TD 01

Exercise 1

1. Solve in $\mathbb{R}^2$ the system
$$\begin{cases} (x-1)(y-2)=0 \\ (x+1)(y+3)=0 \end{cases}$$
2. Prove that the statement
$$\left(\forall n \in \mathbb{N}^* \setminus \{1\} : 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n} \in \mathbb{N}\right) \text{ is false}$$

Exercise 2

1. Give negation of the following propositions :
   - **a)** $\forall x \in \mathbb{R}, \quad 2x > x.$
   - **b)** $\forall x \in \mathbb{R}, \quad x > 0 \Rightarrow \ 2x > x.$
   - **c)** $\exists n \in \mathbb{N}, \quad 5n+11 = 3n+14.$
   - **d)** $\forall \varepsilon \in \mathbb{R}, \quad (\varepsilon > 0) \Rightarrow (\exists n \in \mathbb{N}^* : \frac{1}{n} < \varepsilon)$
2. In the first three cases, indicate which one is true.

Exercise 3

Write the following sentences and negate them using quantifiers :

a) A : "For every two natural numbers a and b, there is a natural number c that satisfies $ac \neq bc$ or $a = b$."
b) B : "The sum of a rational number and an irrational number is an irrational number."
c) C : " If the sum and product of two real numbers belong to the set Q, then these two numbers belong to Q ".
d) D : " For each x and y of $\mathbb{R}$ we have $xy = 0$ is equivalent to $x = 0$ or $y = 0$."

Exercise 4

Are the following sentences true or false ? Justify your answer.

a) P_1 : "$\forall y \in \mathbb{R}, \exists x \in \mathbb{R} : x - y = 1$"
b) P_2 : "$\exists x \in \mathbb{R}, \forall y \in \mathbb{R} : x - y = 1$"
c) P_3 : "$P_1 \Rightarrow P_2$ ".
d) P_4 : "$P_2 \Rightarrow P_1$ ".

Exercise 5

Let f function of $\mathbb{R}$ in $\mathbb{R}$. Translate the following expressions into quantifier terms :

1. f is bounded above.
2. f is bounded.
3. f is even.
4. f never equals zero.
5. f is periodic.
6. f is increasing.
7. f is not the zero function.
8. f attains all values in $\mathbb{N}$.

Exercise 6

Show by recurrence that :

1. $\forall n \in \mathbb{N}^\star : 1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$
2. $\forall n \in \mathbb{N}$, $4^n + 6n - 1$ is a multiple of 9.

Exercise 7

By the absurd show that :

$\forall n \in \mathbb{N}$, n^2 even $\Rightarrow n$ is even.

Solutions of TD 01

Solution of exercise 1

1)

$$\begin{cases} (x-1)(y-2)=0 \\ (x+1)(y+3)=0 \end{cases} \Leftrightarrow (x-1)(y-2)=0 \wedge (x+1)(y+3)=0$$

$$\Leftrightarrow (x-1=0 \vee y-2=0) \wedge (x+1=0 \vee y+3)=0$$

$$\Leftrightarrow (x=1 \vee y=2) \wedge (x=-1 \vee y=-3)$$

$$\Leftrightarrow (x=1 \wedge x=-1) \vee (x=1 \wedge y=-3) \vee (y=2 \wedge x=-1) \vee (y=2 \wedge y=-3)$$

Since Since $(x=1 \wedge x=-1)$ and $(y=2 \wedge y=-3)$ are both false statements, then

$$\begin{cases} (x-1)(y-2)=0 \\ (x+1)(y+3)=0 \end{cases} \Leftrightarrow (x=1 \wedge y=-3) \vee (y=2 \wedge x=-1)$$

$$\Leftrightarrow (x,y)=(1\ ,-3) \vee (x,y)=(-1,2)$$

then the set of solutions to the system is

$S=\{(1\ ,-3),\ (-1,2)\}$.

2)

Show that this statement $(\forall n \in \mathbb{N}^* \setminus \{1\} : 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n} \in \mathbb{N})$ is false

It is sufficient to show that its negation is true

We have the negative of the statement is :

$(\exists n \in \mathbb{N}^* \setminus \{1\} : 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n} \notin \mathbb{N})$

Let $n=2 \in \mathbb{N}^* \setminus \{1\}$.

We have : $1+\frac{1}{2}=\frac{3}{2} \notin \mathbb{N}$

So, the statement

$(\exists n \in \mathbb{N}^* \setminus \{1\} : 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n} \notin \mathbb{N})$ is true.

Then the statement

$(\forall n \in \mathbb{N}^* \setminus \{1\} : 1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n} \in \mathbb{N})$ is false.

Solution of exercise 2

1)

a) the negation is given by

$\exists x \in \mathbb{R} : 2x \leq x$.

b) the negation of proposition (b) is given by

$\exists x \in \mathbb{R} : x > 0 \wedge 2x \leq x$.

c) the negation is given by

$\forall n \in \mathbb{N}, \quad 5n+11 \neq 3n+14$

d) the negation is given by

$\exists \varepsilon \in \mathbb{R} \quad \varepsilon > 0 \wedge \forall n \in \mathbb{N}^* : \frac{1}{n} \geq \varepsilon$

2)

proposition (**a**) is false because its negation is true.

proposition (**b**) is true. For the proof, we proceed as follows

let $x \in \mathbb{R}$,

suppose that $x > 0$ hence $2x = x + x > x$

this completes the proof

3)

proposition (**c**) is false. Solving the equation $5n+11=3n+14$

we find that the only solution is $n=\frac{3}{2}$, or $\frac{3}{2} \notin \mathbb{N}$

Therefore
$\forall n \in \mathbb{N}: \quad 5n + 11 \neq 3n + 14$
is true, or proposition " $\exists n \in \mathbb{N}, \quad 5n + 11 = 3n + 14$" is false.

Solution of exercise 3

a) A **:** "$\forall\ a \in \mathbb{N}, \forall\ b \in \mathbb{N}\ , \exists\ c \in \mathbb{N}: ac \neq bc \vee a = b.$".
Or
A **:** "$\forall\ (a, b) \in \mathbb{N}^2\ , \exists\ c \in \mathbb{N}: ac \neq bc \vee a = b.$".
its negation is : "$\exists\ a \in \mathbb{N}, \exists\ b \in \mathbb{N}\ , \forall\ c \in \mathbb{N}: ac = bc \wedge a \neq b.$".

b) B **:** "$\forall\ x \in \mathbb{Q}, \forall\ y \in \mathbb{R} \setminus \mathbb{Q}\ , x + y \in \mathbb{R} \setminus \mathbb{Q}$ ".
its negation is : "$\exists x \in \mathbb{Q}, \exists\ y \in \mathbb{R} \setminus \mathbb{Q}\ , x + y \notin \mathbb{R} \setminus \mathbb{Q}$ ".

c) C **:** "$\forall\ (x, y) \in \mathbb{R}^2, (x + y \in \mathbb{Q} \wedge xy \in \mathbb{Q}) \Longrightarrow (x \in \mathbb{Q} \wedge y \in \mathbb{Q})$".
its negation is : " $\exists\ (x, y) \in \mathbb{R}^2, (x + y \in \mathbb{Q} \wedge xy \in \mathbb{Q}) \wedge (x \notin \mathbb{Q} \vee y \notin \mathbb{Q})$"

d) D **:** "$\forall\ (x, y) \in \mathbb{R}^2, (xy = 0) \Leftrightarrow (x = 0 \vee y = 0)$".
its negation is :
We know that : $P \Leftrightarrow Q$ means $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ i.e. $(P \vee \overline{Q}) \wedge (Q \vee \overline{P})$
So that, the negation of D is : "$\exists\ (x, y) \in \mathbb{R}^2, (xy = 0 \wedge (x \neq 0 \wedge y \neq 0)) \vee ((x = 0 \vee y = 0) \wedge xy \neq 0)$".

Solution of exercise 4

a) P_1 **:** "$\forall y \in \mathbb{R}, \exists x \in \mathbb{R}: x - y = 1$"
Let $y \in \mathbb{R}$,
The equation $x - y = 1$ with the unknown x is equivalent to $x = y + 1$.
Since 1 and y are in $\mathbb{R}$, then $x \in \mathbb{R}$.
So, "$\forall y \in \mathbb{R}, \exists x = y + 1 \in \mathbb{R}: x - y = 1$".
Then the statement $\mathbf{P_1}$ is true.

b) P_2 **:** "$\exists x \in \mathbb{R}, \forall y \in \mathbb{R}: x - y = 1$"
Suppose there is $x \in \mathbb{R}$ such that for each $y \in \mathbb{R}$
$x - y = 1$
We take $x = y$
Then,
$x - y = x - x = 0 \neq 1$
Therefore, statement $\mathbf{P_2}$ is incorrect
i.e., $\mathbf{P_2}$ is false

c) $\mathbf{P_3}$ **:** "$P_1 \Rightarrow P_2$ ".
Know that statement P_1 is true and statement P_2 is false.
So, the statement "$P_1 \Rightarrow P_2$ " is false.

d) P_4 **:** "$P_2 \Rightarrow P_1$ ".
Know that statement P_2 is false and statement P_1 is true.
So, the statement "$P_2 \Rightarrow P_1$ " is true.

Solution of exercise 5

1. $\exists M \in \mathbb{R}, \forall x \in \mathbb{R}, f(x) \leq M$.
2. $\exists M \in \mathbb{R}, \exists m \in \mathbb{R}, \forall x \in \mathbb{R}, \quad m \leq f(x) \leq M$.
3. $\forall x \in \mathbb{R}, f(x) = f(-x)$.

4. $\forall x \in \mathbb{R},\ f(x) \neq 0.$
5. $\exists \alpha \in \mathbb{R}^*,\ \forall x \in \mathbb{R},\ f(x+\alpha) = f(x).$
6. $\forall x \in \mathbb{R},\ \forall y \in \mathbb{R},\ x \leq y \Rightarrow f(x) \leq f(y).$
7. $\exists x \in \mathbb{R},\ f(x) \neq 0.$
8. $\forall n \in \mathbb{N},\ \exists x \in \mathbb{R},\ f(x) = n.$

Solution of exercise 6

1. Let's show that $P_n : 1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}, \quad \forall n \in \mathbb{N}^\star$

   • For $n = 1$ we have : $1^3 = \frac{1^2(2)^2}{4} = 1$
   So P_1 is true.
   • We suppose that $P_n : 1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$ is true.
   And we show that : $1^3 + 2^3 + ... + n^3 + (n+1)^3 = \frac{(n+1)^2(n+2)^2}{4}$ is true.
   Using P_n we obtain :

$$\begin{aligned} 1^3 + 2^3 + ... + (n+1)^3 &= 1^3 + 2^3 + ... + n^3 + (n+1)^3 \\ &= \frac{n^2(n+1)^2}{4} + (n+1)^3 \\ &= \frac{n^2(n+1)^2 + 4(n+1)^3}{4} \\ &= \frac{(n+1)^2(n^2+4n+4)}{4} \\ &= \frac{(n+1)^2(n+2)^2}{4} \end{aligned}$$

   Thus P_{n+1} is true, then $\forall n \in \mathbb{N}^\star : 1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$.

2. Let's show that $P_n : 4^n + 6n - 1, \quad \forall n \in \mathbb{N}$ is a multiple of 9,
   that's to say $\forall n \in IN\ , \exists k \in \mathbb{Z}\ \ /\ 4^n + 6n - 1 = 9k$
   • For $n = 1$ we have : $\exists k = 1 \in \mathbb{Z},\ 4 + 6 - 1 = 9 = 9(1)$, P_1 is true.
   • We suppose that : $\forall n \in IN\ , \exists k \in \mathbb{Z}\ \ /\ 4^n + 6n - 1 = 9k$ is true..
   And we show that : $\forall n \in IN,\ \exists ?k^{'} \in \mathbb{Z}\ /\ 4^{n+1} + 6(n+1) - 1 = 9k^{'}$is true..

$$\begin{aligned} 4^{n+1} + 6(n+1) - 1 &= 4.4^n + 6n + 6 - 1 \\ &= (9-5)4^n + 6n + 5 \\ &= 9(4^n) - 5.4^n - 5(6n) + 36n + 5 \\ &= -5(4^n + 6n - 1) + 9(4^n) + 36n \\ &= -5(9k) + 9(4^n) + 9(4n) \\ &= 9(-5k + 4^n + 4n) \\ &= 9k^{'}, \quad \text{such as } k^{'} = -5k + 4^n + 4n \end{aligned}$$

Solution of exercise 7

Let $n \in \mathbb{N}$ by the absurd suppose that n^2 is even and n is odd, then $\exists k \in \mathbb{Z}$ such that :
$n = 2k + 1$ hence $n^2 = 2(2k^2 + 2k) + 1 = 2k^{'} + 1,\ k^{'} = (2k^2 + 2k) \in \mathbb{Z}$,
n^2 is odd contradiction because n^2 is even.
What we initially assumed is false i.e. $\forall n \in \mathbb{N}, n^2$ even $\Rightarrow n$ is even.