such as $\vec{V} - \vec{0} = \vec{V}$
  - Magnitude of $\vec{D}$: $D = \sqrt{V_1^2 + V_2^2 + 2V_1V_2cos(\pi - \theta)}$

6. Multiplication of a vector by a scalar

Let $\vec{V}$ be a vector and λ a scalar, the product λ. $\vec{V}$ is a vector with the same direction as $\vec{V}$ if λ is positive, and with the opposite direction of $\vec{V}$ if λ is negative. If $\lambda = 0$, $\lambda.\vec{V} = \vec{0}$.

7. Components of a vector

A vector $\vec{V}$ can be expressed by its components, that is to say by the sum of two or more vectors.



a) **In space**, let $\mathcal{R}\ (O, \vec{i}, \vec{j}, \vec{k})$ be a direct orthonormal system of origin O and base $(\vec{i}, \vec{j}, \vec{k})$. In the base of $(\vec{i}, \vec{j}, \vec{k})$, the vector $\overrightarrow{OM}$ decomposes by unique way in the form:

$$\vec{V} = \overrightarrow{OM} = V_x\vec{i} + V_y\vec{j} + V_z\vec{k}$$

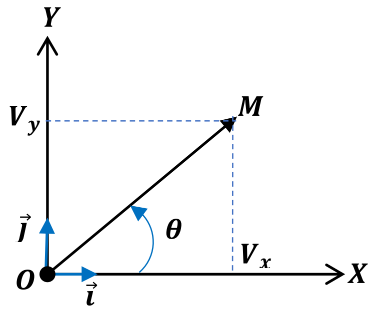
$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$: Are called the components of the vector $\overrightarrow{OM}$ in the base $(\vec{i}, \vec{j}, \vec{k})$, where:

$$\begin{cases} V_x = OM' \cos\varphi = V \sin\theta \cos\varphi \\ V_y = OM' \sin\varphi = V \sin\theta \sin\varphi \\ V_z = V \cos\theta \end{cases}$$

Then: $\vec{V} = V\sin\theta\cos\varphi\,\vec{i} + V \sin\theta\sin\varphi\,\vec{j} + V \cos\theta\,\vec{k}$

And: $\vec{u} = \sin\theta\cos\varphi\,\vec{\imath} + \sin\theta\sin\varphi\,\vec{\jmath} + \cos\theta\,\vec{k}$

The magnitude is written: $\|\vec{V}\| = \sqrt{V_x^2 + V_y^2 + V_z^2}$



b) In plane: in the case of a plane provided with an orthonormal basis $(O, \vec{\imath}, \vec{\jmath})$, the vector $\overrightarrow{OM}$ decomposes as follows:

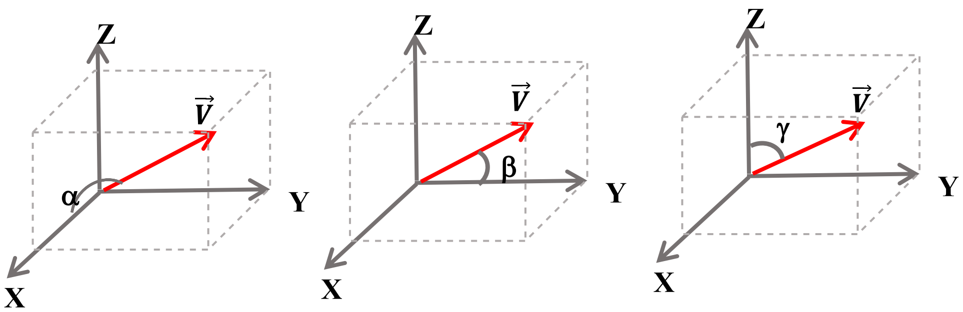
$$\vec{V} = \overrightarrow{OM} = V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k}$$

$\begin{pmatrix} V_x \\ V_y \end{pmatrix}$: Are called the components of the vector $\overrightarrow{OM}$ in the base $(\vec{\imath}, \vec{\jmath})$, where:

$$\begin{cases} V_x = V\cos\theta \\ V_y = V\sin\theta \end{cases} \Rightarrow \vec{V} = V\cos\theta\,\vec{\imath} + V\sin\theta\,\vec{\jmath}$$

8. The directional cosines of a vector

In analytical geometry, the directional cosines also known as direction cosines of a vector is defined as the cosines of the angles α, β and γ between the three coordinate axes *Ox*, *Oy* and *Oz* respectively and the vector $\vec{V}$.



We have:

$$V^2 = V_x^2 + V_y^2 + V_z^2$$

$$V^2 = (V\cos\alpha)^2 + (V\cos\beta)^2 + (V\cos\gamma)^2$$

$$\Rightarrow V^2 = V^2[\cos\alpha^2 + \cos\beta^2 + \cos\gamma^2]$$

Thus:

$$\boldsymbol{\cos\alpha^2 + \cos\beta^2 + \cos\gamma^2 = 1}$$

$\cos\alpha$, $\cos\beta$ *et* $\cos\gamma$ are called the directional cosines of the vector $\vec{V}$

On the other hand,

$$\vec{V} = V_x\vec{i} + V_y\vec{j} + V_z\vec{k}$$

$$\vec{V} = V\cos\alpha\,\vec{i} + V\cos\beta\,\vec{j} + V\cos\gamma\,\vec{k}$$

➢ $\begin{cases}\vec{V} = V(\cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\gamma\,\vec{k}) \\ \vec{V} = V\,\vec{u}\end{cases} \Longrightarrow \boldsymbol{\vec{u} = \cos\alpha\,\vec{i} + \cos\beta\,\vec{j} + \cos\gamma\,\vec{k}}$

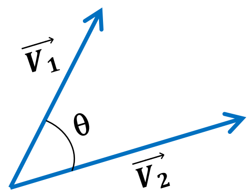
9. Scalar product (Dot product)

a) Definition

The scalar product of two vectors $\vec{V_1}$ and $\vec{V_2}$ in space present the following real number:

$$\vec{V_1}.\vec{V_2} = \|\vec{V_1}\|.\|\vec{V_2}\|.\cos(\vec{V_1}.\vec{V_2})$$

θ, the angle formed between the two vectors $\vec{V_1}$ and $\vec{V_2}$.



b) Properties

- Commutativity: $\vec{V_1}.\vec{V_2} = \vec{V_2}.\vec{V_1}$
- Distributivity: $\vec{V_1}.(\vec{V_2} + \vec{V_3}) = \vec{V_1}.\vec{V_2} + \overrightarrow{\vec{V_1}.V_3}$
- Multiplication by a real: $\lambda\vec{V_1}.\mu\vec{V_2} = \lambda\mu.\vec{V_2}.\vec{V_1}$
- $\vec{V}.\vec{V} = \vec{V}^2 = \|\vec{V}\|^2$
- Case of nullity: $\vec{V_1}.\vec{V_2} = \vec{0}$, $\Leftrightarrow$ $\vec{V_1} = \vec{0}$, ou $\vec{V_2} = \vec{0}$, or $\vec{V_1} \perp \vec{V_2}$
- $(\vec{V_1} + \vec{V_2})^2 = \|\vec{V_1}\|^2 + \|\vec{V_2}\|^2 + 2\|\vec{V_1}\|\|\vec{V_2}\|\cos(\vec{V_1}, \vec{V_2})$
- $(\vec{V_1} - \vec{V_2})^2 = \|\vec{V_1}\|^2 + \|\vec{V_2}\|^2 - 2\|\vec{V_1}\|\|\vec{V_2}\|\cos(\vec{V_1}, \vec{V_2})$

c) Scalar product of unit vectors

According to the properties of the scalar product, we have:

$\vec{i}.\vec{i} = \vec{j}.\vec{j} = \vec{k}.\vec{k} = 1$

$\vec{i}.\vec{j} = \vec{j}.\vec{k} = \vec{i}.\vec{k} = 0$

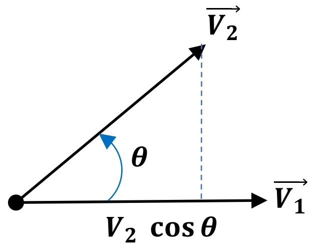
Analytical expression of the scalar product

Let: $\vec{V_1} = x_1\,\vec{i} + y_1\,\vec{j} + z_1\,\vec{k}$ and $\vec{V_2} = x_2\,\vec{i} + y_2\,\vec{j} + z_2\vec{k}$ be two vectors in the base $(\vec{i}, \vec{j}, \vec{k})$.

$$\vec{V_1}.\vec{V_2} = x_1.x_2 + y_1.y_2 + z_1.z_2$$

d) Projection of a vector

If we consider the value $V_2 \cos\theta$ the projection of the vector $\vec{V_2}$ on $\vec{V_1}$, the scalar product is therefore written in this form:



$$\vec{V_1}\,.\,\vec{V_2} = V_1\,.V_2\cos\theta$$

$$\vec{V_1}\,.\,\vec{V_2} = V_1\,.\,proj\,\vec{V_2}/_{\vec{V_1}}$$

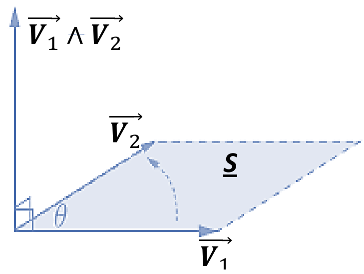
$$\begin{cases} \boldsymbol{proj\,\vec{V_2}/_{\vec{V_1}}} = \boldsymbol{\vec{V_1}\,.\,\vec{V_2}/V_1} \\ \boldsymbol{proj\,\vec{V_1}/_{\vec{V_2}}} = \boldsymbol{\vec{V_1}\,.\,\vec{V_2}/V_2} \end{cases}$$

10. Vector product (Cross product)

a) Definition

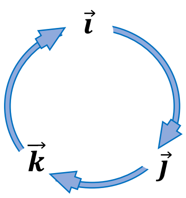
There is another product of vectors, specific to $\mathscr{R}^3$. It is the vector product which defined geometrically as follows:

The vector product of two vectors is defined as a vector perpendicular to both $\vec{V_1}$ and $\vec{V_2}$, whose its magnitude is:



$$\|\vec{V_1}\wedge\vec{V_2}\| = \|\vec{V_1}\|.\|\vec{V_2}\|.\sin(\vec{V_1}\,.\vec{V_2})$$

θ, the angle formed between the two vectors $\vec{V_1}$ and $\vec{V_2}$. And the orientation is obtained by the right-hand rule.



b) Properties

- Antisymmetric: $\vec{V_1}\wedge\vec{V_2} = -\vec{V_2}\wedge\vec{V_1}$
- Distributivity: $\vec{V_1}\wedge(\vec{V_2}+\vec{V_3}) = (\vec{V_1}\wedge\vec{V_2}) + (\vec{V_1}\wedge\vec{V_3})$
- Multiplication by a real: $\lambda\vec{V_1}\wedge\mu\vec{V_2} = \lambda\mu\,\vec{V_1}\wedge\vec{V_2}$
- Case of nullity: $\vec{V_1}\wedge\vec{V_2} = \vec{0}, \iff \vec{V_1} = \vec{0}$, ou $\vec{V_2} = \vec{0}$, where $\vec{V_1}$ and $\vec{V_2}$ are collinear
- $\vec{i}\wedge\vec{i} = \vec{j}\wedge\vec{j} = \vec{k}\wedge\vec{k} = 0$

- Circular permutation: $\vec{i}\wedge\vec{j} = \vec{k},\ \vec{j}\wedge\vec{k} = \vec{i}\ et\ \vec{i}\wedge\vec{k} = -\vec{j}$

c) Analytical expression of the vector product

Let: $\overrightarrow{V_1} = x_1\,\vec{i} + y_1\,\vec{j} + z_1\,\vec{k}$ and $\overrightarrow{V_2} = x_2\,\vec{i} + y_2\,\vec{j} + z_2\vec{k}$ be two vectors in the base $(\vec{i},\ \vec{j}, \vec{k})$.

$$\overrightarrow{V_1}\wedge\overrightarrow{V_2} = \begin{vmatrix} \vec{i} & -\vec{j} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix}\vec{i} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix}\vec{j} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}\vec{k}$$

$$\overrightarrow{V_1}\wedge\overrightarrow{V_2} = (y_1z_2 - y_2z_1)i - (x_1z_2 - x_2z_1)j + (x_1y_2 - x_2y_1)k$$

d) Magnitude of vector product

The magnitude of the vector product of two vectors represents the area of an equilateral included between these two vectors.

$$S = \|\overrightarrow{V_1}\wedge\overrightarrow{V_2}\| = \|\overrightarrow{V_1}\|.\|\overrightarrow{V_2}\|.\sin\overline{(V_1, V_2)}$$

$$S = \|\overrightarrow{V_1}\|.h \quad ; h : height$$

e) Multiple vector product

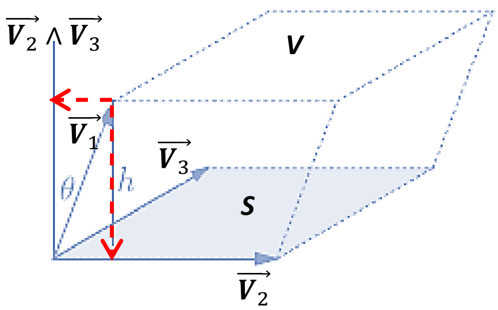
Let be three vectors: $\overrightarrow{V_1},\ \overrightarrow{V_2}$ et $\overrightarrow{V_3}$:

$$\overrightarrow{V_1}\wedge(\overrightarrow{V_2}\wedge\overrightarrow{V_3}) = \overrightarrow{V_1}.(\overrightarrow{V_3}.\overrightarrow{V_2}) - \overrightarrow{V_1}.(\overrightarrow{V_2}.\overrightarrow{V_3})$$

11. Mixed product

a) Definition

We also define the mixed product (scalar and vector product) or triple product of three vectors $\overrightarrow{V_1}.(\overrightarrow{V_2}\wedge\overrightarrow{V_3})$ as follows:



$$\overrightarrow{V_1}.(\overrightarrow{V_2}\wedge\overrightarrow{V_3}) = \|\overrightarrow{V_1}\|.\|\overrightarrow{V_2}\wedge\overrightarrow{V_3}\|.\cos\theta\ ,$$

θ, Angle formed between $\overrightarrow{V_1}$ *and* $(\overrightarrow{V_2}\wedge\overrightarrow{V_3})$

This product can also be expressed by the following manner:

$$\overrightarrow{V_1}\,.\left(\overrightarrow{V_2}\wedge\overrightarrow{V_3}\right) = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Geometrically, the value obtained from the mixed product (three vectors) is equal to the volume of the parallelepiped formed by these three vectors:

We have: $\overrightarrow{V_1}\,.\left(\overrightarrow{V_2}\wedge\overrightarrow{V_3}\right) = \left\|\overrightarrow{V_1}\right\|.\left\|\overrightarrow{V_2}\wedge\overrightarrow{V_3}\right\|.\cos\theta$

After the projection: $\begin{cases} h = \left\|\overrightarrow{V_1}\right\|.\cos\theta \\ S = \left\|\overrightarrow{V_2}\wedge\overrightarrow{V_3}\right\| \end{cases}$ So: $\Longrightarrow V = h\,.S$

b) Properties

- Circular permutation: $\overrightarrow{V_1}\,.\left(\overrightarrow{V_2}\wedge\overrightarrow{V_3}\right) = \overrightarrow{V_2}\,.\left(\overrightarrow{V_3}\wedge\overrightarrow{V_1}\right) = \overrightarrow{V_3}\,.\left(\overrightarrow{V_1}\wedge\overrightarrow{V_2}\right)$
- Multiplication by a real: $\lambda\overrightarrow{V_1}\,.\left(\mu\overrightarrow{V_2}\wedge\varepsilon\overrightarrow{V_3}\right) = \lambda\mu\varepsilon.\overrightarrow{V_1}\,.\left(\overrightarrow{V_2}\wedge\overrightarrow{V_3}\right)$

12. Moment of a vector

a. Moment of a vector about a point

The moment of a vector $\overrightarrow{AB}$ about a fixed-point O is defined by the vector:

$$\vec{M}_O\left(\overrightarrow{AB}\right) = \overrightarrow{OA}\wedge\overrightarrow{AB}$$

A : a point belongs to the axis carrying the vector $\overrightarrow{AB}$

The magnitude of the moment is written:

$$\left\|\vec{M}_O\left(\overrightarrow{AB}\right)\right\| = \left\|\overrightarrow{OA}\wedge\overrightarrow{AB}\right\| = \left\|\overrightarrow{OA}\right\|.\left\|\overrightarrow{AB}\right\|.\sin\left(\overrightarrow{OA},\overrightarrow{AB}\right)$$

b. Moment of a vector about an axis

The moment of a vector $\overrightarrow{AB}$ about an axis (Δ) is defined by the scalar product:

$$M_\Delta\left(\overrightarrow{AB}\right) = \vec{M}_O\left(\overrightarrow{AB}\right).\overrightarrow{u_\Delta}$$

Where, $\overrightarrow{u_\Delta}$, the unit vector of axis (Δ).

13. Applications

☒ Exercise-1

In the Cartesian system (Oxyz) provided with a base $(\vec{\imath},\vec{\jmath},\vec{k})$, we give: $\begin{cases} \overrightarrow{V_1} = 2\,\vec{\imath} + 3\,\vec{\jmath} + 4\,\vec{k} \\ \overrightarrow{V_2} = -\,\vec{\imath} + 3\,\vec{\jmath} + \vec{k} \end{cases}$

1- Represent graphically the vectors $\overrightarrow{V_1}$ and $\overrightarrow{V_2}$ in space (Oxyz).

2- Calculate the vectors: $\vec{S} = \overrightarrow{V_1} + \overrightarrow{V_2}$ and $\vec{D} = \overrightarrow{V_1} - \overrightarrow{V_2}$, then represent them graphically.

3- Calculate the scalar product $\overrightarrow{V_1} \cdot \overrightarrow{V_2}$ as well as the vector product $\overrightarrow{V_1} \wedge \overrightarrow{V_2}$.

4- Give the projection of the vector $\vec{S}$ on the Ox, Oy axis then on Oz.