Exo.1

1) ⓐ $\nabla f_1 = \begin{pmatrix} 2x \\ 3y^2 \\ 4z^2 \end{pmatrix}$ et $\nabla f_1 = 0 \Leftrightarrow x = 0$ et $y = 0$ et $z = 0$

Le point critique est l'origine $O(0,0,0)$

ⓑ $\nabla f_2 = \begin{pmatrix} 2xy^3z^4 \\ 3x^2y^2z^4 \\ 4x^2y^3z^3 \end{pmatrix}$ et $\nabla f_2 = 0 \Leftrightarrow \begin{cases} x=0 \vee y=0 \vee z=0 \\ \wedge \\ x=0 \vee y=0 \vee z=0 \\ \wedge \\ x=0 \vee y=0 \vee z=0 \end{cases}$

Les points critiques sont les trois plans $x=0$ et $y=0$ et $z=0$

ⓒ $\nabla f_3 = \begin{pmatrix} \sin y \ln z \\ \cos y \ln z \\ \sin y / z \end{pmatrix} e^x$ et $\nabla f_3 = 0 \Leftrightarrow \begin{cases} y = k\pi \vee z = 1 \\ \wedge \\ y = \frac{\pi}{2} + k\pi \vee z = 1 \\ \wedge \\ y = k\pi \end{cases}$

$\Leftrightarrow \quad y = k\pi \wedge z = 1.$

2) $\nabla f = \begin{pmatrix} 3x^2+4x+1 \\ -2y \end{pmatrix}$ et $\nabla f = 0 \Leftrightarrow (x,y) = (-1,0) \vee (-1/3, 0).$

3) ⓐ $\mathbb{R}^2 \xrightarrow[\nabla f]{f} \mathbb{R} \xrightarrow[g']{g} \mathbb{R} \Rightarrow g \circ f : \mathbb{R}^2 \xrightarrow[\nabla g \circ f]{} \mathbb{R}$

* $g \circ f(x,y) = g(e^x \cos y) = 4e^x + 4\cos y + 1 \Rightarrow \nabla g \circ f = \begin{pmatrix} 4e^x \\ -4\sin y \end{pmatrix}$
* $g \circ f = g'[f(x)]\, \nabla f(x,y) = 4 \begin{pmatrix} e^x \\ -\sin y \end{pmatrix}.$

ⓑ $\mathbb{R} \xrightarrow[f']{f} \mathbb{R}^2 \xrightarrow[\nabla g]{g} \mathbb{R}$, $g \circ f : \mathbb{R} \xrightarrow[(g \circ f)']{} \mathbb{R}$

* $g \circ f(x) = g(e^x, \cos x) = 4e^x + 2\cos y \Rightarrow (g \circ f)' = 4e^x - 2\sin y$

$(g \circ f)'(x) = \langle \nabla g[f(x)], f'(x) \rangle = (e^x, -\sin y) \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$

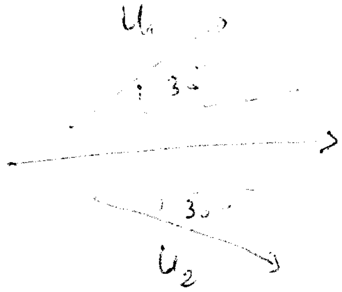
Exo2 :

① $\nabla(f\cdot g) = \begin{pmatrix} \frac{\partial}{\partial x_1} f\cdot g \\ \vdots \\ \frac{\partial}{\partial x_n} f\cdot g \end{pmatrix} = \begin{pmatrix} g\frac{\partial}{\partial x_1} f + f\frac{\partial}{\partial x_1} g \\ \vdots \\ g\frac{\partial}{\partial x_n} f + f\frac{\partial}{\partial x_n} g \end{pmatrix} = g\begin{pmatrix} \frac{\partial}{\partial x_1} f \\ \vdots \\ \frac{\partial}{\partial x_n} f \end{pmatrix} + f\begin{pmatrix} \frac{\partial}{\partial x_1} g \\ \vdots \\ \frac{\partial}{\partial x_n} g \end{pmatrix}$

② la même chose.

③ il a été résolu dans le cour

Exo3



① $u = (r\cos 30°, r\sin 30°) = r\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$\|u\| = 1 \Leftrightarrow r = 1$

$d_u f(1,2) = \frac{\sqrt{3}}{2}\frac{\partial f}{\partial x}(1,2) + \frac{1}{2}\frac{\partial f}{\partial y}(1,2) = 2e^4(\sqrt{3}+1)$.

② le gradient de T au Point (1,3) est $\nabla T(1,3) = (-3,4)$

ⓐ la direction pour T augment le plus rapidement est

$\frac{\nabla T(1,3)}{\|\nabla T(1,3)\|} = \begin{pmatrix} -3/5 \\ 4/5 \end{pmatrix}$ et le taux $\|\nabla T(1,3)\| = 5$

ⓑ la direction pour T diminue le plus rapidement est

$-\frac{\nabla T(1,3)}{\|\nabla T(1,3)\|} = \begin{pmatrix} 3/5 \\ -4/5 \end{pmatrix}$ et le Taux $-\|\nabla T(1,3)\| = -5$

Exo4

a) On a $f(x,y)$ deux fois differentiable sur $\mathbb{R}^2$ alors ($f\in C^2(\mathbb{R}^2)$)

$f(x,y) = f(0,0) + x\frac{\partial}{\partial x}f(0,0) + y\frac{\partial}{\partial y}f(0,0) + \frac{x^2}{2}\frac{\partial^2}{\partial x^2}f(0,0) + \frac{y^2}{2}\frac{\partial^2}{\partial y^2}f(0,0) + \frac{xy}{2}\frac{\partial^2 f}{\partial x\partial y}(0,0)$
$+ (x^2+y^2)\,\varepsilon(x,y)$

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$$f(x,y) = -1 + \frac{x^2}{2} + \frac{y^2}{2} + (x^2+y^2)\,\varepsilon(x,y) \quad tq \quad \varepsilon(x,y) \xrightarrow[(x,y)\to(0,0)]{} 0$$

(b) $$f(x+\pi/2, y+\pi/2) = -xy + (x^2+y^2)\,\varepsilon(x,y) \quad tq \quad \varepsilon(x,y) \xrightarrow[(x,y)\to(0,0)]{} 0$$

(c) on a $f \in C^2(\mathbb{R}^2)$ Alors elle est developpable en serie de Taylor à l'ordre 2 et on a

$$f(x,y) = e\left(1 + x + \frac{x^2}{2} + \frac{y^2}{2}\right) + (x^2+y^2)\,\varepsilon(x,y) \quad tq \quad \varepsilon(x,y) \to (0,0)$$

(d) $f \in C^2$ au voisinage de $(0,0)$ alors

$$f(x,y) = \frac{1}{2} - \frac{1}{8}xy + \frac{1}{8}y^2 + (x^2+y^2)\,\varepsilon(x,y) \quad tq \quad \varepsilon(x,y) \to (0,0)$$

En utilisant que $(f\circ g\circ h)' = (g\circ h)'\, f'(g\circ h)$

$$= h'\, g'(h)\, f'(g\circ h).$$

Exercice 05 :

(1) $\|d\| = 1$ alors

$$d_d f(0,0) = \lim_{t\to 0} \frac{f\left(\frac{\sqrt{2}}{2}t, -\frac{\sqrt{2}}{2}t\right) - f(0,0)}{t} = 0 = d^T \nabla f(0,0)$$

$$= \left(\frac{\sqrt{2}}{2} \;\; -\frac{\sqrt{2}}{2}\right)\begin{pmatrix}1\\1\end{pmatrix}$$

(2) $$v = \frac{d}{\|d\|} = \left(3/5, -\frac{4}{5}\right)^T$$

$$d_v f(1,-2) = \lim_{t\to 0} \frac{f\left(1 + \frac{3t}{5}, -2 - \frac{4t}{5}\right) - f(1,-2)}{t} = \frac{19}{5}$$

$$= \left(\frac{3}{5} \;\; -\frac{4}{5}\right)\begin{pmatrix}1\\-4\end{pmatrix}$$

(3) $$v = \frac{d}{\|d\|} = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)^T$$

$$\frac{d}{v}f(0,0)=\lim_{t\to 0}\frac{f\left(-\frac{t}{\sqrt2},\frac{t}{\sqrt2}\right)-f(0,0)}{t}=\lim_{t\to 0}\frac{e^{-t/2}\cos\frac{t}{\sqrt2}-1}{t}$$

$$=\left(e^{-t/\sqrt2}\cos\frac{t}{\sqrt2}\right)'\Big|_{t=0}=\frac{-1}{\sqrt2}=\begin{pmatrix}\frac{-1}{\sqrt2} & \frac{1}{\sqrt2}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt2}\\ 0\end{pmatrix}=-\frac{1}{\sqrt2}.$$

Exo6

① ⓐ Soit $d\in\mathbb{R}^n$ et $x=(x_1,\dots,x_n)^T\in\mathbb{R}^n$

$$\lim_{t\to 0^+}\frac{f(x+td)-f(x)}{t}=\lim_{t\to 0}\frac{cte-cte}{t}=0=\langle\nabla f_1(x),d\rangle=\langle 0,d\rangle$$

$$\Leftrightarrow\ \nabla f_1(x)=0$$

ⓑ On a $\frac{\partial f}{\partial x_i}(x)=\frac{\partial}{\partial x_i}cte=0,\ \forall i=\overline{1,n}\Rightarrow\nabla f(x)=\begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}=0.$

ⓒ $\frac{\partial}{\partial x_j}\left[\frac{\partial f}{\partial x_i}(x)\right]=\frac{\partial}{\partial x_j}0=0,\ \forall i,j=\overline{1,n}\Rightarrow\nabla^2 f(x)=\begin{pmatrix}0&\cdots&0\\ \vdots&\ddots&\vdots\\ 0&\cdots&0\end{pmatrix}=0$

② ⓐ $$\lim_{t\to 0}\frac{f_2(x+td)-f_2(x)}{t}=\lim_{t\to 0}\frac{\langle a,x+td\rangle+b-\langle a,x\rangle-b}{t}$$

$$=\lim_{t\to 0}\frac{\langle a,x\rangle+t\langle a,d\rangle-\langle a,x\rangle}{t}=\langle a,d\rangle=\langle\nabla f_2(x),d\rangle\Leftrightarrow\nabla f_2(x)=a$$

ⓑ $$\frac{\partial}{\partial x_i}f_2(x)=\frac{\partial}{\partial x_i}\left[\langle a,x\rangle+b\right]=\frac{\partial}{\partial x_i}\left[b+\sum_{j=1}^n a_jx_j\right]=a_i,\ \forall i=\overline{1,n}\ \cdots\ *$$

$$\nabla f_2(x)=\begin{pmatrix}a_1\\ \vdots\\ a_n\end{pmatrix}=a$$

ⓒ de $*$ on a

$$\frac{\partial}{\partial x_k}\left[\frac{\partial f_2}{\partial x_j}(x)\right]=\frac{\partial}{\partial x_k}a_j=0,\ \forall j,k=\overline{1,n}\Leftrightarrow\nabla^2 f_2(x)=0.$$

ⓐ $\frac{d}{d_h} f_3(x) = \lim_{t\to 0} \frac{f(x+th)-f(x)}{t} = \lim_{t\to 0} a\langle b, h\rangle$ ~~$+ t a\langle h, h\rangle$~~

$= \langle ab, h\rangle = \langle \nabla f_3(x), h\rangle \Leftrightarrow \nabla f_3(x) = ab$

ⓑ $\frac{\partial}{\partial x_i} f_3(x) = \frac{\partial}{\partial x_i}\left[a\langle b, x\rangle + c\right] = \frac{\partial}{\partial x_i}\left(c + a\sum_{j=1}^{n} b_j x_j\right) = a b_i$

$\Rightarrow \nabla f_3(x) = a\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = ab$

ⓒ $\frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_i} f_3(x)\right] = \frac{\partial}{\partial x_k}(a b_j) = 0 \quad , \forall\, i, k = \overline{1,n} \Rightarrow \nabla^2 f_3(x) = 0$

④ ⓐ $\lim_{t\to 0} \frac{f(x+th) - f(x)}{t} = \lim_{t\to 0} \frac{a\langle x+th, x+th\rangle - a\langle x, x\rangle - b}{t}$

$= \lim_{t\to 0} 2a\langle x, h\rangle - ta\langle h, h\rangle = \langle 2ax, h\rangle = \langle \nabla f_4(x), h\rangle$

ⓑ $\frac{\partial}{\partial x_i}\left[f_4(x)\right] = \frac{\partial}{\partial x_i}\left[a\langle x, x\rangle + b\right] = \frac{\partial}{\partial x_i}\left(b + a\sum_{j=1}^{n} x_j^2\right)$

$= 2a x_i \quad \forall i = \overline{1,n} \Rightarrow \nabla f_4(x) = \begin{pmatrix} 2ax_1 \\ \vdots \\ 2ax_n \end{pmatrix} = 2a\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = 2ax$

ⓒ $\frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_i} f_4(x)\right] = \frac{\partial}{\partial x_k} 2a\left[x_i\right] \qquad i, k = \overline{1,n}$

$= \begin{cases} 2a & i = k \\ 0 & i \neq k \end{cases} \Rightarrow \nabla^2 f_4(x) = \begin{pmatrix} 2a & 0 & \cdots & 0 \\ 0 & 2a & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 2a \end{pmatrix}$

$= 2a I_n = 2a$

ⓓ $\nabla^2 f_4(x) = \nabla\left[\nabla^T f_4(x)\right] = \nabla(2ax)^T = 2a\nabla x^T = 2a I_n$

$\nabla x^T = \begin{pmatrix} \frac{\partial x_1}{\partial x_1} & \frac{\partial x_2}{\partial x_1} & \cdots & \frac{\partial x_n}{\partial x_1} \\ \frac{\partial x_1}{\partial x_2} & \frac{\partial x_2}{\partial x_2} & & \frac{\partial x_n}{\partial x_2} \\ \vdots & & & \vdots \\ \frac{\partial x_1}{\partial x_n} & \frac{\partial x_2}{\partial x_n} & & \frac{\partial x_n}{\partial x_n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \end{pmatrix} = I_n$

⑤ ⓐ $\lim\limits_{t\to 0} \frac{f_S(x+te_i)-f(x)}{t} = \lim\limits_{t\to 0} \frac{\sum_{i=1}^{m} f_i(x+te_i) - \sum_{i=1}^{m} f_i(x)}{t}$

$$= \lim_{t\to 0} \sum_{i=1}^{m} \frac{f_i(x+te_i)-f_i(x)}{t} = \sum_{i=1}^{m} \lim_{t\to 0} \frac{f_i(x+te_i)-f_i(x)}{t}$$

$$= \sum_{i=1}^{m} \langle \nabla f_i(x), e_i\rangle = \left\langle \sum_{i=1}^{m} \nabla f_i(x), e_i \right\rangle = \langle \nabla f_S(x), e_i\rangle$$

ⓑ $\frac{\partial}{\partial x_j} f_S(x) = \frac{\partial}{\partial x_j} \sum_{i=1}^{m} f_i(x) = \sum_{i=1}^{m} \frac{\partial}{\partial x_j} f_i(x), \ \forall\, j=\overline{1,n}$ ... *

$$\iff \nabla f_S(x) = \sum_{i=1}^{m} \nabla f_i(x)$$

ⓒ de * avem: $\frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_j} f_S(x)\right] = \frac{\partial}{\partial x_k} \sum_{i=1}^{m} \frac{\partial}{\partial x_j} f_i(x) = \sum_{i=1}^{m} \frac{\partial^2}{\partial x_k \partial x_j} f_i(x)$

$$\Rightarrow \nabla^2 f_S(x) = \sum_{i=1}^{m} \nabla^2 f_i(x).$$

⑥ ⓐ $\lim\limits_{t\to 0} \frac{f_C(x+te_i)-f_C(x)}{t} = \lim\limits_{t\to 0} \frac{\sum_{i=1}^{m} f_i^2(x+te_i) - \sum f_i^2(x)}{t}$

$$= \sum_{i=1}^{m} \lim_{t\to 0} \frac{f_i^2(x+te_i)-f_i^2(x)}{t} = \sum_{i=1}^{m} \lim_{t\to 0} \left(\frac{f_i(x+te_i)-f_i(x)}{t}\right)\left(f_i(x+te_i)+f_i(x)\right)$$

$a^2-b^2=(a-b)(a+b)$

$$= \sum_{i=1}^{m} \langle \nabla f_i(x)\cdot 2f_i(x), e_i\rangle = \left\langle 2\sum_{i=1}^{m} f_i(x)\nabla f_i(x), e_i\right\rangle = \langle \nabla f_C(x), e_i\rangle$$

ⓑ $\frac{\partial}{\partial x_j} f_C(x) = \frac{\partial}{\partial x_j} \sum_{i=1}^{m} f_i^2(x) = \sum_{i=1}^{m} \frac{\partial}{\partial x_j} f_i^2(x) = \sum_{i=1}^{m} 2f_i(x) \frac{\partial}{\partial x_j} f_i(x)$, $\forall\, j=\overline{1,n}$

$$\nabla f_C(x) = 2\sum_{i=1}^{m} f_i(x)\nabla f_i(x).$$

ⓒ $\frac{\partial}{\partial x_k}\left[\frac{\partial}{\partial x_j} f_C(x)\right] = \frac{\partial}{\partial x_k}\left[2\sum_{i=1}^{m} f_i(x)\frac{\partial}{\partial x_j} f_i(x)\right]$

$$= 2\sum_{i=1}^{m}\left[\frac{\partial}{\partial x_k} f_i(x)\frac{\partial}{\partial x_j} f_i(x) + f_i(x)\frac{\partial^2}{\partial x_k \partial x_j} f_i(x)\right] \quad \forall\, k,j=\overline{1,n}$$

$$\nabla^2 f_C(x) = 2\sum_{i=1}^{m}\left[\nabla f_i(x)\nabla^T f_i(x) + f_i(x)\nabla^2 f_i(x)\right].$$

(k = exterior ⟷
j = interior ↕)

(d) En utilisant $\nabla g^2 = \nabla g \cdot g = 2g\nabla g$ tel que $g = \sum_{i=1}^{m} r_i(x)$.

$\nabla g^2 = \nabla \sum_{i=1}^{m} r_i^2(x) = 2g\nabla g = 2\sum_{i=1}^{m} r_i(x)\nabla r_i(x)$.

(e) $\nabla^2 g = \nabla(\nabla^T g) \Leftrightarrow \nabla^2 \sum_{i=1}^{m} r_i^2(x) = 2\nabla\left[\nabla^T \sum_{i=1}^{m} r_i^2(x)\right]$

$= 2\nabla\left[\sum_{i=1}^{m} r_i(x)\nabla^T r_i(x)\right] \overset{\nabla f \cdot g}{=} 2\left[\sum_{i=1}^{m} \nabla\left(r_i(x)\nabla^T r_i(x)\right)\right]$

$= 2\sum_{i=1}^{m}\left[\nabla r_i(x)\nabla^T r_i(x) + r_i(x)\nabla^2 r_i(x)\right]$.

Exo 4 $f \in C^1(\mathbb{R}^n)$ donc elle est developpable en serie de Taylor avec reste integrale à l'ordre "1" d'où

$f(x+h) = f(x) + \int_0^1 \langle \nabla f(x+th), h\rangle \, dt$.

$f(x+h) - f(x) - \langle \nabla f(x), h\rangle = \int_0^1 \langle \nabla f(x+th) - \nabla f(x), h\rangle \, dt$

$\left| f(x+h) - f(x) - \langle \nabla f(x), h\rangle \right| = \left| \int_0^1 \langle \nabla f(x+th) - \nabla f(x), h\rangle \, dt \right|$

$\leqslant \int_0^1 \left| \langle \nabla f(x+th) - \nabla f(x), h\rangle \right| dt$

$\overset{C.S}{\leqslant} \int_0^1 \|\nabla f(x+th) - \nabla f(x)\| \, \|h\| \, dt$

$\leqslant \int_0^1 L \|x+th-x\| \, \|h\| \, dt$

$= \int_0^1 L\, t \|h\|^2 dt = L\|h\|^2 \int_0^1 t \, dt$

$= \frac{L}{2}\|h\|^2$.