1 Elements of logic

Notions of Logic

Definition

Any relation P that is either true or false is called a "logical proposition".

* When the proposition is true, it is assigned the value 1.

* When the proposition is false, it is assigned the value 0.

Example

(1) (Every prime number is even), this proposition is false.

(2) ($\sqrt{2}$ is an irrational number), this statement is true.

(3) (2 is less than 4), this statement is true.

Negation: $\overline{P}$

Given a logical proposition P , we call the negation of P the logical proposition $\overline{P}$, which is false when P is true and which is true when P is false, so we can represent it as follows:

P	$\overline{P}$
1	0
0	1

Example (1) Let $E \neq \varnothing$, $P : (a \in E)$, then $\overline{P}$: $(a \notin E)$.

Example 1 *(2) P : the function f is positive, then $\overline{P}$: the function f is not positive.*

(3) $P : x + 2 = 0$, then $\overline{P} : x + 2 \neq 0$.

1.1 Logical connectors

1)The conjunction (**and**), ($\wedge$)

definition

Let P, Q be two propositions, the proposition (P and Q) or ($P \wedge Q$) is the conjunction of the two propositions P, Q.

– ($P \wedge Q$) is true if P and Q are both true.

– ($P \wedge Q$) is false in all other cases.

This is summarized in the following truth table:

P	Q	$P \wedge Q$
1	1	1
1	0	0
0	1	0
0	0	0

Example (1) 2 is an even number and 3 is a prime number, this proposition is true.

(2) $3 \leq 2$ and $4 \geq 2$, this proposition is false.

2) The disjunction (or), ($\vee$)

Definition

the disjunction between P, Q is denoted by (P or Q), ($P \vee Q$).

$P \vee Q$ is false if P and Q are both false, if not $P \vee Q$ is true.
This is summarized in the following truth table:

P	Q	$P \vee Q$
1	1	1
1	0	1
0	1	1
0	0	0

Example (1) 2 is an even number or 3 is a prime number. True.
(2) $3 \leq 2$ or $2 \geq 4$. False.

3)The implication ($\Rightarrow$)

Definition

"$P \Rightarrow Q$" is the logical proposition that is false if P is true and Q is false. We say P implies Q or if P then Q. So its truth table is as follows:

Definition 2

P	Q	$P \Rightarrow Q$
1	*1*	*1*
1	*0*	*0*
0	*1*	*1*
0	*0*	*1*

.

Example

(1) $0 \leq x \leq 9 \Rightarrow \sqrt{x} \leq 3$. True.
(2) It's raining, so I'm taking my umbrella. True, it's a consequence.

4) The reciprocal of an implication

Definition

The reciprocal of an implication ($P \Rightarrow Q$) is an implication $Q \Rightarrow P$.

Example

The reciprocal of: $0 \leq x \leq 9 \Rightarrow \sqrt{x} \leq 3$ is $\sqrt{x} \leq 3 \Rightarrow 0 \leq x \leq 9$.

5) **The contrapositive of implication**

Definition

Let P,Q be two propositions, the contrapositive of ($P \Rightarrow Q$) is ($\overline{Q} \Rightarrow \overline{P}$).

Remark

($P \Rightarrow Q$)and ($Q \Rightarrow P$) have the same truth table, i.e., the same truth value.

6) Equivalence ($\Leftrightarrow$)

Definition

We say that the two logical propositions P and Q are logically equivalent, if they are simultaneously true or simultaneously false, and we note "$P \Leftrightarrow Q$", its truth table is

Definition 3

P	Q	$P \Leftrightarrow Q$
1	***1***	***1***
1	***0***	***0***
0	***1***	***0***
0	***0***	***1***

Example $x+2=0 \Leftrightarrow x=-2$.
Proposition
Let P, Q, R be three logical propositions then,
(1) $\overline{\overline{P}} \Leftrightarrow P$.
(2) $P \wedge P \Leftrightarrow P$.
(3) $P \wedge Q \Leftrightarrow Q \wedge P$. **(Commutativity of $\wedge$).**
(4) $P \vee Q \Leftrightarrow Q \vee P$. **(Commutativity of $\vee$).**
(5) $(P \wedge Q) \wedge R \Longleftrightarrow P \wedge (Q \wedge R)$. **(Associativity of $\wedge$).**
(6) $(P \vee Q) \vee R \Longleftrightarrow P \vee (Q \vee R)$. **(Associativity of $\vee$).**
(7) $P \vee P \Leftrightarrow P$.
(8) $P \wedge (P \vee Q) \Longleftrightarrow P$.
(9) $P \vee (P \wedge Q) \Longleftrightarrow P$.
(10) $(P \wedge Q) \vee R \Longleftrightarrow (P \vee R) \wedge (Q \vee R)$. **($\vee$ is distributive over $\wedge$).**
(11) $(P \vee Q) \wedge R \Longleftrightarrow (P \wedge R) \vee (Q \wedge R)$. **($\wedge$ is distributive over $\vee$).**
(12) $\overline{P \wedge Q} \Leftrightarrow \overline{P} \vee \overline{Q}$. **Morgan's laws.**
(13) $\overline{P \vee Q} \Leftrightarrow \overline{P} \wedge \overline{Q}$. **Morgan's laws.**
(14) $\overline{P \Rightarrow Q} \Leftrightarrow P \wedge \overline{Q}$ **.**
(15) $(P \Rightarrow Q) \Leftrightarrow \left(\overline{P} \vee Q\right) \Longleftrightarrow \overline{Q} \Rightarrow \overline{P}$.
(16) $[(P \Rightarrow Q) \wedge (Q \Rightarrow P)] \Leftrightarrow (P \Longleftrightarrow Q)$.
(17) $[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$.
Proof
(11) $(P \vee Q) \wedge R \Longleftrightarrow (P \wedge R) \vee (Q \wedge R)$.

P	Q	R	$(P \vee Q)$	$(P \vee Q) \wedge R$	$(P \wedge R)$	$(Q \wedge R)$	$(P \wedge R) \vee (Q \wedge R)$
1	**1**	**1**	**1**	**1**	**1**	**1**	**1**
0	**0**	**0**	**0**	**0**	**0**	**0**	**0**
1	**1**	**0**	**1**	**0**	**0**	**0**	**0**
1	**0**	**1**	**1**	**1**	**1**	**0**	**1**
0	**1**	**1**	**0**	**1**	**0**	**1**	**1**
1	**0**	**0**	**1**	**0**	**0**	**0**	**0**
0	**1**	**0**	**1**	**0**	**0**	**0**	**0**
0	**0**	**1**	**0**	**0**	**0**	**0**	**0**

(12) $\overline{P \wedge Q} \Leftrightarrow \overline{P} \vee \overline{Q}$.

P	Q	$\overline{P}$	$\overline{Q}$	$P \wedge Q$	$\overline{P \wedge Q}$	$\overline{P} \vee \overline{Q}$
1	**1**	**0**	**0**	**1**	**0**	**0**
1	**0**	**0**	**1**	**0**	**1**	**1**
0	**1**	**1**	**0**	**0**	**1**	**1**
0	**0**	**1**	**1**	**0**	**1**	**1**

(14) $\overline{P \Rightarrow Q} \Leftrightarrow P \wedge \overline{Q}$ **.**

P	Q	$P \Rightarrow Q$	$\overline{P \Rightarrow Q}$	$\overline{Q}$	$P \wedge \overline{Q}$
1	**1**	**1**	**0**	**0**	**0**
0	**0**	**1**	**0**	**1**	**0**
1	**0**	**0**	**1**	**1**	**1**
0	**1**	**1**	**0**	**0**	**0**

(15) $(P \Rightarrow Q) \Leftrightarrow \left(\overline{P} \vee Q\right) \Longleftrightarrow \overline{Q} \Rightarrow \overline{P}$.

P	Q	$\overline{P}$	$\overline{Q}$	$\overline{P} \vee Q$	$P \Rightarrow Q$	$\overline{Q} \Rightarrow \overline{P}$
1	**1**	**0**	**0**	**1**	**1**	**1**
1	**0**	**0**	**1**	**0**	**0**	**0**
0	**1**	**1**	**0**	**1**	**1**	**1**
0	**0**	**1**	**1**	**1**	**1**	**1**

(16) $[(P \Rightarrow Q) \wedge (Q \Rightarrow P)] \Leftrightarrow (P \Longleftrightarrow Q)$.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$[(P \Rightarrow Q) \wedge (Q \Rightarrow P)]$	$P \Longleftrightarrow Q$
1	**1**	**1**	**1**	**1**	**1**
0	**0**	**1**	**1**	**1**	**1**
1	**0**	**0**	**1**	**0**	**0**
0	**1**	**1**	**0**	**0**	**0**

(17) $[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$.

P	Q	R	$P \Rightarrow Q$	$Q \Rightarrow R$	$[(P \Rightarrow Q) \wedge (Q \Rightarrow R)]$	$P \Rightarrow R$	$[(P \Rightarrow Q) \wedge (Q \Rightarrow R)] \Rightarrow (P \Rightarrow R)$
1	**1**	**1**	**1**	**1**	**1**	**1**	**1**
0	**0**	**0**	**1**	**1**	**1**	**1**	**1**
1	**1**	**0**	**1**	**0**	**0**	**0**	**1**
1	**0**	**1**	**0**	**1**	**0**	**1**	**1**
0	**1**	**1**	**1**	**1**	**1**	**1**	**1**
1	**0**	**0**	**0**	**1**	**0**	**0**	**1**
0	**1**	**0**	**1**	**0**	**0**	**1**	**1**
0	**0**	**1**	**1**	**1**	**1**	**1**	**1**

1.1.1 Negation rules

Let P(x) be a proposition,

(1) the negation of

$$\forall x \in E, P(x)$$

is :

$$\exists x \in E, \overline{P}(x).$$

(2) the negation of

$$\exists x \in E, P(x)$$

is :

$$\forall x \in E, \overline{P}(x).$$

Example

the negation of $\forall \varepsilon > 0$, $\exists x \in \mathbb{Z}$ **such that:** $0 < x < \varepsilon$ **is :** $\exists \varepsilon > 0$, $\forall x \in \mathbb{Z}$ **such that :** $x \leq 0$ **ou** $x \geq \varepsilon$.

2 Reasoning methods

2.1 Direct reasoning

To show that $(P \Rightarrow Q)$ **is true, we assume that** P **is true and demonstrate that** Q **is also true.**

Example

Show that

$$\forall x, y \in \mathbb{R}^{+}, x \leq y \Rightarrow x \leq \frac{x+y}{2} \leq y$$

Proof.

$$x \leq y \Rightarrow x + x \leq x + y$$

$$\begin{aligned} &\Rightarrow \quad 2x \leq x + y \\ &\Rightarrow \quad x \leq \frac{x+y}{2} \qquad \textbf{(1)} \end{aligned}$$

$$x \leq y \Rightarrow x + y \leq y + y$$

$$\begin{aligned} &\Rightarrow \quad x + y \leq 2y \\ &\Rightarrow \quad \frac{x+y}{2} \leq y \qquad \textbf{(2)} \end{aligned}$$

from (1) and (2) we have:

$$x \leq \frac{x+y}{2} \leq y$$

Case by case

In ordrer to prove that a certain proposition $P(x)$ **is true for all** x **in a set** E, **we show that** $P(x)$ **is true for** $x \in A \subset E$, **then for x** $\notin$ **A.**

Example

Show that

$$\forall x \in \mathbb{R}, |x-1| \leq x^2 - x + 1$$

Proof.

If $x \geq 1$ **we have** $|x-1| = x - 1$, **then** $x^2 - x + 1 - |x-1| = x^2 - x + 1 - (x-1) = (x-1)^2 + 1 \geq 0$.

So

$$|x-1| \leq x^2 - x + 1$$

If $x \leq 1$ **we have** $|x-1| = -x + 1$, **then** $x^2 - x + 1 - |x-1| = x^2 - x + 1 - (-x+1) = x^2 \geq 0$.

So

$$|x-1| \leq x^2 - x + 1$$

Conclusion: we have

$$\forall x \in \mathbb{R}, |x-1| \leq x^2 - x + 1$$

Contrapositive reasoning

Contrapositive reasoning is based on the following equivalence: $(P \Rightarrow Q) \Leftrightarrow (\overline{Q} \Rightarrow \overline{P})$, So if we want to show the assertion "$P \Rightarrow Q$" it is sufficient to show that $\overline{Q} \Rightarrow \overline{P}$ is true.

Example

Show that: n^2 is uneven then n is uneven too.

proof

By contrapositive, it is sufficient to show that if n is even$\Rightarrow$ n^2 is even,

$$n \textbf{ is even} \Rightarrow$$

$$\exists k \in \mathbb{N}, n = 2k$$

$$\Rightarrow n.n = 2(2k^2) \Rightarrow n^2 = 2k' \textbf{ with } k' = 2k^2 \in \mathbb{N}$$

so,

$$\exists k' \in \mathbb{N}\textbf{,}\ n^2 = 2k'$$

then n^2 is even. hence the result.

2.2 Reasoning by the absurd

To show that $P \Rightarrow Q$ we assume both that P is true and that Q is false, and look for a contradiction.

Thus, if P is true then Q must be true and so $P \Rightarrow Q$ is true.

Example

Show that:

$$\forall x, y \in \mathbb{R}^+, \textbf{if } \frac{x}{1+y} = \frac{y}{1+x} \textbf{ then } x = y.$$

We assume that $\frac{x}{1+y} = \frac{y}{1+x}$ and $x \neq y$.

we have

$$\frac{x}{1+y} = \frac{y}{1+x}$$

$$\Rightarrow x(1+x) = y(1+y)$$

$$\Rightarrow x + x^2 = y + y^2.$$

$$\Rightarrow x^2 - y^2 = -x + y$$

$$\Rightarrow (x-y)(x+y) = -(x-y)$$

As $x \neq y$ **, then**

Example 4

$$x - y \neq 0$$

and dividing by $x-y$ ***gives :***

$$x + y = -1$$

this is a contradiction (the sum of two positive numbers is positive) and therefore

$$\textbf{if } \frac{x}{1+y} = \frac{y}{1+x} \textbf{ then } \mathbf{x} = \mathbf{y}.$$

2.3 Reasoning by recurrence

To show that $\forall n \in IN$, $n \geq n_0$, $P(n)$ **is true we follow the following steps:**

(a) We show that $P(n_0)$ **is true, (initial value).**

(b) Assume that $P(n)$ **is true and show that** $P(n+1)$ **is true.**

Then P **is true for all** $n \geq n_0$.

Example

Show that:

$$\forall n \in IN^* : 1 + 2 + ... + n = \frac{n(n+1)}{2}.$$

(a) For $n = 1$ **we have** $P(1)$ **is true** $: 1 = \frac{1(2)}{2}$.

(b) We assume that

$$1 + 2 + ... + n = \frac{n(n+1)}{2}$$

is true,

we have :

$$\begin{aligned} 1 + 2 + ... + n + 1 &= (1 + 2 + ... + n) + n + 1 \\ &= \frac{n(n+1)}{2} + n + 1 \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

Then,

$$\forall n \in IN^* : 1 + 2 + ... + n = \frac{n(n+1)}{2}.$$