Chapter I: Dimensional Analysis

Summary

1. Physical quantities

Physics is the science that tries to understand, to model and to explain the natural phenomena of the world. Therefore, to describe these phenomena, it is necessary to define certain physical quantities useful for their understanding. We therefore call physical quantity any measured or calculated physical property.

Each physical quantity corresponds to a unit and all the units are grouped into a universal system which is the international system (SI).

The physical quantities can be classified into two categories, some of them can be considered as independent and thus form a subset called: **basic quantities**. The other quantities are called: **derived quantities**.

For example; Distance, Velocity, Mass, Force etc.

1.1. Basic quantities (fundamental)

The basic quantities are quantities that are independent between them. According to the International System of Units (SI), there are seven (7) basic units associated with the seven (7) basic quantities. The seven basic quantities are: length, mass, time, electric current, temperature, quantity of matter (number of moles), and light intensity. (see **Table 1**):

Table 1

Fundamental quantities	Dimension	Unit S. I.	Symbol S. I.
Length	L	$metre$	m
Mass	M	$Kilogram$	kg
Time	T	$second$	s
Current intensity	I	$Ampere$	A
Temperature	θ	$Kelvin$	K

Quantity of matter	N	$mole$	mol
Light intensity	J	$candela$	cd

1.2. Derived quantities

There are other derived units in (SI) associated with the derived quantities. These derived quantities are obtained by combining the basic quantities. Thus, all the quantities in the world can be expressed in terms of the basic dimensions.

Examples:

- **Surface (Area):** the surface being the product of two lengths, its dimension is: L^2, and its unit is square meter (m^2).
- **Velocity (Speed):** the distance traveled per unit of time $V = dx/dt$, its dimension is: $L.\,T^{-1}$, and its unit is (m/s).
- **Force:** The force applied to a mass that causes its acceleration $F = m\gamma$, its dimension is: MLT^{-2}, and its unit is Newton (N).

2. Dimensional Analysis

2.1. Definition

In engineering and science, dimensional analysis is the analysis of the relationships between different physical quantities by identifying their basic quantities.

2.2. Dimensional equations

The nature of a physical quantity is recognized by its dimension. A dimensional equation is a mathematical relation that expresses the dimension of a derived physical quantity as a function of the dimensions of the basic quantities. Let Q be a physical quantity, its dimension is denoted $[Q]$. The dimension of any physical quantity is written in the following form:

$$\mathbf{dim}\,\boldsymbol{Q} = [\boldsymbol{Q}] = \boldsymbol{L^{\alpha} M^{\beta} T^{\gamma} I^{\delta} \theta^{\varepsilon}}$$

Where α, β, γ, δ, and ε are positive or negative rational exponents.

Knowing the dimension of a quantity, we can assign it the appropriate unit.

Examples:

The dimensional equations of some derived quantities are shown in **Table 2**.

Table 2			
Derived quantities	**Formula (Law)**	**Dimensions**	**S.I Units**
Force	Mass × acceleration	$[F] = MLT^{-2}$	N
Moment of force	Force × distance	$[\mathcal{M}] = ML^2T^{-2}$	$N.m$
Work	Force × distance	$[W] = ML^2T^{-2}$	J

Power	Work / time	$[p] = ML^2T^{-3}$	$Watt$
Energy (all form)	Stored work	$[E] = ML^2T^{-2}$	J
Pressure	Force/area	$[P] = ML^{-1}T^{-2}$	$N.m^{-2}$
Coefficient of viscosity	Force × Distance/ Area × Velocity	$[\mu] = ML^{-1}T^{-1}$	$N.m^{-1}$
Angular velocity	Angle / time	$[\omega] = T^{-1}$	$Rad.s^{-1}$
Frequency	1 / time period	$[f] = T^{-1}$	s^{-1}
Period	Time of period	$[T] = T$	s
Surface (Area)	Length × Breadth	$[S] = L^2$	m^2
Volume	Length × breadth × height	$[v] = L^3$	m^3
Volumetric mass density	Mass/ volume	$[\rho] = ML^{-3}$	$Kg.m^{-3}$
Velocity or speed	Distance/ time	$[V] = LT^{-1}$	$m.s^{-1}$
Acceleration **Molar Concentration**	Velocity/time Number of moles /Volume	$[\gamma] = LT^{-2}$ $[C] = NL^{-3}$	$m.s^{-2}$ $mol.m^{-3}$
Charge	Current × time	$[q] = IT$	$A.s = c$
Electric field	Force/ charge	$[E] = MLT^{-2}I^{-1}$	$V.m^{-1}$
Potential	Electric field × distance	$[V] = ML^2T^{-2}I^{-1}$	V

2.3. Some properties on dimensions

- $[A.B] = [A].[B],\ [A/B] = [A]/[B].$
- $[C^{te}] = 1,\ [tg\ \alpha] = 1,\ [\pi] = 1.$
- $[dA] = [A],\ [\Delta A] = [A],\ [\int A] = [A].$
- $[A + B] = [C] \Rightarrow [A] = [B] = [C]$

2.4. Classification of Physical Quantity

Physical quantity has been classified into following four -04-categories on the basis of dimensional analysis.

a) **Dimensional Constant:** These are the physical quantities which possess dimensions and have constant (fixed) value. **e.g.** Planck’s constant, gas constant, gravitational constant etc.

b) **Dimensional Variable:** These are the physical quantities which possess dimensions and do not have fixed value. **e.g.** velocity, acceleration, force etc.

c) **Dimensionless Constant:** These are the physical quantities which do not possess dimensions but have constant (fixed) value. **e.g.** e, π , numbers like 1,2,3,4,5 etc.

d) **Dimensionless Variable:** These are the physical quantities which do not possess dimensions and have variable value. **e.g.** angle, strain etc.

3. Homogeneity of an equation

The equations or formulas must be homogeneous, that is to say both sides of an equation must have the same physical dimension. Checking the homogeneity of an equation allows us to define a physical law which describes any phenomenon.

Using the homogeneity of an equation has the following uses:

- Eliminate calculation errors,
- Avoid nonsense,
- Allows us to define a physical law which describes any phenomenon.
- Checking the correctness of physical equation.

4. Applications

Exercise-1 :

The volumetric mass density $\boldsymbol{\rho}$ of a cylinder of mass $\boldsymbol{m}$, radius $\boldsymbol{R}$ and length $\boldsymbol{l}$ is given by the following relation:

$$\rho = \frac{m^x}{\pi l^y R^2}$$

1- Using the homogeneity of an equation, find the two constants $\boldsymbol{x}$ and $\boldsymbol{y}$

2- Deduce the exact expression of the volumetric mass density $\boldsymbol{\rho}$.

Solution-1:

The volumetric mass density $\boldsymbol{\rho}$ is by definition the ratio between the mass and the volume, that is to say it admits as dimension:

$$[\rho] = ML^{-3} with\ [\rho] = \frac{[m]^x}{[l]^y[R]^2} = M^x L^{-y-2}$$

1- Where $\begin{cases} x = 1 \\ -y - 2 = -3 \end{cases}$, thus $\begin{cases} x = 1 \\ y = 1 \end{cases}$

2- Expression: $\rho = \dfrac{m}{\pi l R^2}$

Exercise-2 :

The time period **(T)** of oscillation of a simple pendulum of length $\boldsymbol{l}$ and negligible mass $\boldsymbol{m}$ is given by the following relation:

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Check the correctness of the following formulae using the dimensional analysis.

Solution-2:

$$[T] = T\ ; \quad [2\pi] = 1; \quad [l] = L; \quad [g] = \left[\frac{F}{m}\right] = [\gamma] = LT^{-2}$$

$$[T] = [2\pi]\left[\frac{l}{g}\right]^{1/2} \Rightarrow [T] = [2\pi]\frac{[l]^{1/2}}{[g]^{1/2}} \Rightarrow T = \frac{L^{1/2}}{L^{1/2}T^{-1}}\ T$$

Thus, the dimensions of the terms on both sides of the relation are the same. Therefore, the relation is correct.

Exercise-3 :

What are the dimensions of the following constants:

a) h, Planck's constant, where $E = h\upsilon$,

b) εo , Vacuum permittivity, where $K = 1/4\pi\varepsilon o$ and $Fe = Kqq'/r^2$,

c) G, the universal gravitational constant, where $F = Gm_1m_2/d^2$

Solution-3:

The dimensions of constants: $h, \varepsilon_o and\ G$

- $E = h\upsilon \Rightarrow [h] = \frac{[E]}{[\upsilon]} = ML^2T^{-1}$
- $Fe = \frac{qq'}{4\pi\varepsilon_o\, r^2} \Rightarrow \varepsilon_o = \frac{qq'}{4\pi\, r^2\, F_e} \Rightarrow [\varepsilon_o] = \frac{[q]^2}{[4\pi]\,[r]^2\,[F_e]}$ and knowing that:

$$\begin{cases} i = \frac{dq}{dt} \Rightarrow [q] = [i][t] = IT \\ [F] = MLT^{-2} \end{cases}$$

$\Rightarrow [\varepsilon_o] = M^{-1}L^{-3}T^4I^2$

- $F = G\,\frac{m_1m_2}{d^2} \Rightarrow G = \frac{F\,d^2}{m_1m_2} \Rightarrow [G] = M^{-1}L^3T^{-2}$