Course 04 : Surface integrals, Stoks and Ostrogradsky theorems

1) Surface integral :

- Surface integral of a scalar function $f : \mathbb{R}^3 \to \mathbb{R}$

The surface integral of f along the surface (S) is defined by the relation :

$$\iint_S f ds = \iint_D f(\vec{r})|\overrightarrow{r_u} \times \overrightarrow{r_v}| du dv$$

With :

ds : surface element.

$\vec{r}$: $Parametrization\ of\ the\ surface\ (S)$. $\begin{cases} x = x(u,v) \\ y = y(u,v) \\ z = z(u,v) \end{cases}$, $S = \vec{r}(D)$.

$\overrightarrow{r_u} = \frac{\partial \vec{r}}{\partial u}$, $\overrightarrow{r_v} = \frac{\partial \vec{r}}{\partial v}$

Examples :

1) $\iint_S (x^2 + zy)\, ds$? $\quad (S) : \begin{cases} x^2 + y^2 = 4 \\ 0 \le z \le 2 \end{cases}$

$\vec{r} : \begin{cases} x = 2cos(\theta) \\ y = 2sin(\theta) \\ z = z \end{cases}$, $\begin{cases} 0 \le \theta \le 2\pi \\ 0 \le z \le 2 \end{cases}$ $\quad (u = \theta\ ,\ v = z)$

$$\overrightarrow{r_\theta} \times \overrightarrow{r_z} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2sin(\theta) & 2cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2cos(\theta)\vec{i} + 2sin(\theta)\vec{j}$$

$$|\overrightarrow{r_\theta} \times \overrightarrow{r_z}| = 2$$

$$\iint_S (x^2 + zy)\, ds = \iint_D f(\vec{r})|\overrightarrow{r_\theta} \times \overrightarrow{r_z}| d\theta dz = \int_0^{2\pi}\int_0^2 (4cos^2(\theta) + 2zsin(\theta))\, 2 d\theta dz$$

$$= 2\int_0^{2\pi} [4zcos^2(\theta) + z^2 sin(\theta)]_0^2 d\theta = 2\int_0^{2\pi} (8cos^2(\theta) + 4sin(\theta)) d\theta$$

$$= 2\int_0^{2\pi} (4 + 4cos(2\theta) + 4sin(\theta)) d\theta = 8\left[\theta + \frac{1}{2}sin(2\theta) - cos(\theta)\right]_0^{2\pi}$$

$$= 16\pi.$$

2) $\iint_S xy\,ds$ $\quad (S): x^2+y-z=0\ .\quad 0\le x\le 1\,,\quad 0\le y\le 3.$

$$\vec{r}=\begin{pmatrix} x \\ y \\ x^2+y \end{pmatrix}\quad,\quad \begin{cases} 0\le x\le 1 \\ 0\le y\le 3 \end{cases},\ \overrightarrow{r_x}\times\overrightarrow{r_y}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2x \\ 0 & 1 & 1 \end{vmatrix}=-2x\vec{i}-\vec{j}+\vec{k}\,.$$

$$\left|\overrightarrow{r_x}\times\overrightarrow{r_y}\right|=\sqrt{4x^2+2}$$

$$\iint_S xy\,ds=\iint_D f(\vec{r})\left|\overrightarrow{r_x}\times\overrightarrow{r_y}\right|dxdy=\int_0^1\int_0^3 xy\sqrt{4x^2+2}\,dxdy.$$

$$=\left(\int_0^1 x\sqrt{4x^2+2}\,dx\right)\left(\int_0^3 ydy\right)=\left(\frac{1}{12}(4x^2+2)^{\frac{3}{2}}\right)_0^1\left(\frac{1}{2}y^2\right)_0^3=\frac{9}{12}\left(3\sqrt{6}-\sqrt{2}\right).$$

- Surface integral of vector field (Flux) $\quad F:\ \mathbb{R}^3\to\mathbb{R}^3$

The surface integral of F along the oriented surface (S) is given by the relation :

$$\iint_S F dS=\iint_S F.\vec{n}ds=\iint_D F(\vec{r}).(\overrightarrow{r_u}\times\overrightarrow{r_v})dudv$$

With :

$ds:$ scalar surface-area element.

$\vec{n}:$ Unit normal vector to the surface (S) .

$\vec{r}$: Parametrization of the surface (S). $\begin{cases} x=x(u,v) \\ y=y(u,v) \\ z=z(u,v) \end{cases}$, $S=\vec{r}(D)$.

$$\overrightarrow{r_u}=\frac{\partial\vec{r}}{\partial u}\,,\quad \overrightarrow{r_v}=\frac{\partial\vec{r}}{\partial v}$$

Examples :

1) $\iint_S F\,ds$? $\quad F(x,y,z)=(z,y^2,xz)\quad (S):\begin{cases} x^2+y^2=4 \\ 0\le z\le 2 \end{cases}$

$$\vec{r}:\begin{cases} x=2cos(\theta) \\ y=2sin(\theta) \\ z=z \end{cases}\quad,\quad \begin{cases} 0\le\theta\le 2\pi \\ 0\le z\le 2 \end{cases}$$

$$\overrightarrow{r_\theta}\times\overrightarrow{r_z}=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2sin(\theta) & 2cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix}=2cos(\theta)\vec{i}+2sin(\theta)\vec{j}$$

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$$F.(\overrightarrow{r_\theta} \times \overrightarrow{r_z}) = 2zcos(\theta) + 8sin^3(\theta)$$

$$\iint_S F\,dS = \iint_D F(\vec{r}).(\overrightarrow{r_\theta} \times \overrightarrow{r_z})d\theta dz = \int_0^{2\pi}\int_0^{2}(2zcos(\theta) + 8sin^3(\theta))d\theta dz$$

$$\iint_S F\,dS = \int_0^{2\pi}(z^2cos(\theta) + 8zsin^3(\theta))_0^2 d\theta = \int_0^{2\pi}(4cos(\theta) + 16sin^3(\theta))d\theta$$

$$= 4\int_0^{2\pi}\big(cos(\theta) + 8(1 - cos^2(\theta))sin(\theta)\big)d\theta = 4\left(sin(\theta) + 8\left(\frac{1}{3}cos^3(\theta) - cos(\theta)\right)\right)_0^{2\pi} = 0.$$

2) $\iint_S F\,dS = ?\quad F(x,y,z) = x\vec{\imath} - z\vec{\jmath} + y\vec{k}\quad (S):\ x^2 + y^2 + z^2 = 9.$

$$\vec{r}: \begin{cases} x = 3cos(\theta)cos(t) \\ y = 3sin(\theta)cos(t) \\ z = 3sin(t) \end{cases}, \qquad \begin{cases} 0 \le \theta \le 2\pi \\ -\frac{\pi}{2} \le t \le \frac{\pi}{2} \end{cases}$$

$$\overrightarrow{r_\theta} \times \overrightarrow{r_t} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ -3sin(\theta)cos(t) & 3cos(\theta)cos(t) & 0 \\ -3cos(\theta)sin(t) & -3sin(\theta)sin(t) & 3cos(t) \end{vmatrix}$$

$$= 9cos(\theta)cos^2(t)\vec{\imath} + 9sin(\theta)cos^2(t)\vec{\jmath} + 9cos(t)sin(t)\vec{k}.$$

$$F.(\overrightarrow{r_\theta} \times \overrightarrow{r_t}) = 27cos^2(\theta)cos^3(t).$$

$$\iint_S F\,dS = 27\int_0^{2\pi}\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} cos^2(\theta)cos^3(t)d\theta dt = 27\left(\int_0^{2\pi}\frac{1 + cos(2\theta)}{2}d\theta\right)\left(\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} cos(t)(1 - sin^2(t))\,dt\right)$$

$$= \frac{27}{2}\left(\theta + \frac{1}{2}sin(2\theta)\right)_0^{2\pi}\left(sin(t) - \frac{1}{3}sin^3(t)\right)_{\frac{-\pi}{2}}^{\frac{\pi}{2}} = 36\pi.$$

2) Stoks´ theorem

Let us consider a vector field $F: \mathbb{R}^3 \to \mathbb{R}^3$.

And (S) be an oriented surface in $\mathbb{R}^3$, bounded By a closed curve (C). Then we have :

$$\oint_C F.dr = \iint_S Curl(F).dS = \iint_S Curl(F).\vec{n}ds \quad / \quad (Curl(F) = Rot(F)).$$

Example : Let : $F(x,y,z) = x\vec{\imath} + y\vec{\jmath} + xyz\vec{k}$, $(S):\ 2x + y + z = 2$, $x,y,z \ge 0$.

$$z = 2 - 2x - y\ ,\quad 0 \le x \le 1, 0 \le y \le 2 - 2x.$$

$$\vec{r} = \begin{pmatrix} x \\ y \\ 2 - 2x - y \end{pmatrix},\ \vec{r}_x \times \vec{r}_y = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$Curl(F) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & xyz \end{vmatrix} = xz\vec{i} - yz\vec{j}$$

$$Curl(F).\left(\vec{r}_x \times \vec{r}_y\right) = 2xz - yz = 2x(2-2x-y) - y(2-2x-y) = 4x - 4x^2 - 2y + y^2.$$

$$\oint_C F\,.dr = \iint_S Curl(F).\vec{n}ds$$

$$\iint_S Curl(F).\vec{n}ds = \int_0^1 \int_0^{2-2x} (4x - 4x^2 - 2y + y^2)dxdy = \int_0^1 \left(4(x-x^2)y - y^2 + \frac{1}{3}y^3\right)_0^{2-2x} dx$$

$$= \int_0^1 \left(4(x-x^2)(2-2x) - (2-2x)^2 + \frac{1}{3}(2-2x)^3\right)dx$$

$$= \int_0^1 \left(4x(1-x)^2 - 4(1-x)^2 + \frac{8}{3}(1-x)^3\right)dx$$

$$= \int_0^1 \left(8(1-t)t^2 - 4t^2 + \frac{8}{3}t^3\right)dt = \int_0^1 \left(4t^2 - \frac{16}{3}t^3\right)dt$$

$$\oint_C F\,.dr = \left(\frac{4}{3}t^3 - \frac{4}{3}t^4\right)_0^1 = 0$$

2) $F(x,y,z) = y\vec{i} + z\vec{j} + x\vec{k}$, $(S): x + y - z = 0$, $0 \le x \le 1, 0 \le y \le 1$.

$\oint_C F\,.dr = ?$ The curve (C) is a boundary of the surface (S).

3) Ostrogradsky´s theorem

Let us consider a vector field $F: \mathbb{R}^3 \to \mathbb{R}^3$.

And (S) be an oriented surface in $\mathbb{R}^3$. Then we have :

$$\iint_S FdS = \iint_S F.\vec{n}ds = \iiint_V Div(F)dxdydz \quad / \quad (V)\ region\ bounded\ by\ a\ surface\ (S).$$

Example : 1) $F(x,y,z) = xy\vec{i} + 2y\vec{j} + yz\vec{k}$, $(S): x^2 + y^2 = 4$, $-1 \le z \le 1$.

$$Div(F) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 2y + 2.$$

$$\iint_S FdS = \iiint_V Div(F)dxdydz = 2\iiint_V (y+1)dxdydz$$

$$\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ z = z \end{cases} , \quad \begin{cases} 0 \le \theta \le 2\pi \\ 0 \le r \le 2 \\ -1 \le z \le 1 \end{cases}$$

$$\iint_S FdS = 2\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{2} r(rsin(\theta)+1)drd\theta dz = 4\int_{0}^{2\pi}\left(\frac{r^2}{2}sin(\theta)+r\right)_0^2 d\theta$$

$$= 8\int_{0}^{2\pi}(sin(\theta)+1)d\theta = 8(-cos(\theta)+\theta)_0^{2\pi} = 16\pi.$$