Tutorial n°1: Dimensional analysis

Exercise 1

A ball of mass (m) at height (h) falls vertically in the air. The experience has shown that the air resistance force (F) is given as a function of the volumetric mass density (ρ), the ball section (S), the drag coefficient (C) and the velocity (V).

1- Determine the dimensional equations of the following quantities: velocity ($\boldsymbol{V}$), acceleration ($\boldsymbol{\gamma}$), volumetric mass density ($\boldsymbol{\rho}$), force ($\boldsymbol{F}$), moment of force ($\boldsymbol{M}$), work ($\boldsymbol{W}$), kinetic energy ($\boldsymbol{E_c}$), potential energy ($\boldsymbol{E_p}$) and surface area ($\boldsymbol{S}$).

2- Check that: $[\mathbf{W}/\boldsymbol{mg}] = [\mathrm{V}^2/2\mathbf{g}] = [\boldsymbol{h}] = \mathbf{L}$.

3- Find the expression of the air resistance force, which takes the form:

$$\boldsymbol{F} = \frac{1}{2}.\boldsymbol{C}.\ \boldsymbol{\rho}^{\boldsymbol{\alpha}}.\boldsymbol{S}^{\boldsymbol{\beta}}.\boldsymbol{V}^{\boldsymbol{\delta}}.$$

(C : is a numerical coefficient without dimension).

Exercise 2

The frequency ($\boldsymbol{\nu}$) of an elastic spring is given as a function of the mass ($\boldsymbol{m}$) of the suspended body and the stiffness constant ($\boldsymbol{k}$). Give the physical law of frequency $\boldsymbol{\nu}$.

Exercise 3

A closed container of volume (V) has (n) moles of gas, at pressure (P) and at temperature (T). The properties of this gas depend on the universal constant (R): $R \approx 8{,}315\, J\, mol^{-1} K^{-1}$.

1- Give the dimension of the pressure ($\boldsymbol{P}$), the volume ($\boldsymbol{V}$) and the ideal gas constant ($\boldsymbol{R}$).

2- In the case of an ideal gas, the quantities are related by the equation of state: $\boldsymbol{PV} = \boldsymbol{nRT}$. Check the homogeneity of this relationship.

3- We now assume that the gas is described by the van der Waals equation:

$$(\boldsymbol{P} + \boldsymbol{a}\frac{\boldsymbol{n}^2}{\boldsymbol{V}^2})(\boldsymbol{V} - \boldsymbol{nb}) = \boldsymbol{nRT}$$

Where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constants. Find the dimensions and units of $\boldsymbol{a}$ and $\boldsymbol{b}$.

Exercise 4

The drift velocity of an electron in an electric conductor is given as a function of its mass (m), its electric charge (e), the mean free time (τ) and the electric field (E).

1- Establish the dimensional equations of the following quantities: the electric charge ($\boldsymbol{q}$), the electric field ($\boldsymbol{E}$), the electric potential difference ($\boldsymbol{U}$), the electric resistance ($\boldsymbol{R}$), the electric capacitance ($\boldsymbol{C}$) and the electric power ($\boldsymbol{P}$).

2- Find the expression of the drift velocity, which takes the form: $\boldsymbol{V} = \boldsymbol{e}^{\boldsymbol{x}}\ \boldsymbol{m}^{\boldsymbol{y}}\ \boldsymbol{\tau}^{\boldsymbol{z}}\ \boldsymbol{E}^{\boldsymbol{t}}$ **.**

Solution of Tutorial n°1: Dimensional analysis

Exo 01 :

1°/ Dimensional equations:

- Velocity V : $[V] = \frac{[dx]}{[dt]} = LT^{-1}$
- Acceleration γ : $[\gamma] = \frac{[d^2x]}{[dt^2]} = \frac{[x]}{[t]^2} = LT^{-2}$
- Mass density ρ : $[\rho] = \frac{[m]}{[V]} = ML^{-3}$
- Force F : $[F] = [m][\gamma] = MLT^{-2}$
- Moment M : $[M] = [F][r] = ML^2T^{-2}$
- Work W : $[W] = [\int F.dl] = ML^2T^{-2}$
- Kinetic energy: $[E_c] = \left[\frac{1}{2}mV^2\right] = ML^2T^{-2}$
- Potential energy: $[E_p] = [mgh] = ML^2T^{-2}$
- Surface: $[S] = [4\pi R^2] = [R^2] = L^2$

2°/ Check that : $[W/mg] = [V^2/2g] = [h] = L$

g : gravitational acceleration, i.e.: $[g] = [\gamma]$

$$\rightarrow \begin{cases} \left[\frac{W}{mg}\right] = \frac{[W]}{[m][g]} = L \\ \left[\frac{V^2}{2g}\right] = \frac{[V]^2}{[2][g]} = L \\ [h] = L \end{cases}$$

So the relationship is verified

3°/ Expression of the air resistance force

Using dimensional homogeneity, it is necessary that:

$$[F] = \left[\frac{1}{2}.C.\ \rho^\alpha.S^\beta.V^\delta\right]$$

$$\Rightarrow [F] = \left[\frac{1}{2}\right][C].[\rho]^\alpha.[S]^\beta.[V]^\delta \Rightarrow [F]$$

$$= [\rho]^\alpha.[S]^\beta.[V]^\delta$$

$$\Rightarrow MLT^{-2} = M^\alpha L^{-3\alpha+2\beta+\delta} T^{-\delta}$$

$$\begin{cases} \alpha = 1 \\ -3\alpha + 2\beta + \delta = 1 \\ \delta = 2 \end{cases} \Rightarrow \begin{cases} \alpha = 1 \\ \beta = 1 \\ \delta = 2 \end{cases}$$

Expression: $\boldsymbol{F = \frac{1}{2}.C.\rho.S.V^2}$

Exo 02 :

1°/ The physical law : $v = f(m, k)$

We have: $v = c.\ m^\alpha.k^\beta$

- $[v] = T^{-1}$; $[k] = \left[\frac{F}{x}\right] = \frac{[F]}{[X]} = MT^{-2}$
- $[m] = M$; $[c] = 1$

For the equation of v to be homogeneous, it is necessary that:

$$[v] = [\ c.\ m^\alpha.k^\beta] \Rightarrow [v] = [m]^\alpha[k]^\beta$$

$$\Rightarrow \begin{cases} \alpha = -1/2 \\ \beta = +1/2 \end{cases} \Rightarrow \boldsymbol{v = c\sqrt{\frac{k}{m}}} \Rightarrow \boldsymbol{v = \frac{1}{2\pi}\sqrt{\frac{k}{m}}}$$

Exo 03 :

1°/ Dimensional equations:

- Pressure P : $[P] = \frac{[F]}{[s]} = ML^{-1}T^{-2}$
- Volume V : $[V] = [l.l.l] = L^3$
- Constant R : $[R] = \frac{[E]}{N.\theta} = ML^2T^{-2}N.\theta$

2°/ Check the homogeneity of : $PV = nRT$

$$[\boldsymbol{PV}] = [\boldsymbol{nRT}] \Rightarrow ML^2T^{-2} = ML^2T^{-2}$$

The formula is therefore homogeneous.

3°/ The dimensions and units of a and b.

In a sum or a difference, the dimensions of the quantities must be equal, so we must have:

$$\begin{cases} [P] = \left[a\frac{n^2}{V^2}\right] \Rightarrow [a] = [P].[V]^2.[n]^{-2} \\ [V] = [nb] \Rightarrow [b] = [V].[n]^{-1} \end{cases}$$

$$\Rightarrow \begin{cases} [\boldsymbol{a}] = \boldsymbol{ML^5T^{-2}N^{-2}} \\ [\boldsymbol{b}] = \boldsymbol{L^3N^{-1}} \end{cases}$$

The associated units are therefore:

- $\boldsymbol{a}$ is expressed in: $\boldsymbol{Kg.m^5.s^{-2}.mol^{-2}}$
- $\boldsymbol{b}$ is expressed in: $\boldsymbol{m^3.mol^{-1}}$

Exo 04 :

1°/ The dimensions of the following quantities:

- $dq = i.dt \Rightarrow [q] = [i][t] = IT$
- $F_e = q.E \Rightarrow [E] = \frac{[Fe]}{[q]} = MLT^{-3}I^{-1}$
- $U = E.d \Rightarrow [U] = [E][d] = ML^2T^{-3}I^{-1}$
- $U = R.i \Rightarrow [R] = \frac{[U]}{[i]} = ML^2T^{-3}I^{-2}$
- $c = \frac{q}{U} \Rightarrow [c] = \frac{[q]}{[U]} = M^{-1}L^{-2}T^4I^2$
- $P = U.i \Rightarrow [P] = [U].[i] = ML^2T^{-3}$

2°/ Expression of the drift velocity (V)

Using dimensional homogeneity, it is necessary that:

$$[V] = [e^x\ m^y\ \tau^z\ E^t] \Rightarrow [V] = [e]^x\,[m]^y\,[\tau]^z\,[E]^t$$

- $[V] = LT^{-1}$; $[\tau] = T$

$$\Rightarrow LT^{-1} = I^{x-t}.T^{x+z-3t}.M^{y+t}.\ L^t$$

$$\begin{cases} x - t = 0 \\ x + z - 3t = -1 \\ y + t = 0 \\ t = 1 \end{cases} \Rightarrow \begin{cases} x = 1 \\ z = 1 \\ y = -1 \\ t = 1 \end{cases} \Rightarrow \boldsymbol{V = \frac{e.\tau.E}{m}}$$