Chapter 1

Numerical Series

1 Definitions

Definition 1 *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. We call the series of general term u_n the infinite sum $\sum_{n=1}^{+\infty} u_n$ (or $\sum_{n\geq 1} u_n$). The sequence $(S_m)_{m\geq 1}$, where $S_m = \sum_{n=1}^{m} u_n$ for all $m \in \mathbb{N}$, is called the sequence of partial sums associated with the series $\sum_{n=1}^{+\infty} u_n$.*

Remark 2 *The above sum begins with u_1, but we will often begin with u_0, u_2, etc...*

Example 3 (Some classical series)

1. $\sum_{n=0}^{+\infty} q^n$ *($q \in \mathbb{R}$), is called the geometric series.*
2. $\sum_{n=0}^{+\infty} \frac{1}{n^\alpha}$ *($\alpha \in \mathbb{R}$), is called the Riemann series or the harmonic series.*
3. $\sum_{n=2}^{+\infty} \frac{1}{n^\alpha (\ln n)^\beta}$ *($\alpha, \beta \in \mathbb{R}$), is called the Bertrand series.*
4. $\sum_{n=1}^{+\infty} \frac{\cos(n\beta)}{n^\alpha}$ *and* $\sum_{n=1}^{+\infty} \frac{\sin(n\beta)}{n^\alpha}$ *($\alpha, \beta \in \mathbb{R}$), are called Abel's series.*

Definition 4 *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of real numbers. and let $(S_m)_{m\geq 1}$ be the associated sequence of partial sums. The series $\sum_{n=1}^{+\infty} u_n$ is said to be*

1. *convergent (cv for short) if the sequence $(S_m)_{m\geq 1}$ is. In such a situation, if $\lim S_m = S$ we say that the series $\sum_{n=1}^{+\infty} u_n$ has the sum equals to S and we write $\sum_{n=1}^{+\infty} u_n = S$. Moreover, the series $R_m = \sum_{n=m+1}^{+\infty} u_n = S - S_m$ is called the rest of order m.*
2. *divergent (div for short) if if the sequence $(S_m)_{m\geq 1}$ is.*

The nature of a series is the fact that it converges or diverges. Two series are said to have the same nature if both converge or both diverge.

Example 5 (Geometric series) *Let $q \in \mathbb{R}$. It is well known that for any $m \in \mathbb{N}$*

$$S_m = \sum_{n=0}^{m} q^n = \begin{cases} m+1, \text{ if } q = 1, \\ \dfrac{q^{m+1}-1}{q-1}, \text{ if } q \neq 1. \end{cases}$$

Since

$$\lim q^m = \begin{cases} +\infty, \text{ if } q > 1, \\ 1, \text{ if } q = 1, \\ 0, \text{ if } q \in (-1,1), \\ \text{does not exist, if } q \leq -1, \end{cases}$$

we conclude that the series $\sum_{n=0}^{+\infty} q^n$ *cv if and only if* $q \in (-1,1)$, *and in such a situation we have*

$$\sum_{n=0}^{+\infty} q^n = \frac{1}{1-q}.$$

Proposition 6 *Let* $\sum_{n=1}^{+\infty} u_n$ *and* $\sum_{n=1}^{+\infty} v_n$ *be two series. Then*

1. $\sum_{n=1}^{+\infty} u_n$ *and* $\sum_{n=1}^{+\infty} v_n$ *cv* $\Longrightarrow \sum_{n=1}^{+\infty} (u_n + v_n)$ *cv.*
2. $\sum_{n=1}^{+\infty} u_n$ *cv* $\Longrightarrow \sum_{n=1}^{+\infty} \lambda u_n$ *cv for any* $\lambda \in \mathbb{R}$.
3. $\sum_{n=1}^{+\infty} u_n$ *cv and* $\sum_{n=1}^{+\infty} v_n$ *div* $\Longrightarrow \sum_{n=1}^{+\infty} (u_n + v_n)$ *div.*

Theorem 7 (Necessary condition) *Let* $\sum_{n=1}^{+\infty} u_n$ *be a numerical series. If* $\sum_{n=1}^{+\infty} u_n$ *cv, then* $\lim u_m = 0$.

Proof. Indeed, for any integer $m \geq 2$ we have $u_m = S_m - S_{m-1}$ where $(S_p)_{p\geq 1}$ is the sequence of partial sums associated with the series $\sum_{n=1}^{+\infty} u_n$. ■

Example 8 *In practice we use the contrapositive* $\lim u_m \neq 0 \Longrightarrow \sum_{n=1}^{+\infty} u_n$ *div. For example, because* $\lim \sin(n) \neq 0$ *the serie* $\sum_{n=1}^{+\infty} \sin(n)$ *div and because* $\lim \frac{n}{n+1} = 1$, *the series* $\sum_{n=1}^{+\infty} \frac{n}{n+1}$ *div.*

Remark 9 *The inverse implication is not true (i.e.* $\lim u_m = 0 \not\Rightarrow \sum_{n=1}^{+\infty} u_n$ *cv). For instance,* $\lim \frac{1}{n} = 0$ *but* $\sum_{n=1}^{+\infty} \frac{1}{n}$ *div.*

Theorem 10 (Cauchy) *Let* $\sum_{n=1}^{+\infty} u_n$ *be a numerical series.* $\sum_{n=1}^{+\infty} u_n$ *cv if and only if for all* $\epsilon > 0$ *there exists* $n_\epsilon \in \mathbb{N}$ *such that for all* $m, p \in \mathbb{N},\ m \geq p \geq n_\epsilon \Longrightarrow \left|\sum_{n=p+1}^{m} u_n\right| \leq \epsilon$.

Proof. Let $(S_k)_{k\geq 1}$ be the sequence of partial sums associated with the series $\sum_{n=1}^{+\infty} u_n$. Then

$$\sum_{n=1}^{+\infty} u_n \text{ cv} \iff (S_k)_{k\geq 1} \text{ cv} \iff (S_k)_{k\geq 1} \text{ is a Cauchy sequence}$$

$$\iff \begin{cases} \text{for all } \epsilon > 0 \text{ there exists } n_\epsilon \in \mathbb{N} \text{ such that} \\ \text{for all } m, p \in \mathbb{N},\ m \geq p \geq n_\epsilon \Longrightarrow |S_m - S_p| = \left|\sum_{n=p+1}^{m} u_n\right| \leq \epsilon. \end{cases}$$

■

Corollary 11 *Let* $\sum_{n=1}^{+\infty} u_n$ *be a numerical series and let* $p \in \mathbb{N}$. *Then* $\sum_{n=1}^{+\infty} u_n$ *cv if and only if* $\sum_{n=p}^{+\infty} u_n$ *cv.*

Proof. Let $(S_m)_{m\geq 1}$ and $(U_m)_{m\geq p}$ be the sequences of partial sums associated respectively with the series $\sum_{n=1}^{+\infty} u_n$ and $\sum_{n=p}^{+\infty} u_n$. The corollary follows from the fact that for all $m \geq p$ we have $|S_m - S_p| = |U_m - U_p|$. ■

Theorem 12 (telescopic series) *Let $(u_n)_{n\geq 1}$ be a sequence of real numbers. Then the series $\sum_{n=1}^{+\infty}(u_{n+1}-u_n)$ cv if and only if $(u_n)_{n\geq 1}$ cv.*

Proof. Let $(S_m)_{m\geq 1}$ be the sequence of partial sums associated with the series $\sum_{n=1}^{+\infty}(u_{n+1}-u_n)$. We have then $S_m=\sum_{n=1}^{m}(u_{n+1}-u_n)=u_{m+1}-u_1$. This shows that the series $\sum_{n=1}^{+\infty}(u_{n+1}-u_n)$ and the sequence $(u_n)_{n\geq 1}$ have the same nature. ■

Example 13 *For all $n\in\mathbb{N}$, we have*

$$\frac{1}{n(n+1)(n+2)}=-\frac{1}{2}\left(\frac{1}{n}-\frac{1}{n+1}\right)+\frac{1}{2}\left(\frac{1}{n+2}-\frac{1}{n+1}\right).$$

Hnece, for any $m\in\mathbb{N}$, we have

$$\begin{aligned}\sum_{n=1}^{m}\frac{1}{n(n+1)(n+2)} &= -\frac{1}{2}\sum_{n=1}^{m}\left(\frac{1}{n+1}-\frac{1}{n}\right)+\frac{1}{2}\sum_{n=1}^{m}\left(\frac{1}{n+2}-\frac{1}{n+1}\right)\\ &= -\frac{1}{2}\left(\frac{1}{m+1}-\frac{1}{1}\right)+\frac{1}{2}\left(\frac{1}{m+2}-\frac{1}{2}\right)\to\frac{1}{4}\ \text{as } m\to+\infty.\end{aligned}$$

We conclude that $\sum_{n=1}^{+\infty}\frac{1}{n(n+1)(n+2)}$ cv and

$$\sum_{n=1}^{+\infty}\frac{1}{n(n+1)(n+2)}=\lim_{m\to\infty}\sum_{n=1}^{m}\frac{1}{n(n+1)(n+2)}=\frac{1}{4}.$$

2 Positive series

Definition 14 *A numerical series $\sum_{n=1}^{+\infty}u_n$ is said to be positive if there exists $n_0\in\mathbb{N}$ such that $u_n>0$ for all $n\geq n_0$.*

Example 15 *Let $u_n=\frac{n+(-1)^n-3}{n^3+1}$ for all $n\geq 0$. Althought $u_0,...u_3$ are negative real numbers the series $\sum_{n=0}^{+\infty}u_n$ is considered as a positive series since $u_n>0$ for all $n\geq 4$.*

Theorem 16 (Boundedness principle) *Let $\sum_{n=1}^{+\infty}u_n$ be a positive series and let $(S_m)_{m\geq 1}$ be the corresponding sequence of partial sums. Then $\sum_{n=1}^{+\infty}u_n$ cv if and only if the sequence $(S_m)_{m\geq 1}$ is upper bounded.*

Proof. Because $S_{m+1}-S_m=u_{m+1}>0$ for all $m\geq n_0=\sup\{n\in\mathbb{N}:u_n\leq 0\}$, we have

$$\sum_{n=1}^{+\infty}u_n \text{ cv}\Longleftrightarrow(S_m)_{m\geq 1}\text{ cv}\Longleftrightarrow(S_m)_{m\geq 1}\text{ is upper bounded.}$$

■

One needs a lot of technicality to prove that the sequence of partial sums associated to a particular series is upper bounded. So it is very hard to use Theorem 16 in practice. However, Theorem 16 remains useful for showing theoretical results.

Some classical series have the form $\sum_{n=1}^{+\infty}f(n)$, where $f:[1,+\infty)\to(0,+\infty)$ is a continuous function. The following theorem provide sufficient conditions for the convergence of such a kind of series.

Theorem 17 (comparaison with an integral) *Let $f : [1, +\infty) \to (0, +\infty)$ be a nonincreasing continuous function. Then*

(i) $\sum_{n=1}^{+\infty} f(n)$ *cv* $\iff \int_1^{+\infty} f(x)dx$ *cv.*

(ii) *For all $m \in \mathbb{N}$, we have*

$$\int_{m+1}^{+\infty} f(x)dx \leq \sum_{n=m+1}^{+\infty} f(n) \leq \int_m^{+\infty} f(x)dx.$$

Proof. Since the function f is nonincreasing for all $n \in \mathbb{N}$ we have

$$f(n+1) \leq \int_n^{n+1} f(x)dx \leq f(n).$$

Thus, if $(S_m)_{m\geq 1}$ is the sequence of partial sums associated with the series $\sum_{n=1}^{+\infty} f(n)$, then

$$\int_1^{m+1} f(x)dx = \sum_{n=1}^{m} \int_n^{n+1} f(x)dx \leq S_m = \sum_{n=1}^{m} f(n) \leq f(1) + \sum_{n=2}^{m} \int_n^{n+1} f(x)dx = f(1) + \int_2^{m+1} f(x)dx.$$

Clearly, the above estimates show that the series $\sum_{n=1}^{+\infty} f(n)$ and the integral $\int_1^{+\infty} f(x)dx$ have the same nature. Moreover, if the series $\sum_{n=1}^{+\infty} f(n)$ cv then

$$\int_{m+1}^{+\infty} f(x)dx = \sum_{n=m+1}^{+\infty} \int_n^{n+1} f(x)dx \leq \sum_{n=m+1}^{+\infty} f(n) \leq \sum_{n=m+1}^{+\infty} \int_n^{n+1} f(x)dx = \int_m^{+\infty} f(x)dx.$$

■

Example 18 *We apply Theorem 17 to the Riemann series $\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$, $\alpha \in \mathbb{R}$. We distinguish two cases*

- *$\alpha \leq 0$, in this case we have $\lim \frac{1}{n^\alpha} \neq 0$. By Theorem 7 the series $\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$ div.*
- *$\alpha > 0$, in this case setting $f(x) = \frac{1}{x^\alpha}$ we have f decreases on $[1, +\infty)$ and $\int_1^{+\infty} \frac{dx}{x^\alpha}$ cv if and only if $\alpha > 1$.*

At the end, we conclude that $\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$ cv $\iff \alpha > 1$.

Example 19 *Similarly we show that the Bertrand series $\sum_{n=1}^{+\infty} \frac{1}{n^\alpha (\ln n)^\beta}$ cv $\iff \alpha > 1$ or $\alpha = 1$ and $\beta > 1$.*

Theorem 20 (comparaison by inequality) *Let $\sum_{n=1}^{+\infty} u_n$ and $\sum_{n=1}^{+\infty} v_n$ be two positive series and suppose that there exists $n_0 \in \mathbb{N}$ such that $u_n \leq v_n$ for all $n \geq n_0$. Then*

1. $\sum_{n=1}^{+\infty} v_n$ *cv* $\Longrightarrow \sum_{n=1}^{+\infty} u_n$ *cv.*

2. $\sum_{n=1}^{+\infty} u_n$ *div* $\Longrightarrow \sum_{n=1}^{+\infty} v_n$ *div.*

Proof. Let for all $m \geq n_0$ $U_m = \sum_{n=n_0}^{m} u_n$ and $V_m = \sum_{n=n_0}^{m} v_n$. We have then

$$\sum_{n=1}^{+\infty} v_n \text{ cv} \implies \sum_{n=n_0}^{+\infty} v_n \text{ cv} \Longrightarrow (V_m) \text{ cv} \Longrightarrow (U_m) \text{ is bounded } (U_m \leq V_m)$$
$$\Longrightarrow \sum_{n=n_0}^{+\infty} u_n \text{ cv} \Longrightarrow \sum_{n=1}^{+\infty} u_n \text{ cv}.$$

■

Example 21 *Since* $e^{-n^2} \leq e^{-n}$ *and* $\sum_{n=0}^{+\infty} e^{-n} = \dfrac{e}{e-1}$ *(cv), the series* $\sum_{n=0}^{+\infty} e^{-n^2}$ *cv.*

Corollary 22 *Let* $\sum_{n=1}^{+\infty} u_n$ *and* $\sum_{n=1}^{+\infty} v_n$ *be two positive series and suppose that there exist* $a, b > 0$ *and* $n_0 \in \mathbb{N}$ *such that* $a \leq \dfrac{u_n}{v_n} \leq b$ *for all* $n \geq n_0$. *Then the series* $\sum_{n=1}^{+\infty} u_n$ *and* $\sum_{n=1}^{+\infty} v_n$ *have the same nature.*

Proof. We have $av_n \leq u_n \leq bv_n$ for all $n \geq n_0$. Hence, applying Theorem 20 we get

$$\sum_{n=1}^{+\infty} v_n cv \Longrightarrow \sum_{n=1}^{+\infty} bv_n \text{ cv} \Longrightarrow \sum_{n=1}^{+\infty} u_n \text{ cv}$$

and

$$\sum_{n=1}^{+\infty} u_n \text{ cv} \Longrightarrow \sum_{n=1}^{+\infty} av_n \text{ cv} \Longrightarrow \sum_{n=1}^{+\infty} v_n \text{ cv}.$$

■

Corollary 23 *Let* $\sum_{n=1}^{+\infty} u_n$ *and* $\sum_{n=1}^{+\infty} v_n$ *be two positive series and suppose* $\lim \dfrac{u_n}{v_n} = l$, *with* $l \in (0, +\infty)$. *Then the series* $\sum_{n=1}^{+\infty} u_n$ *and* $\sum_{n=1}^{+\infty} v_n$ *have the same nature.*

Proof. By the definition of the limit for $\epsilon = l/2$, there exists $n_0 \in \mathbb{N}$ such that $\dfrac{l}{2} \leq \dfrac{u_n}{v_n} \leq \dfrac{3l}{2}$ for all $n \geq n_0$. Thus the conclusion in Corollary 23 follows from Corollary 22. ■

Corollary 24 *Let* $\sum_{n=1}^{+\infty} u_n$ *and* $\sum_{n=1}^{+\infty} v_n$ *be two positive series and suppose* $u_n \sim v_n$ *at* $+\infty$. *Then the series* $\sum_{n=1}^{+\infty} u_n$ *and* $\sum_{n=1}^{+\infty} v_n$ *have the same nature.*

Proof. Since $u_n \sim v_n$ at $+\infty$ is equivalent to $\lim \dfrac{u_n}{v_n} = 1$. We conclude by Corollary 23 that the series $\sum_{n=1}^{+\infty} u_n$ and $\sum_{n=1}^{+\infty} v_n$ have the same nature. ■

Example 25 *Since* $\sin\left(\dfrac{1}{n}\right) \backsim \dfrac{1}{n}$ *at* $+\infty$ *and* $\sum_{n=1}^{+\infty} \dfrac{1}{n}$ *div, the series* $\sum_{n=1}^{+\infty} \sin\left(\dfrac{1}{n}\right)$ *div.*

Corollary 26 (Riemann criterion) *Let $\sum_{n=1}^{+\infty} u_n$ be a positive series and suppose that there exists $\alpha \in \mathbb{R}$ such that $\lim n^\alpha u_n = l$. Then*

1. *If $l \in [0, +\infty)$ and $\alpha > 1$ then $\sum_{n=1}^{+\infty} u_n$ cv.*
2. *If $l \in (0, +\infty)$ or $l = +\infty$ and $\alpha \leq 1$ then $\sum_{n=1}^{+\infty} u_n$ div.*

Proof. We distinguish the following three cases.

- $l \in (0, +\infty)$, in this case we have $u_n \backsim \dfrac{l}{n^\alpha}$ at $+\infty$ and the series $\sum_{n=1}^{+\infty} u_n$ and $\sum_{n=1}^{+\infty} \dfrac{1}{n^\alpha}$ have the same nature. That is $\sum_{n=1}^{+\infty} u_n$ cv if $\alpha > 1$ and $\sum_{n=1}^{+\infty} u_n$ div if $\alpha \leq 1$.
- $l = 0$ and $\alpha > 1$, in this case there exists $n_1 \in \mathbb{N}$ such that $u_n \leq \dfrac{1}{n^\alpha}$ for all $n \geq n_1$. Since $\sum_{n=1}^{+\infty} \dfrac{1}{n^\alpha}$ cv, the series $\sum_{n=1}^{+\infty} u_n$ cv.
- $l = +\infty$ and $\alpha \leq 1$, in this case there exists $n_1 \in \mathbb{N}$ such that $u_n \geq \dfrac{1}{n^\alpha}$ for all $n \geq n_1$. Since $\sum_{n=1}^{+\infty} \dfrac{1}{n^\alpha}$ div, the series $\sum_{n=1}^{+\infty} u_n$ div.

■

Example 27 *Consider the series $\sum_{n=1}^{+\infty} \dfrac{n^2 + 2 + \sin(n)}{n^4 + n + 1}$. Since $\dfrac{n^2 + 2 + \sin(n)}{n^4 + n + 1} \backsim \dfrac{1}{n^2}$ at $+\infty$ and the series $\sum_{n=1}^{+\infty} \dfrac{1}{n^2}$ cv, the series $\sum_{n=1}^{+\infty} \dfrac{n^2 + 2 + \sin(n)}{n^4 + n + 1}$ cv.*

Theorem 28 (logarithmic comparaison) *Let $\sum_{n=1}^{+\infty} u_n$ and $\sum_{n=1}^{+\infty} v_n$ be two positive series and suppose that there exists $n_0 \in \mathbb{N}$ such that $\dfrac{u_{n+1}}{u_n} \leq \dfrac{v_{n+1}}{v_n}$ for all $n \geq n_0$. Then*

1. *$\sum_{n=1}^{+\infty} v_n$ cv $\Longrightarrow \sum_{n=1}^{+\infty} u_n$ cv.*
2. *$\sum_{n=1}^{+\infty} u_n$ div $\Longrightarrow \sum_{n=1}^{+\infty} v_n$ div.*

Proof. For all $n > n_0$ we have

$$\frac{u_n}{u_{n_0}} = \frac{u_n}{u_{n-1}} \frac{u_{n-1}}{u_{n-2}} \cdots \frac{u_{n_0+1}}{u_{n_0}} \leq \frac{v_n}{v_{n-1}} \frac{v_{n-1}}{v_{n-2}} \cdots \frac{v_{n_0+1}}{v_{n_0}} = \frac{v_n}{v_{n_0}},$$

leading to $u_n \leq \dfrac{u_{n_0}}{v_{n_0}} v_n$ for all $n \geq n_0$.

By Theorem 20, we deduce the implication in 1 and its contrapositive in 2. ■

Theorem 29 (d'Alembert criterion) *Let $\sum_{n=1}^{+\infty} u_n$ be a positive series and suppose $\lim \dfrac{u_{n+1}}{u_n} = l$. Then*

1. *$l < 1 \Longrightarrow \sum_{n=1}^{+\infty} u_n$ cv.*

2. $l > 1 \Longrightarrow \sum_{n=1}^{+\infty} u_n$ *div.*

Proof. First, suppose that $l < 1$ and let $\epsilon > 0$ be such that $l + \epsilon < 1$. Then there exists $n_0 \in \mathbb{N}$ such that $\dfrac{u_{n+1}}{u_n} \leq (l+\epsilon) = \dfrac{v_{n+1}}{v_n}$ for all $n \geq n_0$, where for $v_n = (l+\epsilon)^n$. Since the series $\sum_{n=1}^{+\infty} v_n$ cv, we deduce from Theorem 28 that the series $\sum_{n=1}^{+\infty} u_n$ cv.

If $l > 1$, then choosing $\epsilon > 0$ such that $l - \epsilon > 1$ there exists $n_0 \in \mathbb{N}$ such that $\dfrac{u_{n+1}}{u_n} \geq (l-\epsilon) = \dfrac{v_{n+1}}{v_n}$ for all $n \geq n_0$, where for $v_n = (l-\epsilon)^n$. Since the series $\sum_{n=1}^{+\infty} v_n$ div, we deduce from Theorem 28 that the series $\sum_{n=1}^{+\infty} u_n$ div. ■

Example 30 *Consider the series* $\sum_{n=1}^{+\infty} u_n$, *where* $u_n = \dfrac{n!}{n^n}$. *We have*

$$\lim \frac{u_{n+1}}{u_n} = \lim \frac{(n+1)!}{(n+1)^{n+1}} \frac{n^n}{n!} = \lim \left(1 + \frac{1}{n}\right)^{-n} = e^{-1} < 1.$$

By d'Alembert criterion the series $\sum_{n=1}^{+\infty} u_n$ *cv.*

Theorem 31 (Cauchy criterion) *Let* $\sum_{n=1}^{+\infty} u_n$ *be a positive series and suppose* $\lim \sqrt[n]{u_n} = l$. *Then*

1. $l < 1 \Longrightarrow \sum_{n=1}^{+\infty} u_n$ *cv.*

2. $l > 1 \Longrightarrow \sum_{n=1}^{+\infty} u_n$ *div.*

Proof. First, suppose that $l < 1$ and let $\epsilon > 0$ be such that $l + \epsilon < 1$. Then there exists $n_0 \in \mathbb{N}$ such that $u_n \leq (l+\epsilon)^n$ for all $n \geq n_0$. Since the series $\sum_{n=1}^{+\infty} (l+\epsilon)^n$ cv, we deduce from Theorem 20 that the series $\sum_{n=1}^{+\infty} u_n$ cv.

If $l > 1$, then choosing $\epsilon > 0$ such that $l - \epsilon > 1$ there exists $n_0 \in \mathbb{N}$ such that $u_n \geq (l-\epsilon)^n$ for all $n \geq n_0$. Since the series $\sum_{n=1}^{+\infty} (l-\epsilon)^n$ div, we deduce from Theorem 20 that the series $\sum_{n=1}^{+\infty} u_n$ div. ■

Example 32 *Consider the series* $\sum_{n=1}^{+\infty} u_n$, *where* $u_n = \left((-1)^n + 2\right) 2^{-n}$. *We have*

$$\lim \sqrt[n]{u_n} = \frac{1}{2} \lim \sqrt[n]{(-1)^n + 2} = \frac{1}{2} \lim \exp\left(\frac{\ln\left(2 + (-1)^n\right)}{n}\right) = \frac{1}{2} < 1.$$

By Cauchy criterion the series $\sum_{n=1}^{+\infty} u_n$ *cv.*

Comments. Let $\sum_{n=1}^{+\infty} u_n$ be a positive series.

1. If $\lim \dfrac{u_{n+1}}{u_n} = l$, then $\lim \sqrt[n]{u_n} = l$. Indeed, let $\epsilon > 0$ be arbitrary. There exists $n_\epsilon \in \mathbb{N}$ such that $(l-\epsilon)\, u_n \leq u_{n+1} \leq (l+\epsilon)\, u_n$ for all $n \geq n_\epsilon$. Therefore, we have

$$(l-\epsilon)^{n-n_\epsilon} u_{n_\epsilon} \leq u_n \leq (l+\epsilon)^{n-n_\epsilon} u_{n_\epsilon} \text{ for all } n \geq n_\epsilon.$$

This leads to

$$l-\epsilon = \liminf (l-\epsilon)^{1-\frac{n_\epsilon}{n}} \sqrt[n]{u_{n_\epsilon}} \leq \liminf \sqrt[n]{u_n} \leq \limsup \sqrt[n]{u_n} \leq \limsup (l+\epsilon)^{1-\frac{n_\epsilon}{n}} \sqrt[n]{u_{n_\epsilon}} = l+\epsilon.$$

Since ϵ is arbitrary, we conclude that

$$\liminf \sqrt[n]{u_n} = \limsup \sqrt[n]{u_n} = \lim \sqrt[n]{u_n} = l.$$

2. The inverse implication is false. Indeed, considering the series $\sum_{n=1}^{+\infty} u_n$ where $u_n = \begin{cases} a^k b^k, \text{ if } n = 2k, \\ a^k b^{k+1}, \text{ if } n = 2k+1, \end{cases}$ a and b are distinctive positive real numbers. We have $\lim \sqrt[n]{u_n} = \sqrt{ab}$, however $\limsup \dfrac{u_{n+1}}{u_n} = \max(a,b) \neq \liminf \dfrac{u_{n+1}}{u_n} = \min(a,b)$.

3. Arguing as in the proofs of CorTheorems 29 and 31 we prove the following implications

$$\begin{aligned} \limsup \frac{u_{n+1}}{u_n} &< 1 \Longrightarrow \sum_{n=1}^{+\infty} u_n \text{ cv}, \\ \liminf \frac{u_{n+1}}{u_n} &> 1 \Longrightarrow \sum_{n=1}^{+\infty} u_n \text{ div}, \\ \limsup \sqrt[n]{u_n} &< 1 \Longrightarrow \sum_{n=1}^{+\infty} u_n \text{ cv}, \\ \liminf \sqrt[n]{u_n} &> 1 \Longrightarrow \sum_{n=1}^{+\infty} u_n \text{ div}. \end{aligned}$$

D'Alembert criterion do not answer to the case $\lim \dfrac{u_{n+1}}{u_n} = 1$. The following two results concerns this particular situation.

Theorem 33 (Duhamel-Raabe criterion) *Let $\sum_{n=1}^{+\infty} u_n$ be a positive series and suppose that*

$$\frac{u_{n+1}}{u_n} = 1 - \frac{l}{n} + \circ\left(\frac{1}{n}\right) \text{ at } +\infty.$$

Then

1. $l > 1 \Longrightarrow \sum_{n=1}^{+\infty} u_n$ *cv,*
2. $l < 1 \Longrightarrow \sum_{n=1}^{+\infty} u_n$ *div.*

Proof. Suppose that $l > 1$. Let $\alpha \in (1, l)$ and set $v_n = \dfrac{1}{n^\alpha}$. We have then

$$\frac{v_{n+1}}{v_n} = \left(1 + \frac{1}{n}\right)^{-\alpha} = 1 - \frac{\alpha}{n} + \circ\left(\frac{1}{n}\right) \text{ at } +\infty$$

and

$$\lim n \left(\frac{v_{n+1}}{v_n} - \frac{u_{n+1}}{u_n}\right) = l - \alpha > 0.$$

Hence, there exists $n_0 \in \mathbb{N}$ such that $\dfrac{u_{n+1}}{u_n} \leq \dfrac{v_{n+1}}{v_n}$ for all $n \geq n_0$. Since $\sum_{n=1}^{+\infty} v_n$ cv, Theorem 28 ensures the convergence of the series $\sum_{n=1}^{+\infty} u_n$.

Similarly, if $l<1$ considering $v_n=\dfrac{1}{n^\alpha}$ with $\alpha\in(l,1)$, we get and

$$\lim n\left(\frac{u_{n+1}}{u_n}-\frac{v_{n+1}}{v_n}\right)=\alpha-l>0.$$

Hence, there exists $n_0\in\mathbb{N}$ such that $\dfrac{v_{n+1}}{v_n}\leq\dfrac{u_{n+1}}{u_n}$ for all $n\geq n_0$. Since $\sum_{n=1}^{+\infty}v_n$ div, Theorem 28 ensures the divergence of the series $\sum_{n=1}^{+\infty}u_n$. ■

Example 34 *Consider the series* $\sum_{n=1}^{+\infty}u_n$, *where* $u_n=\dfrac{n!}{(a+1)(a+2)\cdots(a+n)}$ *and* $a>0$. *We have*

$$\begin{aligned}\frac{u_{n+1}}{u_n} &= \frac{(n+1)!}{(a+1)(a+2)\cdots(a+n+1)}\frac{(a+1)(a+2)\cdots(a+n)}{n!}=\left(1-\frac{a}{a+n+1}\right)\\ &= 1-\frac{a}{n}+\circ\left(\frac{1}{n}\right).\end{aligned}$$

By Duhamel-Raabe criterion we conclude that the series $\sum_{n=1}^{+\infty}u_n$ *cv if* $a>1$ *and div* $a<1$.

Theorem 35 (Gauss criterion) *Let* $\sum_{n=1}^{+\infty}u_n$ *be a positive series and suppose that there exists* $\alpha>1$ *such that*

$$\frac{u_{n+1}}{u_n}=1-\frac{l}{n}+O\left(\frac{1}{n^\alpha}\right)\ \ at\ +\infty.$$

Then

1. $l>1\Longrightarrow\sum_{n=1}^{+\infty}u_n$ *cv,*
2. $l\leq1\Longrightarrow\sum_{n=1}^{+\infty}u_n$ *div.*

Proof. Let $v_n=n^lu_n$. We have

$$\begin{aligned}\frac{v_{n+1}}{v_n} &= \left(1-\frac{l}{n}+O\left(\frac{1}{n^2}\right)\right)\left(1-\frac{l}{n}+O\left(\frac{1}{n^\alpha}\right)\right)\\ &= 1+O\left(\frac{1}{n^\beta}\right),\ \text{where}\ \beta=\min(2,\alpha).\end{aligned}$$

This leads to

$$\ln\left(\frac{v_{n+1}}{v_n}\right)=O\left(\frac{1}{n^\beta}\right)$$

and there exist $M>0$ and $n_0\in\mathbb{N}$ such that $\left|\ln\left(\dfrac{v_{n+1}}{v_n}\right)\right|\leq\dfrac{M}{n^\beta}$ for all $n\geq n_0$. Therefore, $\sum_{n=1}^{+\infty}\ln\left(\dfrac{v_{n+1}}{v_n}\right)$ and obviously, $\sum_{n=1}^{m}\ln\left(\dfrac{v_{n+1}}{v_n}\right)=\ln(v_{m+1})-\ln(v_1)\to\theta$ as $m\to+\infty$. This shows that there is $k>0$ such that $u_n\backsim\dfrac{k}{n^l}$. The conclusion of Theorem 35 Riemann criterion (Corollary 26). ■

Example 36 *Consider the series* $\sum_{n=1}^{+\infty} u_n$, *where* $u_n = \dfrac{n!e^n}{n^{n+p}}$ *and* $p \in \mathbb{R}$. *We have*

$$\begin{aligned}\frac{u_{n+1}}{u_n} &= \frac{(n+1)!e^{n+1}}{(n+1)^{n+p+1}}\frac{n^{n+p}}{n!e^n} = e\left(1+\frac{1}{n}\right)^{-n-p} \\ &= e\exp\left(-(n+p)\ln\left(1+\frac{1}{n}\right)\right) \\ &= 1+\frac{p-\frac{1}{2}}{n}+O\left(\frac{1}{n^2}\right).\end{aligned}$$

By Gauss criterion we conclude that the series $\sum_{n=1}^{+\infty} u_n$ *cv if* $p > \dfrac{3}{2}$ *and div if* $p \leq \dfrac{3}{2}$.

Example 37 *Consider the series* $\sum_{n=1}^{+\infty} u_n$, *where* $u_n = \dfrac{\prod_{k=1}^{n}(2k-1)}{\prod_{k=1}^{n}2k}\dfrac{1}{\sqrt{n}}$ *and* $p \in \mathbb{R}$. *We have*

$$\begin{aligned}\frac{u_{n+1}}{u_n} &= \frac{\prod_{k=1}^{n+1}(2k-1)}{\prod_{k=1}^{n+1}2k}\frac{\sqrt{n}}{\sqrt{n+1}}\frac{\prod_{k=1}^{n}2k}{\prod_{k=1}^{n}(2k-1)} = \frac{2n+1}{2n}\left(1+\frac{1}{n}\right)^{-1/2} \\ &= 1-\frac{1}{n}+O\left(\frac{1}{n^2}\right).\end{aligned}$$

By Gauss criterion we conclude that the series $\sum_{n=1}^{+\infty} u_n$ *div.*

3 Alternating series

Definition 38 *A numerical series* $\sum_{n=1}^{+\infty} u_n$ *is said to be an alternating series if its general term* u_n *changes sign infinetly many times.*

Example 39 $\sum_{n=1}^{+\infty} \dfrac{(-1)^n}{n}$ *and* $\sum_{n=1}^{+\infty} \dfrac{\cos(n)}{n}$ *are alternating series.*

Definition 40 *A numerical series* $\sum_{n=1}^{+\infty} u_n$ *is said to be absolutely converging (acv for short), if the positive series* $\sum_{n=1}^{+\infty} |u_n|$ *cv.*

Theorem 41 *Let* $\sum_{n=1}^{+\infty} u_n$ *be a numerical series. Then*

$$\sum_{n=1}^{+\infty} u_n \ acv \implies \sum_{n=1}^{+\infty} u_n \ cv.$$

Proof. Taking in consideration that for all $m, p \in \mathbb{N}$ with $m > p$

$$\left|\sum_{n=p+1}^{m} u_n\right| \leq \sum_{n=p+1}^{m} |u_n|,$$

we obtain the following hierarchy,

$$\sum_{n=1}^{+\infty} u_n \text{ acv} \implies \sum_{n=1}^{+\infty} |u_n| \text{ cv} \implies \left(\sum_{n=1}^{m} |u_n|\right)_{m\geq 1} \text{ is of Cauchy} \implies \left(\sum_{n=1}^{m} u_n\right)_{m\geq 1} \text{ is of Cauchy}$$
$$\implies \left(\sum_{n=1}^{m} u_n\right)_{m\geq 1} \text{ cv} \implies \left(\sum_{n=1}^{+\infty} u_n\right)_{m\geq 1} \text{ cv.}$$

■

Example 42 *Consider the series* $\sum_{n=1}^{+\infty} u_n$, *where* $u_n = \dfrac{\cos(n)}{n^2}$. *Since*

$$|u_n| = \frac{|\cos(n)|}{n^2} \leq \frac{|\cos(n)|}{n^2} \text{ and } \sum_{n=1}^{+\infty} \frac{1}{n^2} \text{ cv,}$$

Theorem 20 ensures the convergence of the series $\sum_{n=1}^{+\infty} u_n$.

Theorem 43 (Leibniz criterion) *Let* $(u_n)_{n\geq 1}$ *be a positive sequence such that* $(u_n)_{n\geq 1}$ *is non-increasing with* $\lim u_n = 0$. *Then* $\sum_{n=1}^{+\infty} (-1)^n u_n$ *cv.*

Proof. Consider the subsequences $(S_{2m})_{m\geq 1}$ and $(S_{2m+1})_{m\geq 0}$ where $(S_m)_{m\geq 1}$ is the sequence of partial sums associated with the series $\sum_{n=1}^{+\infty} (-1)^n u_n$. The sequences $(S_{2m})_{m\geq 1}$ and $(S_{2m+1})_{m\geq 0}$ are adjacent since

$$\begin{aligned} S_{2m+2} - S_{2m} &= u_{2m+2} - u_{2m+1} \leq 0, \\ S_{2m+3} - S_{2m+1} &= -u_{2m+3} + u_{2m+2} \geq 0 \text{ and} \\ S_{2m+1} - S_{2m} &= -u_{2m+1} \to 0 \text{ as } m \to +\infty. \end{aligned}$$

Hence, $(S_{2m})_{m\geq 1}$ and $(S_{2m+1})_{m\geq 0}$ are convergent and have the same limit. This proves that $(S_m)_{m\geq 1}$ cv and so is $\sum_{n=1}^{+\infty} (-1)^n u_n$. ■

Example 44 *Since for any* $\alpha > 0$, $\dfrac{1}{n^\alpha}$ *decreases and* $\lim \dfrac{1}{n^\alpha} = 0$ *if and only if* $\alpha > 0$, *we have*

$$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^\alpha} \text{ cv} \iff \alpha > 0.$$

Theorem 45 (Abel criterion) *Let* $(u_n)_{n\geq 1}$ *and* $(v_n)_{n\geq 1}$ *be two sequences satisfying*

1. $\exists M > 0$ *such that for all* $m \in \mathbb{N}$ *we have* $|\sum_{n=1}^{m} u_n| \leq M$,
2. $\sum_{n=1}^{+\infty} |v_n - v_{n+1}|$ *cv,*
3. $\lim v_n = 0$.

Then the series $\sum_{n=1}^{+\infty} u_n v_n$ *cv.*

Proof. Let $(U_m)_{m\geq 1}$ be the sequence of partial sums associated with $(u_m)_{m\geq 1}$. For all $k, p \in \mathbb{N}$ we have

$$\begin{aligned}\sum_{n=k+1}^{k+p} u_n v_n &= \sum_{n=k+1}^{k+p} (U_n - U_{n-1}) v_n = \sum_{n=k+1}^{k+p} U_n v_n - \sum_{n=k}^{k+p-1} U_n v_{n+1} \\ &= \sum_{n=k+1}^{k+p} U_n (v_n - v_{n+1}) + U_{k+p} v_{k+p} - U_k v_{k+1}.\end{aligned}$$

Let $\epsilon > 0$, there exists $n_\epsilon \in \mathbb{N}$ such that for all $k \geq n_\epsilon$ and all $p \in \mathbb{N}$ we have

$$\left| \sum_{n=k+1}^{k+p} (v_n - v_{n+1}) \right| \leq \frac{\epsilon}{3M}, \ \ |v_{k+p}| \leq \frac{\epsilon}{3M} \text{ and } |v_{k+1}| \leq \frac{\epsilon}{3M}.$$

This together with Condition 1 in Theorem 45 leads to

$$\left| \sum_{n=k+1}^{k+p} u_n v_n \right| \leq M \left| \sum_{n=k+1}^{k+p} (v_n - v_{n+1}) \right| + M |v_{k+p}| + M |v_{k+1}| \leq \epsilon \text{ for all } k \geq n_\epsilon \text{ and all } p \in \mathbb{N}.$$

Hence, we conclude by Theorem 10 that the series $\sum_{n=1}^{+\infty} u_n v_n$ cv. ■

Example 46 *Consider the Abel series* $\sum_{n=1}^{+\infty} u_n v_n$, *where* $u_n = \cos(n\beta)$ *and* $v_n = \dfrac{1}{n^\alpha}$ *with* $\beta \neq (2k+1)\pi$. *We have*

$$\begin{aligned}\left| \sum_{n=1}^{m} u_n \right| &\leq \frac{1}{|\cos(\beta/2)|} \text{ for all } m \in \mathbb{N}, \\ \lim \frac{1}{n^\alpha} &= 0 \Longleftrightarrow \alpha > 0, \\ \lim \frac{\cos(n\beta)}{n^\alpha} &\neq 0 \Longleftrightarrow \alpha \leq 0, \\ |v_n - v_{n+1}| &\backsim \frac{\alpha}{n^{\alpha+1}} \text{ at } +\infty.\end{aligned}$$

All the above calculations show that $\sum_{n=1}^{+\infty} u_n v_n$ *cv if and only if* $\alpha > 0$.

Similarly for the series $\sum_{n=1}^{+\infty} \dfrac{\sin(n\beta)}{n^\alpha}$ *with* $\beta \neq 2k\pi$, *we prove that the series* $\sum_{n=1}^{+\infty} \dfrac{\sin(n\beta)}{n^\alpha}$ *cv if and only if* $\alpha > 0$.

Corollary 47 *Let* $(u_n)_{n\geq 1}$ *and* $(v_n)_{n\geq 1}$ *be two sequences satisfying*

Theorem 48 *1.* $\sum_{n=1}^{+\infty} u_n$ *cv,*

2. $(v_n)_{n\geq 1}$ *is monotonic and cv.*

Then the series $\sum_{n=1}^{+\infty} u_n v_n$ *cv.*

Proof. Set $v = \lim v_n$. By Abel criterion the series $\sum_{n=1}^{+\infty} u_n (v_n - v)$ cv. So

$$\sum_{n=1}^{+\infty} u_n v_n = \sum_{n=1}^{+\infty} u_n (v_n - v) + v \sum_{n=1}^{+\infty} u_n \text{ cv.}$$

■

Using asymptotic developpement.

Consider the series $\sum_{n=1}^{+\infty} u_n$ where $u_n = \dfrac{(-1)^n}{\sqrt{n} + (-1)^n}$. We have $u_n = \dfrac{\frac{(-1)^n}{\sqrt{n}}}{1 + \frac{(-1)^n}{\sqrt{n}}} = f\left(\frac{(-1)^n}{\sqrt{n}}\right)$, where $f(x) = \dfrac{x}{1+x}$.

Since $\frac{(-1)^n}{\sqrt{n}} \to 0$ as $n \to +\infty$ and near 0, $f(x) = x - x^2 + x^3 + \circ(x^3)$, we have

$$u_n = \frac{(-1)^n}{\sqrt{n}} - \frac{1}{n} + \frac{(-1)^n}{n\sqrt{n}} + \circ\left(\frac{1}{n\sqrt{n}}\right).$$

Because

- $\sum_{n=1}^{+\infty} \frac{(-1)^n}{\sqrt{n}}$ cv (by Leibniz criterion),
- $\sum_{n=1}^{+\infty} \frac{1}{n}$ div (Riemann series with $\alpha = 1$),
- $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n\sqrt{n}}$ acv ($\left|\frac{(-1)^n}{n\sqrt{n}}\right| \leq \frac{1}{n\sqrt{n}}$ and $\sum_{n=1}^{+\infty} \frac{1}{n\sqrt{n}}$ cv $) \Longrightarrow \sum_{n=1}^{+\infty} \frac{(-1)^n}{n\sqrt{n}}$ cv,
- $\sum_{n=1}^{+\infty} \circ\left(\frac{(-1)^n}{n\sqrt{n}}\right)$ acv ($\left|\circ\left(\frac{(-1)^n}{n\sqrt{n}}\right)\right| \leq \frac{1}{n\sqrt{n}}$ and $\sum_{n=1}^{+\infty} \frac{1}{n\sqrt{n}}$ cv $) \Longrightarrow \sum_{n=1}^{+\infty} \circ\left(\frac{(-1)^n}{n\sqrt{n}}\right)$ cv,

the series $\sum_{n=1}^{+\infty} u_n$ cv.

Exercise 49 *Determine the nature of the series* $\sum_{n=0}^{+\infty} \dfrac{\cos(n)}{n + \cos(n)}$.