University of Oum El Bouaghi — Academic year: 2026/2027
Introduction to Topology — License 2 - Mathematics

Sheet of exercises N°1

Exercise 1 *Let $X =]0, +\infty[$. For $x, y \in X$, we set*

$$d(x,y) = \left| \frac{1}{x} - \frac{1}{y} \right|.$$

1. *Show that d is a distance on X.*
2. *Show that, in the metric space (X,d), $\mathbb{N}^*$ is a bounded subset and $\left\{ \frac{1}{n} \ ; \ n \in \mathbb{N}^* \right\}$ is an unbounded subset.*

Exercise 2 *We equip a set X with the* discrete *distance. For $a \in X$ and $r > 0$, determine $B(a,r)$, $\overline{B}(a,r)$ and $S(a,r)$.*
[Hint: Distinguish cases according to the value of r].

Exercise 3 *Let $[a,b]$ be a closed bounded interval of $\mathbb{R}$.* On the set

$$C([a,b],\mathbb{R}) = \{ f : [a,b] \to \mathbb{R} \ ; \ f \text{ continuous} \},$$

we define:

$$\begin{aligned} d_\infty(f,g) &= \sup_{t \in [a,b]} |f(t) - g(t)|, \\ d_1(f,g) &= \int_a^b |f(t) - g(t)| \, dt. \end{aligned}$$

1. *Show that d_∞ and d_1 are distances on $C([a,b],\mathbb{R})$ (d_∞ is called the uniform convergence distance or the Sup metric and d_1 the L^1-distance).*
2. *Give a geometric interpretation of the distance $d_\infty(f,g)$ between two functions $f, g \in C([a,b],\mathbb{R})$, then describe the ball $B_\infty(f_0, r)$ with centre $f_0 \in C([a,b],\mathbb{R})$ and radius $r > 0$.*
3. *Give a geometric interpretation of the distance $d_1(f,g)$ between two functions $f, g \in C([a,b],\mathbb{R})$.*

Exercise 4 (Construction of distances) *We call a gauge a map $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ which is non-decreasing, only vanishing at 0 and sub-additive (i.e. $\varphi(s+t) \leq \varphi(s) + \varphi(t)$, $\forall s, t \in \mathbb{R}_+$).*

1. *Let (X,d) be a metric space and $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a gauge. Show that $d' = \varphi \circ d$ is a distance on X.*

2. *Application: Deduce that*

$$d_1 = \frac{d}{1+d}, \qquad d_2 = \min\{1, d\}, \qquad d_3 = \ln(1+d), \qquad \text{and} \qquad d_4 = d^\alpha \text{ with } 0 < \alpha < 1$$

are distances on X.

3. *Show that d and $d_1 = \frac{d}{1+d}$ are topologically equivalent.*

4. *Show that d and $d' = \varphi \circ d$ are metrically equivalent if there exists a real constant $C > 0$ such that*

$$C^{-1}t \leq \varphi(t) \leq Ct, \qquad \forall t \in \mathbb{R}^+.$$

Exercise 5 *Show that in a metric space every finite subset is closed.*

Exercise 6 *We equip $\mathbb{R}$ with the usual distance. Determine the interior $\mathring{A}$ and the closure $\overline{A}$ of the set A in each of the following cases:*

$$(a)\ A = \left\{ \frac{1}{n} \ ;\ n \in \mathbb{N}^* \right\}, \qquad (b)\ A =]-\infty, -1[\cup \left\{ 0, \frac{\pi}{4}, \sqrt{3} \right\}.$$

Exercise 7 *Let (X, d) be a metric space and let A and B be two non-empty subsets of X. Show that:*

1. $\forall x, y \in X : \quad |d(x, A) - d(y, A)| \leq d(x, y),$
2. $\forall x \in X : \quad x \in \overline{A} \iff d(x, A) = 0,$
3. $\operatorname{diam} \overline{A} = \operatorname{diam} A.$
4. $\operatorname{diam}(A \cup B) \leq \operatorname{diam} A + \operatorname{diam} B + d(A, B)$ *where* $d(A, B) := \inf_{(a,b) \in A \times B} d(a, b).$

Exercise 8 *Show that in a metric space,*

1. *Every convergent sequence is bounded. The converse is false.*
2. *Every subsequence of a convergent sequence is convergent.*

Exercise 9 *Let d and d' be two distances on a set X. Show that:*

1. *If d and d' are metrically equivalent then (X, d) and (X, d') have the same bounded parts.*
2. *d and d' are topologically equivalent if and only if (X, d) and (X, d') have the same convergent sequences.*

Supplementary exercises

Exercise 10 Let p be a real number such that $p > 1$; we define a real number $q > 1$ by $\frac{1}{p} + \frac{1}{q} = 1$ (q is called the conjugate of p).

1. Let a, b be real numbers ≥ 0. Prove that

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \qquad \text{[Young inequality]}.$$

[Hint: You may use the concavity of the function ln*].*

2. Let $a_1, ..., a_n$ and $b_1, ..., b_n$ be real numbers. Prove the two inequalities

$$\sum_{i=1}^{n} |a_i b_i| \leq \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} \left(\sum_{i=1}^{n} |b_i|^q \right)^{1/q} \qquad \text{[Hölder inequality]},$$

$$\left(\sum_{i=1}^{n} |a_i + b_i|^p \right)^{1/p} \leq \left(\sum_{i=1}^{n} |a_i|^p \right)^{1/p} + \left(\sum_{i=1}^{n} |b_i|^p \right)^{1/p} \qquad \text{[Minkowski inequality]}.$$

3. Deduce that a distance d_p is defined on the space $\mathbb{R}^n$ by setting, for $x = (x_1, ..., x_n),\ y = (y_1, ..., y_n) \in \mathbb{R}^n$:

$$d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p \right)^{1/p} \qquad \text{[Hölder distance of order } p\text{]}.$$

4. Show that in $\mathbb{R}^n$, all Hölder distances are equivalent to one another.

5. Show that

$$\forall x, y \in \mathbb{R}^n : \quad \lim_{p \to \infty} d_p(x, y) = d_\infty(x, y)$$

where $d_\infty(x, y) = \max\limits_{1 \leq i \leq n} |x_i - y_i|$.

Exercise 11 Let f and g be continuous functions on $[a, b]$ and $1 < p, q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

1. Show that

$$\left| \int_a^b f(t) g(t) dt \right| \leq \left(\int_a^b |f(t)|^p \, dt \right)^{1/p} \left(\int_a^b |g(t)|^q \, dt \right)^{1/q} \qquad \text{[Hölder inequality]},$$

$$\left(\int_a^b |f(t) + g(t)|^p \, dt \right)^{1/p} \leq \left(\int_a^b |f(t)|^p \, dt \right)^{1/p} + \left(\int_a^b |g(t)|^p \, dt \right)^{1/p} \qquad \text{[Minkowski inequality]}.$$

[Hint: Adapt the method used in the previous exercise].

2. Deduce that a distance is defined on the space $C([a, b])$ by setting:

$$d_p(f, g) = \left(\int_a^b |f(t) - g(t)|^p \, dt \right)^{1/p},$$

called the L^p-*distance.*