

TD 03

💡 Exercise 1

Let P^* be the set of prime numbers strictly greater than 2.
We consider the relation \mathcal{R} between two elements of P^* defined as :

$$p\mathcal{R}q \Leftrightarrow \frac{p+q}{2} \in P^*.$$

Is the relation \mathcal{R} reflexive, symmetric, and transitive ?

💡 Exercise 2

We define the relation \mathcal{R} on \mathbb{R}^2 by :

$$(x, y)\mathcal{R}(x', y') \Leftrightarrow x + y = x' + y'$$

1. Show that \mathcal{R} is an equivalence relation.
2. Find the equivalence class of the couple $(0, 0)$.

💡 Exercise 3

Let R is a relation defined as

$$\forall x, y \in \mathbb{R}, xRy \Leftrightarrow x^2 - x = y^2 - y,$$

- 1) Show that R is an equivalence relation.
- 2) Find the following equivalence classes : $\dot{0}$, $\dot{1}$, $\dot{2}$, $\dot{\frac{1}{2}}$.

💡 Exercise 4

We define the relation \mathcal{T} on \mathbb{R}^2 by

$$(x, y)\mathcal{T}(x', y') \Leftrightarrow |x - x'| \leq y' - y.$$

1. Verify that \mathcal{T} is an order relation. Is this order total ?
2. Let $(a, b) \in \mathbb{R}^2$ represent the set $\{(x, y) \in \mathbb{R}^2 / (x, y)\mathcal{T}(a, b)\}$.

Solutions of TD 03

Solution of exercise 1

1. For all $p \in P^*$, $\frac{p+p}{2} = p \in P^*$, so $p\mathcal{R}p$; hence, \mathcal{R} is reflexive.
2. For all $p, q \in P^*$, $p\mathcal{R}q$ implies $\frac{p+q}{2} \in P^*$, which in turn implies $\frac{q+p}{2} \in P^*$, so $q\mathcal{R}p$, making \mathcal{R} symmetric.
3. For all $p, q, r \in P^*$, if $p\mathcal{R}q$ and $q\mathcal{R}r$, then it follows that $\frac{p+q}{2} \in P^*$ and $\frac{q+r}{2} \in P^*$. However, we cannot always deduce that $\frac{p+r}{2} \in P^*$.

To illustrate, consider a counterexample in the form of a table. In this table, the first row contains values of q , the first column contains values of p , and the values inside represent $\frac{p+q}{2}$:

$p \backslash q$	3	5	7	11	13	17
3	3	4	5	7	8	10
5	4	5	6	8	9	11
7	5	6	7	9	10	12
11	7	8	9	11	12	14
13	8	9	10	12	13	15
17	10	11	12	14	15	17

In this table, for example, $11\mathcal{R}3$ and $3\mathcal{R}7$, but 11 is not in relation with 7. The cited example in this table is the only counterexample. To find more, you would need to create a larger table. Thus, \mathcal{R} is not transitive.

Solution of exercise 2

1. \mathcal{R} is an equivalence relation if and only if it is reflexive, symmetric and transitive

1)

a) \mathcal{R} is reflexive if and only if $\forall (x, y) \in \mathbb{R}^2 : (x, y)\mathcal{R}(x, y)$

$$(x, y)\mathcal{R}(x, y) \Leftrightarrow x + y = x + y$$

Hence \mathcal{R} is reflexive.

b) \mathcal{R} is symmetric if and only if $\forall (x, y), (x', y') \in \mathbb{R}^2 : (x, y)\mathcal{R}(x', y') \Rightarrow (x', y')\mathcal{R}(x, y)$

$$(x, y)\mathcal{R}(x', y') \Leftrightarrow x + y = x' + y' \Rightarrow x' + y' = x + y \Leftrightarrow (x', y')\mathcal{R}(x, y)$$

Hence \mathcal{R} is symmetric.

c) \mathcal{R} is transitive if and only if $\forall (x, y), (x', y'), (x'', y'') \in \mathbb{R}^2 : (x, y)\mathcal{R}(x', y') \wedge (x', y')\mathcal{R}(x'', y'') \Rightarrow (x, y)\mathcal{R}(x'', y'')$

$$\begin{aligned} (x, y)\mathcal{R}(x', y') \wedge (x', y')\mathcal{R}(x'', y'') &\Leftrightarrow \begin{cases} x + y = x' + y' \\ \wedge \\ x' + y' = x'' + y'' \end{cases} \\ &\Rightarrow x + y = x'' + y'' \\ &\Leftrightarrow (x, y)\mathcal{R}(x'', y'') \end{aligned}$$

Hence \mathcal{R} is transitive.

Thus \mathcal{R} is an equivalence relation.

2)

Let's find the equivalence class of the couple $(0, 0)$.

$$\begin{aligned} \widehat{(0, 0)} &= \{(x, y) \in \mathbb{R}^2 / (x, y)\mathcal{R}(0, 0)\} \\ &= \{(x, y) \in \mathbb{R}^2 / x + y = 0 + 0\} \end{aligned}$$

$$= \{(x, y) \in \mathbb{R}^2 / x + y = 0\}$$

$$= \{(x, y) \in \mathbb{R}^2 / y = -x\} = \{(x, -x) / x \in \mathbb{R}\}.$$

Solution of exercise 3

1)

- a) $\forall x \in \mathbb{R}, x^2 - x = x^2 - x \Leftrightarrow xRx \Leftrightarrow R$ is reflexive.
 b) $\forall x, y \in \mathbb{R}, xRy \Leftrightarrow x^2 - x = y^2 - y \Leftrightarrow y^2 - y = x^2 - x \Leftrightarrow yRx$, R is symmetric.
 c) $\forall x, y, z \in \mathbb{R}, xRy \Leftrightarrow x^2 - x = y^2 - y \wedge y^2 - y = z^2 - z \Leftrightarrow x^2 - x = z^2 - z \Leftrightarrow zRx$, R is transitive.

2) equivalence class of : $\dot{0}, \dot{1}, \dot{2}, \dot{\frac{1}{2}}$

- a) $\dot{0} = \{y \in E / 0Ry\}, 0Ry \Leftrightarrow y^2 - y = 0$, ainsi $\dot{0} = \{0, 1\}$.
 b) $\dot{1} = \{y \in E / 1Ry\}, y^2 - y = 1 - 1 = 0$, ainsi $\dot{1} = \{0, 1\}$.
 c) $\dot{2} = \{y \in E / 2Ry\}, y^2 - y = 2$, ainsi $\dot{2} = \{-1, 2\}$.
 d) $\dot{\frac{1}{2}} = \{y \in E / \frac{1}{2}Ry\}, y^2 - y = \frac{1}{4} - \frac{1}{2} = -\frac{1}{4}$, ainsi $\dot{\frac{1}{2}} = \{\frac{1}{2}\}$.

Solution of exercise 4

1. \mathcal{T} is an order relation if and only if it is reflexive and antisymmetric and transitive.a) \mathcal{T} is reflexive if and only if $\forall (x, y) \in \mathbb{R}^2, (x, y)\mathcal{T}(x, y)$

$$(x, y)\mathcal{T}(x, y) \Leftrightarrow |x - x| \leq y - y \Rightarrow 0 \leq 0$$

Hence \mathcal{T} is reflexive.b) \mathcal{T} is anti-symmetric if and only if

$$\forall (x, y), (x', y') \in \mathbb{R}^2, ((x, y)\mathcal{T}(x', y') \wedge (x', y')\mathcal{T}(x, y)) \Rightarrow (x, y) = (x', y')$$

$$(x, y)\mathcal{T}(x', y') \wedge (x', y')\mathcal{T}(x, y) \Rightarrow |x - x'| \leq y' - y \text{ and } |x' - x| \leq y - y'$$

$$\Rightarrow 2|x - x'| \leq 0$$

$$\Rightarrow |x - x'| = 0$$

$$\Rightarrow x = x'$$

$$\Rightarrow y' - y \geq 0 \text{ and } y - y' \geq 0$$

$$\Rightarrow y' - y \geq 0 \text{ and } y' - y \leq 0$$

$$\Rightarrow y' - y = 0 \Rightarrow y' = y$$

Hence $(x, y) = (x', y')$, then \mathcal{T} is anti-symmetric.c) \mathcal{T} is transitive if and only if

$$\forall (x, y), (x', y'), (x'', y'') \in \mathbb{R}^2, ((x, y)\mathcal{T}(x', y') \wedge (x', y')\mathcal{T}(x'', y'')) \Rightarrow (x, y)\mathcal{T}(x'', y'')$$

$$((x, y)\mathcal{T}(x', y') \wedge (x', y')\mathcal{T}(x'', y'')) \Rightarrow |x - x'| \leq y' - y \text{ and } |x' - x''| \leq y'' - y'$$

$$\Rightarrow (y - y' \leq x - x' \leq y' - y) \text{ and } (y' - y'' \leq x' - x'' \leq y'' - y')$$

$$\Rightarrow y - y'' \leq x - x'' \leq y'' - y$$

$$\Rightarrow |x - x''| \leq y'' - y$$

$$\Rightarrow (x, y)\mathcal{T}(x'', y'')$$

Hence \mathcal{T} is transitive, then it is an order relation.

The order is not total because $\exists (x, y) = (1, 7)$ and $(x', y') = (3, 7)$ such that if we assumes that $(x, y)\mathcal{T}(x', y') \Rightarrow |1 - 3| \leq 7 - 7 = 0$ which is absurd.

Moreover $(x', y')\mathcal{T}(x, y) \Rightarrow |3 - 1| \leq 7 - 7 = 0$ false

2. Let $(a, b) \in \mathbb{R}^2$, let us determine the set $\{(x, y) \in \mathbb{R}^2 / (x, y)\mathcal{T}(a, b)\}$.

$$(x, y)\mathcal{T}(a, b) \Leftrightarrow |x - a| \leq b - y$$

$$\Leftrightarrow (x - a)^2 - (y - b)^2 \leq 0$$

$$\Leftrightarrow [(x - a) + (y - b)][(x - a) - (y - b)] \leq 0$$

$$\Leftrightarrow [(x - a + y - b) \geq 0 \wedge (x - a) - (y - b) < 0]$$

$$\vee [(x - a + y - b) < 0 \wedge (x - a) - (y - b) \geq 0].$$

we put :

D_1 : the closed half-plane of equations $(x - y - a + b) \geq 0$.

D_2 : the open half-plane of equations $(x + y - a - b) < 0$.

D_3 : the open half-plane of equations $(x - y - a + b) < 0$.

D_4 : the closed half-plane of equations $(x + y - a - b) \geq 0$.

From where

$$(a, b) = \{(x, y) \in \mathbb{R}^2 / (x, y) \mathcal{T}(a, b)\} = (D_1 \cap D_2) \cup (D_3 \cap D_4).$$