Chapter 2

Sequences of functions

In all this chapter, we let I be a real interval and we denote by $\mathcal{F}(I,\mathbb{R})$ the set of all real functions defined on I. For any $f \in \mathcal{F}(I,\mathbb{R})$, the symbol $\|f\|$ denotes the supremum of the function $|f|$ on the interval I, that is

$$\|f\| = \sup_{x\in I} |f(x)|.$$

1 Definitions

Definition 1.1 *We call a sequence of functions any map $\mathbb{N} \to \mathcal{F}(I,\mathbb{R})$ associating for each integer n a function $f_n \in \mathcal{F}(I,\mathbb{R})$. Usually, a sequence of function is denoted by $(f_n)_{n\geq 1}$, $(g_n)_{n\geq 1}$ etc...*

Example 1.2

1. $f_n(x) = x^n,\ x \in \mathbb{R}$.
2. $f_n(x) = \exp(nx),\ x \in \mathbb{R}$.

Definition 1.3 *Let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}(I,\mathbb{R})$ be a sequence of functions and let $f \in \mathcal{F}(I,\mathbb{R})$. We say that $(f_n)_{n\in\mathbb{N}}$ converge pointwise to f on I, if for all $x \in I$ we have $\lim_{n\to\infty} f_n(x) = f(x)$.*

Pointwise convergence defines the convergence of functions in terms of the convergence of their values at each point of their domain. We say that the sequence $(f_n)_{n\geq 1}$ converges pointwise if it converges pointwise to some function.

Example 1.4

1. *The sequence $(f_n)_{n\geq 1}$ defined by $f_n(x) = \dfrac{x}{n}$, converges pointwise to 0 in $\mathbb{R}$.*
2. *The sequence $(f_n)_{n\geq 1}$ defined by $f_n(x) = 1 + \exp(-nx)$, converges pointwise to the 1 in $(0,+\infty)$.*

Pointwise convergence is the natural way to define the convergence for sequences of functions. Unfortunately, this mode of convergence does not preserve certain properties of the sequence. The following examples illustrate this situation.

Example 1.5 *Consider the sequence of functions $(f_n)_{n\geq 1}$ defined on the interval $(0,\pi/2)$ by $f_n(x) = \dfrac{nx}{nx^2+\cos x}$. It is easy to see that $(f_n)_{n\geq 1}$ converges pointwise to $f(x) = 1/x$. Notice that for all $n \in \mathbb{N}$, f_n is bounded and integrable on $(0,\pi/2)$ but $f(x) = 1/x$ is not. This shows that the pointwise convergence do not preserve the boundedness and the integrability.*

Example 1.6 *Consider the sequence of functions* $(f_n)_{n\geq 1}$ *defined on the interval* $[0,2]$ *by* $f_n(x)=\dfrac{nx^2}{nx^2+\cos(nx)}$. *It is easy to see that* $(f_n)_{n\geq 1}$ *converges pointwise to*

$$f(x)=\begin{cases} 1, & \text{if } x\neq 0,\\ 0, & \text{if } x=0.\end{cases}$$

Notice that for all $n\in\mathbb{N}$, f_n *is continuous on* $[0,2]$ *but* f *is not. This shows that the pointwise convergence do not preserve the continuity.*

Example 1.7 *Consider the sequence of functions* $(f_n)_{n\geq 1}$ *defined on* $\mathbb{R}$ *by* $f_n(x)=\dfrac{x^2}{\sqrt{x^2+1/n}}$. *It is easy to see that* $(f_n)_{n\geq 1}$ *converges pointwise to* $f(x)=|x|$.

Notice that for all $n\in\mathbb{N}$, f_n *is differentiable at* 0 *on* $[0,2]$ *but* f *is not. This shows that the pointwise convergence do not preserve the differentiability.*

2 Uniform convergence

In this section we introduce a mode of convergence stronger than the pointwise one, called uniform convergence. The difference between the pointwise convergence and the uniform convergence is analogous to the difference between the continuity and the uniform continuity.

Definition 2.1 *Let* $(f_n)_{n\in\mathbb{N}}\subset\mathcal{F}(I,\mathbb{R})$ *be a sequence of functions and let* $f\in\mathcal{F}(I,\mathbb{R})$. *We say that* $(f_n)_{n\in\mathbb{N}}$ *converge uniformely to* f *on* I, *if for all* $\epsilon>0$ *there exists* $n_\epsilon\in\mathbb{N}$ *such that for all* $n\in\mathbb{N}$, $n\geq n_\epsilon\Longrightarrow|f_n(x)-f(x)|\leq\epsilon$ *for all* $x\in I$.

Remark 2.2 *Notice that a sequence* $(f_n)_{n\in\mathbb{N}}\subset\mathcal{F}(I,\mathbb{R})$ *converge uniformely to* $f\in\mathcal{F}(I,\mathbb{R})$ *in* I *if and only if*

$$\lim_{n\to\infty}\left(\sup_{x\in I}|f_n(x)-f(x)|\right)=\lim_{n\to\infty}\|f_n-f\|=0.$$

It is easy to prove the following proposition.

Proposition 2.3 *Let* $(f_n)_{n\in\mathbb{N}}\subset\mathcal{F}(I,\mathbb{R})$ *be a sequence of functions and let* $f\in\mathcal{F}(I,\mathbb{R})$. *If* $(f_n)_{n\in\mathbb{N}}$ *converge uniformely to* f *in* I, *then* $(f_n)_{n\in\mathbb{N}}$ *converge pointwise to* f *in* I.

Example 2.4 *Consider the sequence* $(f_n)_{n\geq 1}$ *defined by* $f_n(x)=x^n$ *on* $[0,a]$ *with* $a\in(0,1)$. *For all* $x\in[0,a]$ *we have* $\lim_{n\to\infty}f_n(x)=0$. *Moreover, we have*

$$\lim_{n\to\infty}\left(\sup_{x\in I}|f_n(x)-0|\right)=\lim_{n\to\infty}\|f_n\|=\lim_{n\to\infty}a^n=0.$$

This shows that the sequence $(f_n)_{n\geq 1}$ *converges uniformely to* 0 *on each interval* $[0,a]$ *with* $a\in(0,1)$.

Example 2.5 *Consider the sequence of functions* $(f_n)_{n\in\mathbb{N}}$ *defined on* $[0,1]$ *by*

$$f_n(x)=\begin{cases} 2x, \ \text{if } 0\leq x\leq \dfrac{1}{2n},\\ -2x+\dfrac{2}{n}, \ \text{if } \dfrac{1}{2n}\leq x\leq \dfrac{1}{n},\\ 0, \ \text{if } \dfrac{1}{n}\leq x\leq 1. \end{cases}$$

We have $f_n(0)=0$ *for all* $n\in\mathbb{N}$ *and for* $x>0$, *we have* $\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty} 0=0$. *Hence,* $(f_n)_{n\in\mathbb{N}}$ *is pointwise convergent to* 0. *Moreover, we have* $\sup_{x\in[0,1]}|f_n(x)-0|=\frac{1}{n}\to 0$ *as* $n\to\infty$. *We conclude that* $(f_n)_{n\in\mathbb{N}}$ *is uniformely convergent to* 0.

Exercise 2.6 *Let* $(f_n)_{n\geq 1}$ *be the sequence defined by* $f_n(x)=\left(1+\dfrac{x}{n}\right)^n$ *on* $\mathbb{R}$. *Study the uniform convergence of* $(f_n)_{n\geq 1}$ *on* $\mathbb{R}$.

Solution 2.7 *Clearly, for all* $x\in\mathbb{R}$ *we have* $\lim f_n(x)=e^x$. *Set* $g_n(x)=\left|\left(1+\dfrac{x}{n}\right)^n-e^x\right|$. *We have* $g_n(n)=e^n-2^n\to+\infty$. *This shows that* $(f_n)_{n\geq 1}$ *do not converge uniformely to* e^x *on* $\mathbb{R}$.

Theorem 2.8 (Cauchy) *Let* $(f_n)_{n\in\mathbb{N}}\subset\mathcal{F}(I,\mathbb{R})$ *be a sequence of functions. Then* $(f_n)_{n\in\mathbb{N}}$ *converge uniformely if and only if for all* $\epsilon>0$ *there exists* $n_\epsilon\in\mathbb{N}$ *such that for all* $m,n\in\mathbb{N}$, $m,n\geq n_\epsilon\Longrightarrow\|f_m-f_n\|\leq\epsilon$.

Proof.

$\Rightarrow$) Suppose that $(f_n)_{n\in\mathbb{N}}$ converges uniformely to a function $f\in\mathcal{F}(I,\mathbb{R})$. Then for any $\epsilon>0$, there is $n_\epsilon\in\mathbb{N}$ such that for all $n\in\mathbb{N}$, $n\geq n_\epsilon\Longrightarrow\|f_n-f\|\leq\epsilon/2$. Hence, for any $m,n\geq n_\epsilon$, we have

$$\begin{aligned}\|f_m-f_n\| &= \sup_{x\in I}|f_m(x)-f_n(x)|\leq\sup_{x\in I}\left(|f_m(x)-f(x)|+|f(x)-f_n(x)|\right)\\ &\leq \sup_{x\in I}\left(|f_m(x)-f(x)|\right)+\sup_{x\in I}\left(|f(x)-f_n(x)|\right)\\ &= \|f_m-f\|+\|f_n-f\|\leq\epsilon/2+\epsilon/2=\epsilon.\end{aligned}$$

$\Leftarrow$) Suppose that for all $\epsilon>0$ there exists $n_\epsilon\in\mathbb{N}$ such that for all $m,n\in\mathbb{N}$, $m,n\geq n_\epsilon\Longrightarrow\|f_m-f_n\|=\sup_{x\in I}|f_m(x)-f_n(x)|\leq\epsilon$. Then, for all $x\in I$, $(f_n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence and the sequence converges pointwise to some function $f\in\mathcal{F}(I,\mathbb{R})$. Let $\epsilon>0$ and n_ϵ as supposed, then for any $x\in I$ and all $m,n\geq n_\epsilon$ we have

$$\begin{aligned}|f_n(x)-f(x)| &\leq |f_n(x)-f_m(x)|+|f_m(x)-f(x)|\\ &\leq \|f_m-f_n\|+|f_m(x)-f(x)|\\ &\leq \epsilon+|f_m(x)-f(x)|.\end{aligned}$$

Letting $m\to+\infty$, we get $|f_n(x)-f(x)|\leq\epsilon$ for all $x\in I$. Proving the uniform convergence of $(f_n)_{n\in\mathbb{N}}$ to f. ■

3 Properties of the uniform convergence

In all this section, we let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}(I,\mathbb{R})$ be a sequence of functions and $f \in \mathcal{F}(I,\mathbb{R})$.

Theorem 3.1 *Suppose that $(f_n)_{n\in\mathbb{N}}$ converge uniformely to f on I and there is $n_0 \in \mathbb{N}$ such that f_n is bounded on I for all $n \geq n_0$. Then the function f is bounded on I.*

Proof. Let $\epsilon > 0$ be given, there is $n_1 \in \mathbb{N}$ such that for all $n \geq \max(n_0, n_1)$ we have

$$\|f_n - f\| = \sup_{x\in I} |f_n(x) - f(x)| \leq \epsilon.$$

Taking $n_* \in \mathbb{N}$ with $n_* \geq \max(n_0, n_1)$, we obtain

$$\|f\| = \sup_{x\in I} |f(x)| \leq \sup_{x\in I} |f_{n_*}(x) - f(x)| + \sup_{x\in I} |f_{n_*}(x)| \leq \epsilon + \|f_{n_*}\|.$$

Proving that f is bounded. ■

Theorem 3.2 *Suppose that $(f_n)_{n\in\mathbb{N}}$ converge uniformely to f on $[a,b] \subset I$ and there is $n_0 \in \mathbb{N}$ such that f_n is integrable on $[a,b] \subset I$ for all $n \geq n_0$. Then the function f is integrable on $[a,b]$ and we have $\lim_{n\to\infty} \int_a^b f_n(t)dt = \int_a^b f(t)dt$.*

Proof. Let $\epsilon > 0$, by the uniform convergence of (f_n) to f, there is $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have

$$f_n(x) - \frac{\epsilon}{4(b-a)} \leq f(x) \leq f_n(x) + \frac{\epsilon}{4(b-a)} \text{ for all } x \in [a,b].$$

Also, there is asubdivision $\{x_0, x_1, \ldots x_k\}$ of $[a,b]$ such that

$$\sum_{i=0}^{i=k} (M_{n,i} - m_{n,i})(x_i - x_{i-1}) \leq \epsilon/2,$$

where $M_{n,i} = \sup_{x\in[x_{i-1},\ x_i]} f_n(x)$ and $m_{n,i} = \inf_{x\in[x_{i-1},\ x_i]} f_n(x)$.

Hence, if $M_i = \sup_{x\in[x_{i-1},\ x_i]} f(x)$ and $m_i = \inf_{x\in[x_{i-1},\ x_i]} f(x)$, we have then

$$m_{n,i} - \frac{\epsilon}{4(b-a)} \leq m_i \leq M_i \leq M_{n,i} + \frac{\epsilon}{4(b-a)},$$

leading to

$$\sum_{i=0}^{i=k} (M_i - m_i)(x_i - x_{i-1}) \leq \sum_{i=0}^{i=k} (M_{n,i} - m_{n,i})(x_i - x_{i-1}) + \epsilon/2 \leq \epsilon.$$

This proves that f is integrable on $[a,b]$. At the end, from

$$\left| \int_a^b f_n(t)dt - \int_a^b f(t)dt \right| \leq \int_a^b |f_n(t) - f(t)|\, dt \leq (b-a)\|f_n - f\|,$$

we conclude that $\lim_{n\to\infty} \int_a^b f_n(t)dt = \int_a^b f(t)dt$. ■

Corollary 3.3 *Suppose that $(f_n)_{n\in\mathbb{N}}$ converge uniformely to f on $[a,b]\subset I$ and there is $n_0\in\mathbb{N}$ such that f_n is integrable on $[a,b]$ for all $n\geq n_0$. Then the function f is integrable on $[a,b]$ and the sequence of fonctions $(F_n)_{n\geq 1}$, where for all $n\in\mathbb{N}$ $F_n(x)=\int_a^x f_n(t)dt$ converges uniformely to the function F on $[a,b]$, where $F(x)=\int_a^x f(t)dt$ for all $x\in[a,b]$.*

Proof. Indeed, we have

$$\begin{aligned}\|F_n-F\| &= \sup_{x\in[a,b]}\left|\int_a^x (f_n(t)-f(t))\,dt\right| \leq \sup_{x\in[a,b]}\int_a^x |f_n(t)-f(t)|\,dt\\ &\leq \int_a^x \sup_{t\in[a,b]}|f_n(t)-f(t)|\,dt\\ &\leq (b-a)\,\|f_n-f\|\to 0 \text{ as } n\to\infty.\end{aligned}$$

■

Theorem 3.4 *Let $a\in I$ and suppose that $(f_n)_{n\in\mathbb{N}}$ converge uniformely to f on I and there is $n_0\in\mathbb{N}$ such that $\lim_{x\to a} f_n(x)$ exists and is finite for all $n\geq n_0$. Then $\lim_{x\to a} f(x)$ exists and is finite and we have*

$$\lim_{n\to\infty}\lim_{x\to a} f_n(x)=\lim_{x\to a}\lim_{n\to\infty} f_n(x)=\lim_{x\to a} f(x).$$

Proof. First, set $\lim_{x\to a} f_n(x)=l_n$ for all $n\geq n_0$. Because $(f_n)_{n\in\mathbb{N}}$ converges uniformely, $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence and for all $\epsilon>0$, there exists $n_\epsilon\in\mathbb{N}$ such that for all $m,n,\geq n_\epsilon$ we have

$$|f_m(x)-f_n(x)|\leq\epsilon \text{ for all } x\in I.$$

Letting $x\to a$, we obtain $|l_m-l_n|\leq\epsilon$, proving that $(l_k)_{k\geq 1}$ is a Cauchy sequence.

Set $l=\lim l_n$ and let $\epsilon>0$ be given, there is $n_1\in\mathbb{N}$ such that for all $n\geq\max(n_0,n_1)$ we have

$$\|f_n-f\|=\sup_{x\in I}|f_n(x)-f(x)|\leq\epsilon/3$$

and

$$|l_n-l|\leq\epsilon/3.$$

Let n_* be given with $n_*\geq\max(n_0,n_1)$, there exists $\delta_*>0$ such that for all $x\in I$,

$$|x-a|\leq\delta_*\Longrightarrow|f_{n_*}(x)-l_{n_*}|\leq\epsilon/3.$$

Therefore, for all $x\in I$ with $|x-a|\leq\delta_*$ we have

$$\begin{aligned}|f(x)-l| &\leq |f_{n_*}(x)-f(x)|+|f_{n_*}(x)-l_{n_*}|+|l_{n_*}-l|\\ &\leq \epsilon/3+\epsilon/3+\epsilon/3=\epsilon.\end{aligned}$$

The proof is complete. ■

Corollary 3.5 *Let $a\in I$ and suppose that $(f_n)_{n\in\mathbb{N}}$ converge uniformely to f on I and there is $n_0\in\mathbb{N}$ such that f_n is continuous at a for all $n\geq n_0$. Then the function f is continuous at a.*

Proof. We have by Theorem 3.4

$$f(a) = \lim_{n\to\infty} f_n(a) = \lim_{n\to\infty} \lim_{x\to a} f_n(x) = \lim_{x\to a} \lim_{n\to\infty} f_n(x) = \lim_{x\to a} f(x).$$

■

The foolwing corollary is a direct consequence of Corollary 3.5.

Corollary 3.6 *Suppose that $(f_n)_{n\in\mathbb{N}}$ converge uniformely to f on I and there is $n_0 \in \mathbb{N}$ such that f_n is continuous on I for all $n \geq n_0$. Then the function f is continuous on I.*

Theorem 3.7 *Suppose that there is $n_0 \in \mathbb{N}$ such that f_n is continuously differentiable on $[a,b] \subset I$ for all $n \geq n_0$. If $(f_n')_{n\in\mathbb{N}}$ converge uniformely to g on $[a,b]$ and there is $x_0 \in [a,b]$ such that $(f_n(x_0))_{n\in\mathbb{N}}$ converges, then $(f_n)_{n\in\mathbb{N}}$ converges uniformely to a continuously differentiable function f on $[a,b]$ and we have $f' = g$.*

Proof. For all $x \in [a,b]$ we have

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(t)dt.$$

We conclude then by Corollary 3.3 that $(f_n)_{n\geq 1}$ converge uniformely to f and $f(x) = f(x_0) + \int_{x_0}^{x} g(t)dt$. Proving that $f' = g$. ■

The following is a consequence of Theorem 3.7.

Corollary 3.8 *Suppose that there is $n_0 \in \mathbb{N}$ such that $f_n \in C^k([a,b])$ for all $n \geq n_0$ where $[a,b] \subset I$. If $\left(f_n^{(k)}\right)_{n\in\mathbb{N}}$ converge uniformely to g on $[a,b]$ and there is $x_0 \in [a,b]$ such that $\left(f_n^{(i)}(x_0)\right)_{n\in\mathbb{N}}$ converges for all $i \in \{0,1,\ldots(k-1)\}$. Then $(f_n)_{n\in\mathbb{N}}$ converges uniformely to a function $f \in C^k([a,b])$ and we have $f^{(k)} = g$.*