

TD 02

💡 Exercise 1

Let $A, B, C \in P(E)$, proving the following :

1. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.
2. $A \subset B \Leftrightarrow A \cup B = B$
3. $A \cup B = A \cap C \Leftrightarrow B \subset A \subset C$

💡 Exercise 2

Given A, B and C three parts of a set E ,

1. Show that :
 - (a) $(A \cap B) \cup \complement_E B = A \cup \complement_E B$.
 - (b) $(A \setminus B) \setminus C = A \setminus (B \cup C)$.
2. Simplify :
 - (a) $\overline{(A \cup B) \cap (C \cup \overline{A})}$.
 - (b) $\overline{(A \cap B) \cup (C \cap \overline{A})}$.

💡 Exercise 3

Let E, F two sets, $f : E \rightarrow F$ is an application, proving the following :

1. $\forall A, B \in P(E), f(A \cap B) \subset f(A) \cap f(B)$.
2. $\forall A, B \in P(E), f(A \cup B) = f(A) \cup f(B)$.
3. $\forall A \in P(F), f^{-1}(F \setminus A) = E \setminus f^{-1}(A)$.

💡 Exercise 4

1. Describe the direct image of \mathbb{R} by the exponential function
2. Find $f([0, 1[), f(\mathbb{R}), f(]-1, 2[), f^{-1}(\{3\})$, par la fonction $f : x \mapsto x^2$, defined on \mathbb{R} .

💡 Exercise 5

Let $f : [1; +\infty[\rightarrow [0; +\infty[$ is a function such that $f(x) = x^2 - 1$ bijective ?

💡 Exercise 6

Let E, F, G three sets, $f : E \rightarrow F, g : F \rightarrow G$ and $h : G \rightarrow E$

Proving that is $h \circ g \circ f$ is injective and if $g \circ f \circ h$ and $f \circ h \circ g$ are surjectives. Then, f, g and h are bijectives.

Solutions of TD 02

Solution of exercise 1

$$1. A \setminus (B \cap C) = A \cap \complement_E(B \cap C) = (A \cap \complement_E B) \cup (A \cap \complement_E C) = (A \setminus B) \cup (A \setminus C).$$

(a) " \Rightarrow "

Posing that $A \subset B$, we have $B \subset (A \cup B)$, for $x \in A \cup B$ then $x \in A$ or $x \in B$ we have $x \in B$, therefore $(A \cup B) \subset B$, hence $A \cup B = B$.

(b) " \Leftarrow "

Posing that $A \cup B = B$, because $A \subset (A \cup B)$, on a $A \subset B$.

2. (a) " \Rightarrow "

Posing that $A \cup B = A \cap C$; we have $B \subset (A \cup B) = (A \cap C) \subset A \subset (A \cup B) = (A \cap C) \subset C$.

(b) " \Leftarrow "

Posing that $B \subset A \subset C, A \cup B = A = A \cap C$.

Solution of exercise 2

1. Let us show that :

$$(a) (A \cap B) \cup \complement_E B \stackrel{?}{=} A \cup \complement_E B.$$

$$\begin{aligned} \text{Let } x \in (A \cap B) \cup \complement_E B &\Leftrightarrow x \in (A \cap B) \vee x \in \complement_E B, \\ &\Leftrightarrow (x \in A \wedge x \in B) \vee (x \notin B) \\ &\Leftrightarrow (x \in A \vee x \notin B) \wedge (x \in B \vee x \notin B) \\ &\Leftrightarrow x \in (A \cup \complement_E B) \wedge x \in (B \cup \complement_E B) \\ &\Leftrightarrow x \in (A \cup \complement_E B) \cap E \\ &\Leftrightarrow x \in A \cup \complement_E B \end{aligned}$$

Because $E = \complement_E B \cup B$ and $A \cup \complement_E B$ is a subset of E .


$$(b) (A \setminus B) \setminus C \stackrel{?}{=} A \setminus (B \cup C).$$

$$\begin{aligned} x \in (A \setminus B) \setminus C &\Leftrightarrow (x \in A \wedge x \notin B) \wedge (x \notin C) \\ &\Leftrightarrow x \in A \wedge (x \notin B \wedge x \notin C) \\ &\Leftrightarrow x \in A \wedge x \in (\complement_E B \cap \complement_E C) \\ &\Leftrightarrow x \in A \wedge x \notin (B \cup C) \quad (\text{Laws of Morgan}) \\ &\Leftrightarrow x \in A \setminus (B \cup C) \end{aligned}$$


2. Let's simplify

$$\begin{aligned} (a) \overline{(A \cup B) \cap (C \cup \overline{A})} \\ \overline{(A \cup B) \cap (C \cup \overline{A})} &= (\overline{A \cap B}) \cap (\overline{C \cap \overline{A}}) \\ &= (\overline{A \cap B}) \cap (\overline{C} \cap \overline{\overline{A}}) \\ &= (\overline{A \cap B}) \cap (\overline{C} \cap A) \\ &= (\overline{A} \cap \overline{B}) \cap (\overline{C} \cap A) \\ &= (\overline{A} \cap A) \cap (\overline{C} \cap \overline{B}) \\ &= (\phi) \cap (\overline{C} \cap \overline{B}) \\ &= \phi \end{aligned}$$


$$\begin{aligned} (b) \overline{(A \cap B) \cup (C \cap \overline{A})} \\ \overline{(A \cap B) \cup (C \cap \overline{A})} &= (\overline{A \cap B}) \cup (\overline{C \cap \overline{A}}) \\ &= (\overline{A} \cup \overline{B}) \cup (\overline{C} \cup \overline{\overline{A}}) \\ &= (\overline{A} \cup \overline{B}) \cup (\overline{C} \cup A) \\ &= E \cup (\overline{C} \cup \overline{B}) \\ &= E \end{aligned}$$

 **Solution of exercise 3**


1. Let $y \in f(A \cap B)$. There exists $x \in (A \cap B)$ such that $y = f(x)$. Since $x \in A$, we have $y = f(x) \in f(A)$, and similarly, as $x \in B$, we have $y \in f(B)$. Therefore, $y \in f(A) \cap f(B)$, which implies $f(A \cap B) \subset f(A) \cap f(B)$.
2. • " \subset " : Let $y \in f(A \cup B)$. There exists $x \in A \cup B$ such that $y = f(x)$. If $x \in A$, then $y \in f(A)$. If $x \in B$ and $y \in f(B)$, then in both cases $y \in f(A) \cup f(B)$, so $f(A \cup B) \subset f(A) \cup f(B)$.
• " \supset " : Let $y \in f(A) \cup f(B)$. If $y \in f(A)$, there exists $x \in A$ such that $y = f(x)$. Since $x \in A \subset (A \cup B)$, we have $y \in f(A \cup B)$. The same argument applies if $y \in f(B)$. Therefore, $f(A) \cup f(B) \subset f(A \cup B)$.
By showing both inclusions, we have established the equality.
3. $x \in f^{-1}(F \setminus A) \Leftrightarrow f(x) \in F \setminus A \Leftrightarrow f(x) \notin A \Leftrightarrow x \notin f^{-1}(A) \Leftrightarrow x \in E \setminus f^{-1}(A)$.

 **Solution of exercise 4**

- 1) $\exp(\mathbb{R}) = \mathbb{R}^{*+}$, $f([0, 1]) = [0, 1[$, $f(\mathbb{R}) = \mathbb{R}^+$, $f(] - 1, 2]) = [0, 4[$, $f^{-1}(\{3\}) = \{-\sqrt{3}, \sqrt{3}\}$.

 **Solution of exercise 5**

1. f is injective because, for $x, y \in [1, +\infty[$, $f(x) = f(y)$ implies $x^2 - 1 = y^2 - 1$, which leads to $x = \pm y$. Since $x, y \in [1, +\infty[$, they have the same sign, so $x = y$.
2. f is surjective because, for $y \in [0, +\infty[$, we're looking for an element $x \in [1, +\infty[$ such that $y = f(x) = x^2 - 1$. The real $x = \sqrt{y + 1}$ works, so f is surjective. Therefore, f is bijective.

 **Solution of exercise 6**

We have the following implications :

1. $g \circ f$ injective $\Rightarrow f$ injective.
2. $g \circ f$ surjective $\Rightarrow g$ surjective.

Suppose $h \circ g \circ f$ is injective and both $g \circ f \circ h$ and $f \circ h \circ g$ are surjective.

Since $(h \circ g) \circ f$ is injective, we have f is injective. And since $f \circ (h \circ g)$ is surjective, we have f is surjective.

Therefore, f is bijective, and we can introduce f^{-1} .

By composition, $h \circ g = (h \circ g \circ f) \circ f^{-1}$ is injective, and therefore, g is injective.

On the other hand, $g \circ f \circ h$ is surjective, and therefore, g is surjective as well. In the end, g is bijective.

By composition, $h = (h \circ g) \circ g^{-1}$ is injective, and $h = f^{-1} \circ (f \circ h \circ g) \circ g^{-1}$ is surjective. Thus, h is bijective.