

Course 02 : Differential operators

Scalar and Vector fields

Scalar field : Is defined us

$$f : \mathbb{R}^3 \rightarrow \mathbb{R}$$

$$(x, y, z) \mapsto f(x, y, z), \text{ Scalar}$$

Example : $f(x, y, z) = 3x^2z + yz$, $f(1, -1, 3) = 6$.

Vector field : Is defined us

$$F : \mathbb{R}^3 \rightarrow \mathbb{R}^3$$

$$(x, y, z) \mapsto F(x, y, z) = (F_1, F_2, F_3), \text{ Vector}$$

Example : $F(x, y, z) = xy^2 \vec{i} + 2yxz \vec{j} - 3xz \vec{k}$, $F(1, -1, 1) = \vec{i} - 2\vec{j} - 3\vec{k}$.

Differential operators

Nabla operator : Is defined as

$$\nabla = \frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k}$$

Gradient operator : Let f a scalar field.

The gradient of f is a vector defined us $\overrightarrow{\text{Grad}}f = \nabla f$

$$\overrightarrow{\text{Grad}}f = \frac{\partial f}{\partial x} \vec{i} + \frac{\partial f}{\partial y} \vec{j} + \frac{\partial f}{\partial z} \vec{k}$$

Example : $f(x, y, z) = xy + 2y - 3x + 4$, $\overrightarrow{\text{Grad}}f(x, y, z) = (y - 3)\vec{i} + (x + 2)\vec{j}$
 $\overrightarrow{\text{Grad}}f(0, 1, 1) = -2\vec{i} + 2\vec{j}$.

Divergence : Let F a vector field. $F(x, y, z) = (F_1, F_2, F_3)$.

The divergence of the vector F is a scalar defined us $\text{Div}F = \nabla \cdot F$.

$$\text{Div}F = \left(\frac{\partial}{\partial x} \vec{i} + \frac{\partial}{\partial y} \vec{j} + \frac{\partial}{\partial z} \vec{k} \right) \cdot (F_1, F_2, F_3) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Example : $F(x, y, z) = xy^2 \vec{i} + 2yxz \vec{j} - 3x^3z \vec{k}$

$$\text{Div}F(x, y, z) = \nabla \cdot F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = y^2 + 2xz - 3x^3$$

$$\text{Div}F(-2, 0, -1) = 7.$$

Curl (Rotational) : Let F a vector field. $F(x, y, z) = (F_1, F_2, F_3)$.

The curl of the vector F is a vector defined us $\text{Rot}F = \nabla \times F$.

$$\text{Rot}F = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \vec{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \vec{k}.$$

Example : Let $F(x, y, z) = xy^2 \vec{i} + 2yxz \vec{j} - 3x^3z \vec{k}$.

$$\begin{aligned} \text{Rot}F(x, y, z) &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} \right) \vec{i} - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z} \right) \vec{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) \vec{k} \\ &= -2xy \vec{i} + 9x^2z \vec{j} + 2y(z - x) \vec{k}. \\ \text{Rot}F(0,1,1) &= 2\vec{k}. \end{aligned}$$

Laplacian :

If f is a scalar field, the Laplacian is defined us $\Delta f = \nabla^2 f = \nabla \cdot \nabla f$.

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Example : Let $f(x, y, z) = x^3y + 5yz - 2y^3 - xz + 2$.

$$\Delta f(x, y, z) = 6xy - 12y.$$

If F is a vector field, the Laplacian is defined us $\Delta F = \nabla^2 F = (\nabla^2 F_1, \nabla^2 F_2, \nabla^2 F_3)$.

Scalar potential :

If we have the relation : $F = -\text{Grad}f$, we say that F is a vector field derived from a scalar potential f .

The scalar potential exist if the vector field F is irrotational. ($\text{Rot}F = 0$).

Exemple : $F(x, y, z) = (2xy + z^2) \vec{i} + (x^2 + 3y^2) \vec{j} + 2xz \vec{k}$, $f ? F = -\text{Grad}f$.

Vector potential :

If we have the relation : $F = \text{Rot}G$, we say that F is a vector field derived from a vector potential G .

The vector potential exist if $\text{Div}F = 0$.