TD 04

Exercise 1

We equip $\mathbb{R}$ with the internal composition law defined as :

$$\forall x, y \in \mathbb{R}^+ : x \star y = \sqrt{x^2 + y^2}.$$

1. Show that $\star$ is commutative, associative, and has a neutral element.
2. Determine the symmetrizable elements.

Exercise 2

Show that $(G, \star)$ is a group and specify whether it is abelian (commutative) :

$$x \star y = \frac{x + y}{1 + xy}, \text{ on } G = (-1, 1).$$

Exercise 3

Let be the set $\mathbb{R}^* \times \mathbb{R}$ provided with internal law $\star$ such that

$$(a, b) \star (\alpha, \beta) = (a\alpha, \frac{\beta}{a} + b\alpha)$$

Show that $(\mathbb{R}^* \times \mathbb{R}, \star)$ is a group, is it commutative ?

Exercise 4

Let $(G, +)$ be a commutative group. We denote $End(G)$ as the set of endomorphisms of G on which we define the operation $+$ as :

$$\begin{aligned} f + g : \quad G &\to G \\ x &\mapsto f(x) + g(x). \end{aligned}$$

Prove that $(End(G), +, \circ)$ is a ring.

Exercise 5

1. Determine if part H is a subgroup of group G.
   (a) $G = (\mathbb{Z}, +)$; $H = \{\text{even numbers}\}$
   (b) $G = (\mathbb{Z}, +)$; $H = \{\text{odd numbers}\}$.
2. Show that $U = \{z \in \mathbb{C}, |z| = 1\}$ equipped with multiplication is a subgroup of $(\mathbb{C}^*, \times)$.

Solutions of TD 4

Solution of exercise 1

1. It is clear that $\star$ is commutative and associative, and accepts a neutral element $e = 0$,
$$x \star 0 = \sqrt{x^2 + 0^2} = |x| = x, \text{ as } x > 0.$$
2. Since $\star$ admits a neutral element, we look for symmetrizable elements if they exist. Suppose y is the symmetric of x. Then,
$$\forall x \in \mathbb{R}^+; \quad x \star y = 0 \Rightarrow \sqrt{x^2 + y^2} = 0 \Rightarrow x^2 = -y^2,$$
which is impossible. Thus, $\nexists y \in \mathbb{R}^+$ such that $x \star y = 0$. Therefore, no element of $\mathbb{R}^{+*}$ accepts a symmetric element. The only symmetrizable element with respect to $\star$ is the neutral element $e = 0$ (the neutral element is symmetrical to itself).

Solution of exercise 2

$$x \star y = \frac{x + y}{1 + xy} \quad \text{on} \quad G =]-1, 1[.$$

1. According to the question, it's not announced that $\star$ is an internal composition law on G. Therefore, we first verify that $\star$ is indeed a well-defined internal composition law on G, i.e., if $x, y \in G$, then $x \star y \in G$:
Let's study the function defined on $]-1; 1[$ by $f(t) = \frac{t+y}{1+ty}$. It's differentiable on $[-1; 1]$, and its derivative satisfies $f'(t) = \dfrac{1 - y^2}{(1 + ty)^2} > 0$ on $]-1; 1[$. Thus, f is strictly increasing on $[-1; 1]$.
Since
$$f(-1) = \frac{(-1 + y)}{(1 - y)} = -1,$$
and
$$f(1) = \frac{(1 + y)}{(1 + y)} = 1.$$
Then,
$$f(-1) < x \star y = f(x) < f(1),$$
hence $x \star y \in G$.
2. The law is associative : for all
$$x, y, z \in G, \quad x \star (y \star z) = \frac{x + (y \star z)}{1 + x(y \star z)} = \frac{x + \frac{y+z}{1+yz}}{1 + x\frac{y+z}{1+yz}} = \frac{x + y + z + xyz}{1 + xy + xz + yz},$$
and a similar calculation yields the same result for $(x \star y) \star z$.
3. e is a neutral element for the law, so for all $x \in G$, $x \star e = x \Rightarrow x + e = x + xe^2$, which implies $e(1 - x^2) = 0$, hence $e = 0$. Thus, it admits a neutral element $e = 0$.
4. Every element $x \in G$ is symmetrizable, and the symmetrical element is $-x$. Indeed, $x \star (-x) = (-x) \star x = 0$. Moreover, the law is clearly abelian because for all $x; y \in G$, $x \star y = \frac{x+y}{1+xy} = y \star x$. Therefore, $(G; \star)$ is an abelian group.

Solution of exercise 3

1)

Let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in \mathbb{R}^* \times \mathbb{R}$
$[(a_1, b_1) \star (a_2, b_2)] \star (a_3, b_3) = (a_1a_2, \frac{b_2}{a_1} + b_1a_2) \star (a_3, b_3)$

$= (a_1a_2a_3, \frac{b_3}{a_1a_2} + (\frac{b_2}{a_1} + b_1a_2)a_3)$
$= (a_1a_2a_3, \frac{b_3}{a_1a_2} + \frac{b_2a_3}{a_1} + b_1a_2a_3)$

$(a_1, b_1) \star [(a_2, b_2) \star (a_3, b_3)] = (a_1, b_1) \star (a_2a_3, \frac{b_3}{a_2} + b_2a_3)$
$= (a_1a_2a_3, \frac{\frac{b_3}{a_2} + b_2a_3}{a_1} + b_1a_2a_3)$
$= (a_1a_2a_3, \frac{b_3}{a_1a_2} + \frac{b_2a_3}{a_1} + b_1a_2a_3)$

Then, $[(a_1, b_1) \star (a_2, b_2)] \star (a_3, b_3) = (a_1, b_1) \star [(a_2, b_2) \star (a_3, b_3)]$.
therefore the law $\star$ is an associative
2)
$\exists?(\alpha, \beta) \in \mathbb{R}^* \times \mathbb{R}, \forall (a, b) \in \mathbb{R}^* \times \mathbb{R} : (a, b) \star (\alpha, \beta) = (a, b)$
$(a, b) \star (\alpha, \beta) = (a, b) \Rightarrow (a\alpha, \frac{\beta}{a} + b\alpha) = (a, b)$

$$\Rightarrow \begin{cases} a\alpha = a \\ \wedge \\ \frac{\beta}{a} + b\alpha = b \end{cases}$$
$$\Rightarrow \begin{cases} \alpha = \frac{a}{a} = 1 \\ \wedge \\ \beta + b = b \end{cases}$$
$$\Rightarrow \begin{cases} \alpha = \frac{a}{a} = 1 \\ \wedge \\ \beta = 0 \end{cases}$$

Then $e = (1, 0)$ is a neutral element in $(\mathbb{R}^* \times \mathbb{R}, \star)$
3) $\forall (a, b) \in \mathbb{R}^* \times \mathbb{R}, \exists?(\alpha, \beta) \in \mathbb{R}^* \times \mathbb{R} : (a, b) \star (\alpha, \beta) = (1, 0)$
$(a, b) \star (\alpha, \beta) = (1, 0) \Rightarrow (a\alpha, \frac{\beta}{a} + b\alpha) = (1, 0)$

$$\Rightarrow \begin{cases} a\alpha = 1 \\ \wedge \\ \frac{\beta}{a} + b\alpha = 0 \end{cases}$$
$$\Rightarrow \begin{cases} \alpha = \frac{1}{a} \\ \wedge \\ \frac{\beta}{a} + \frac{b}{a} = 0 \end{cases}$$
$$\Rightarrow \begin{cases} \alpha = \frac{1}{a} \\ \wedge \\ \beta = -b \end{cases}$$

Then $(\mathbb{R}^* \times \mathbb{R}, \star)$ is a group.
4) $\star$ is not commutative because :
$\exists\ (1, 3), (2, 0) \in \mathbb{R}^* \times \mathbb{R} : (1, 3) \star (2, 0) \neq (2, 0) \star (1, 3)$.

Solution of exercise 4

First, notice that $(+)$ and $(\circ)$ are indeed internal composition laws on $End(G)$.

1. $(End(G), +)$ is a commutative group. Indeed, the law $(+)$ is associative, the function $0_G : G \to G;\ g \to 0$ is a neutral element for $(+)$, and every element $f \in End(G)$ has an inverse $-f : G \to G;\ x \mapsto -f(x)$.
2. The law is associative.
3. The law $(\circ)$ is distributive with respect to $(+)$. For all $f, g, h \in End(G)$ and all $x \in G$,

$$((f + g) \circ h)(x) = (f + g)(h(x)) = f(h(x)) + g(h(x)) = (f \circ h + g \circ h)(x).$$

Thus, $(End(G); +; \circ)$ forms a ring.

Solution of exercise 5

1. **(a)**
H is a subgroup of G. Indeed, $0 \in H$, if $x, y \in H$, then $-x$ and $x+y$ are two even integers and therefore $-x \in H$, $x + y \in H$
The subgroup characterization theorem tells us that H is a subgroup of G.
(b)
$0 \notin H$, and therefore H is not a subgroup of G.

2. $|z| = 1$ then $1 \in U \Rightarrow U \neq \phi$
   Let's $z_{1,}\ z_2\ \ \in U$ and therefore $|z_1| = 1$ and $|z_2| = 1$
   $|z_1 z_2^{-1}| = |\frac{z_1}{z_2}| = |\frac{1}{1}| = 1.$
   then $z_1 z_2^{-1} \in U$
   So, $(U, \times)$ is a subgroup of $(\mathbb{C}^*, \times\ )$