Course03 : Curvilinear integrals (Line integrals)

1) Curvilinear integrals of scalar field

Let us consider the scalar field f in space $\mathbb{R}^3$.
And $\phi : t \longmapsto (x(t), y(t), z(t))$ a parametrization of the curve (path) (C).
We define the line(curvilinear) integral of the scalar field f along the path (C)by :

$$\int_C f(x,y,z)dl = \int_a^b f(x(t),y(t),z(t))\frac{dl}{dt}\,dt.$$

dl : Is the arc-length differential.

$$dl = \sqrt{(dx)^2+(dy)^2+(dz)^2} \Rightarrow \frac{dl}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}$$

$$\int_C f(x,y,z)dl = \int_a^b f(x(t),y(t),z(t))\sqrt{x'^2+y'^2+z'^2}\,dt\,.$$

Examples :

1) $\int_C x(1+4y)\,dl$, $C = \{(x,y) \in R^2, \quad y = x^2 \ , \ 0 \le x \le 1\}$.

$Parametrization\ of\ C :$ $\begin{cases} x(t) = t \\ y(t) = t^2 \end{cases}$, $0 \le t \le 1$

$$\int_C x(1+4y)dl = \int_0^1 x(1+4y)\frac{dl}{dt}\,dt$$

$$\frac{dl}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2} = \sqrt{1+(2t)^2}$$

$$\int_C x(1+4y)dl = \int_0^1 t(1+4t^2)\sqrt{1+4t^2}\,dt = \int_0^1 t(1+4t^2)^{\frac{3}{2}}\,dt$$

$Put : u = 1+4t^2 \Rightarrow du = 8tdt.$

$$\int_0^1 t(1+4t^2)^{\frac{3}{2}}\,dt = \frac{1}{8}\int_1^5 u^{\frac{3}{2}}\,du = \frac{1}{20}\left(u^{\frac{5}{2}}\right)_1^5 = \frac{1}{20}\left(25\sqrt{5}-1\right) = \frac{5}{4}\sqrt{5}-\frac{1}{20}.$$

$$\int_C x(1+4y)\,dl = \frac{5}{4}\sqrt{5}-\frac{1}{20}.$$

2) $\int_C (x+y)dl \quad , \qquad C: x^2+y^2-2y=0$

$Parametrization\ of\ \ C: x^2+(y-1)^2=1 \begin{cases} x(t)=cos(t) \\ y(t)=1+sin(t) \end{cases} , \ 0 \le t \le 2\pi$

$$\int_C (x+y)dl = \int_0^{2\pi} (cos(t)+1+sin(t))\frac{dl}{dt}dt \quad , \quad \frac{dl}{dt} = \sqrt{(-sin(t))^2+cos^2(t)} = 1.$$

$$\int_C (x+y)dl = \int_0^{2\pi} (cos(t)+1+sin(t))dt = (sin(t)-cos(t)+t)_0^{2\pi} = 2\pi.$$

Remarks :

1) The line integral of scalar field does not depend on the orientation of the curve C .

2) The length of curve C is : $L = \int_C dl$.

2) Curvilinear integral of vector field

We consider the vector field $\vec{F}(x,y,z) = P(x,y,z)\vec{\imath} + Q(x,y,z)\vec{\jmath} + R(x,y,z)\vec{k}$.

And $\vec{r}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$, $with\ \ a \le t \le b$, is a parametrization of curve (C).

The line integral of the vector field F along the curve (C) is defined us :

$$\int_C \vec{F}.\overrightarrow{dr} = \int_C Pdx + Qdy + Rdz = \int_a^b \vec{F}.\frac{\overrightarrow{dr}}{dt}dt$$

$$\frac{\overrightarrow{dr}}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}$$

$$\int_C \vec{F}.\overrightarrow{dr} = \int_a^b (Px' + Qy' + Rz')dt.$$

Examples :

1) $\int_C \vec{F}.\overrightarrow{dr} = ? \quad \vec{F}(x,y,z) = xz\vec{\imath} - yz\vec{k}$, $(C): The\ segment\ from\ A(-1,2,0)\ to\ B(3,0,1).$

$$Parametrization\ of\ (C): \begin{cases} x(t) = (1-t)x_A + tx_B = 4t-1 \\ y(t) = (1-t)y_A + ty_B = 2(1-t) \\ z(t) = (1-t)z_A + tz_B = t \end{cases} , \ 0 \le t \le 1 .$$

$$\vec{F}(x,y,z) = (4t^2 - t, 0, 2t^2 - t)$$

$$\begin{cases} x'(t) = 4 \\ y'(t) = -2 \\ z'(t) = 1 \end{cases}$$

$$\int_C \vec{F}.\overrightarrow{dr} = \int_0^1 \left(4(4t^2 - t) + (2t^2 - 2t)\right) dt = \int_0^1 (18t^2 - 6t)\, dt = (6t^3 - 3t^2)_0^1 = 3.$$

2) $F(x,y) = y\vec{\imath} + 3x\vec{\jmath}$, $(C): \dfrac{x^2}{4} + y^2 = 1$ *From* $(2,0)$ *to* $(-2,0)$.

Parametrization of (C): $\begin{cases} x(t) = 2cos(t) \\ y(t) = sin(t) \end{cases}$, $0 \le t \le \pi$

$$F(x,y) = sin(t)\vec{\imath} + 6cos(t)\vec{\jmath} \quad , \begin{cases} x'(t) = -2sin(t) \\ y'(t) = cos(t) \end{cases}$$

$$\int_C \vec{F}.\overrightarrow{dr} = \int_0^\pi (sin(t)\vec{\imath} + 6cos(t)\vec{\jmath}).(-2sin(t)\vec{\imath} + cos(t)\vec{\jmath})dt = \int_0^\pi (-2sin^2(t) + 6cos^2(t))dt$$

$$= \int_0^\pi (-2 + 8cos^2(t))dt = \int_0^\pi (2 + 4cos(2t))dt = (2t + 2sin(2t))_0^\pi = 2\pi.$$

Remarks :

1) The curvilinear integral of vector field depends on the orientation of the curve (C).
2) If the orientation of the curve $(C)$is in the positive direction, we write

$$\oint_C F.dr \; instead \int_C F.dr.$$