

FORMULAIRE DE MÉCANIQUE DES FLUIDES

Opérateurs classiques

en coordonnées cartésiennes

$$\overrightarrow{\text{grad}} f(x, y, z) = \vec{\nabla} f(x, y, z) = \frac{\partial f}{\partial x} \vec{e}_x + \frac{\partial f}{\partial y} \vec{e}_y + \frac{\partial f}{\partial z} \vec{e}_z$$

$$\text{div } \vec{A} = \vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\text{rot } \vec{A} = \vec{\nabla} \wedge \vec{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \vec{e}_x + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \vec{e}_y + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \vec{e}_z$$

$$\Delta f = \vec{\nabla} \cdot (\vec{\nabla} f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\Delta \vec{A} = \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{A}) = \Delta A_x \vec{e}_x + \Delta A_y \vec{e}_y + \Delta A_z \vec{e}_z$$

en coordonnées cylindriques

$$\vec{\nabla} f(r, \theta, z) = \frac{\partial f}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \vec{e}_\theta + \frac{\partial f}{\partial z} \vec{e}_z$$

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r} \frac{\partial}{\partial r} (r A_r) + \frac{1}{r} \frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\vec{\nabla} \wedge \vec{A} = \frac{1}{r} \left(\frac{\partial A_z}{\partial \theta} - \frac{\partial}{\partial z} (r A_\theta) \right) \vec{e}_r + \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) \vec{e}_\theta + \frac{1}{r} \left(\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right) \vec{e}_z$$

$$\Delta f = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial f}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\Delta \vec{A} = \left(\Delta A_r - \frac{1}{r^2} A_r - \frac{2}{r^2} \frac{\partial A_\theta}{\partial \theta} \right) \vec{e}_r + \left(\Delta A_\theta - \frac{1}{r^2} A_\theta + \frac{2}{r^2} \frac{\partial A_r}{\partial \theta} \right) \vec{e}_\theta + \Delta A_z \vec{e}_z$$

en coordonnées sphériques

$$\vec{\nabla} f(r, \theta, \varphi) = \frac{\partial f}{\partial r} \vec{e}_r + \frac{1}{r} \frac{\partial f}{\partial \theta} \vec{e}_\theta + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi} \vec{e}_\varphi$$

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{r} \left(\frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{\sin \theta} \frac{\partial A_\varphi}{\partial \varphi} \right)$$

$$\vec{\nabla} \wedge \vec{A} = \frac{1}{r \sin \theta} \left(\frac{\partial}{\partial \theta} (A_\varphi \sin \theta) - \frac{\partial A_\theta}{\partial \varphi} \right) \vec{e}_r + \frac{1}{r} \left(\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r} (r A_\varphi) \right) \vec{e}_\theta + \frac{1}{r} \left(\frac{\partial}{\partial r} (r A_\theta) - \frac{\partial A_r}{\partial \theta} \right) \vec{e}_\varphi$$

$$\Delta f = \frac{1}{r^2} \left(\frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2} \right)$$

$$\begin{aligned} \Delta \vec{A} = & \left(\Delta A_r - \frac{2}{r^2} A_r - \frac{2}{r^2 \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) - \frac{2}{r^2 \sin^2 \theta} \frac{\partial A_\varphi}{\partial \varphi} \right) \vec{e}_r \\ & + \left(\Delta A_\theta + \frac{2}{r^2} \frac{\partial A_r}{\partial \theta} - \frac{1}{r^2 \sin^2 \theta} A_\theta - \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial A_\varphi}{\partial \varphi} \right) \vec{e}_\theta \\ & + \left(\Delta A_\varphi + \frac{2}{r^2 \sin \theta} \frac{\partial A_r}{\partial \varphi} + \frac{2 \cos \theta}{r^2 \sin^2 \theta} \frac{\partial A_\theta}{\partial \varphi} - \frac{1}{r^2 \sin^2 \theta} A_\varphi \right) \vec{e}_\varphi \end{aligned}$$

propriétés générales

$$\vec{\nabla} \cdot (f \vec{A}) = \vec{A} \cdot \vec{\nabla} f + f \vec{\nabla} \cdot \vec{A}$$

$$\vec{\nabla} \wedge (f \vec{A}) = f \vec{\nabla} \wedge \vec{A} + \vec{\nabla} f \wedge \vec{A}$$

$$\vec{\nabla} \cdot (\vec{\nabla} \wedge \vec{A}) = 0$$

$$\vec{\nabla} \wedge (\vec{\nabla} f) = \vec{0}$$

$$\vec{A} \wedge (\vec{B} \wedge \vec{C}) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

Transformation d'intégrales multiples

Formule d'Ostrogradski

$$\iiint_{(\tau)} \vec{\nabla} \cdot \vec{A} \cdot d\tau = \iint_{(\sigma)} \vec{A} \cdot \vec{n} d\sigma$$

(σ) est une surface fermée, frontière d'un domaine (σ) ;

$\vec{n}$ est le vecteur unitaire normal à (σ) dirigé vers l'extérieur.

Conséquences :

$$\text{Formule du gradient} \quad \iiint_{(\tau)} \vec{\nabla} f d\tau = \iint_{(\sigma)} f \vec{n} d\sigma$$

$$\text{Formule du rotationnel} \quad \iiint_{(\tau)} \vec{\nabla} \wedge \vec{A} d\tau = \iint_{(\sigma)} \vec{n} \wedge \vec{A} d\sigma$$

$$\iint_{(\sigma)} \vec{n} \wedge \vec{\nabla} f d\sigma = \int_{(L)} f d\vec{r}$$

Formule de Stokes-Ampère

$$\iint_{(\sigma)} (\vec{\nabla} \wedge \vec{A}) \cdot \vec{n} d\sigma = \int_{(L)} \vec{A} \cdot d\vec{r}$$

La circulation d'un vecteur $\vec{A}$ le long d'un contour fermé (L) est égale au flux de $\vec{\nabla} \wedge \vec{A}$ à travers une surface (σ) admettant (L) comme frontière.

Fonction de courant et potentiel des vitesses

$$v_x = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \quad v_r = \frac{\partial \varphi}{\partial r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta}$$

$$v_y = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x} \quad v_\theta = \frac{1}{r} \frac{\partial \varphi}{\partial \theta} = -\frac{\partial \psi}{\partial r}$$

Potentiel complexe des vitesses :

$$f(z) = \varphi + i\psi \quad \text{avec} \quad z = x + iy \quad \text{ou} \quad z = r e^{i\theta}$$

Tenseur des contraintes pour un fluide newtonien incompressible

en coordonnées cartésiennes

$$\begin{aligned}\sigma_{xx} &= -p + 2\mu \frac{\partial v_x}{\partial x} & \tau_{xy} &= \tau_{yx} = \mu \left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x} \right) \\ \sigma_{yy} &= -p + 2\mu \frac{\partial v_y}{\partial y} & \tau_{yz} &= \tau_{zy} = \mu \left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y} \right) \\ \sigma_{zz} &= -p + 2\mu \frac{\partial v_z}{\partial z} & \tau_{xz} &= \tau_{zx} = \mu \left(\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right)\end{aligned}$$

en coordonnées cylindriques

$$\begin{aligned}\sigma_{rr} &= -p + 2\mu \frac{\partial v_r}{\partial r} & \tau_{r\theta} &= \tau_{\theta r} = \mu \left(r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right) \\ \sigma_{\theta\theta} &= -p + 2\mu \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) & \tau_{\theta z} &= \tau_{z\theta} = \mu \left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r} \frac{\partial v_z}{\partial \theta} \right) \\ \sigma_{zz} &= -p + 2\mu \frac{\partial v_z}{\partial z} & \tau_{zr} &= \tau_{rz} = \mu \left(\frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right)\end{aligned}$$

Equation de Navier-Stokes pour un fluide newtonien incompressible

en coordonnées cartésiennes

$$\begin{aligned}\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) &= -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) \\ \rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) &= -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right) \\ \rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right) - \rho g\end{aligned}$$

en coordonnées cylindriques

$$\begin{aligned}\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) &= -\frac{\partial p}{\partial r} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_r}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{v_r}{r^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right) \\ \rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) &= -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} - \frac{v_\theta}{r^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right) \\ \rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) &= -\frac{\partial p}{\partial z} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) - \rho g\end{aligned}$$