

## Chapter 3: Taylor's Formulas, Limited Development

### 3.1 Taylor's Formulas

#### 3.1.1 Taylor's formula with Lagrange remainder

##### Theorem 3.1

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a function of class  $C^n$  on the interval  $[a, b]$  and  $f^{(n)}$  is differentiable over the interval  $]a, b[$  then:

For every two numbers  $x, x_0$  of the interval  $[a, b]$  where  $x \neq x_0$  we have:

Where  $c$  is a real number from  $]x_0, x[$  (or  $]x, x_0[$ , if  $x < x_0$ ). The remainder  $R_n = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$  called Lagrange remainder.

##### Proof

Let us define the functions  $g$  and  $\varphi$  on the interval  $[a, b]$  by

$$g(t) = f(x) - \sum_{j=0}^n \frac{f^{(j)}(t)}{j!} (x - t)^j,$$

and

$$\varphi(t) = g(t) - \frac{g(x_0)}{(x_0 - x)^{n+1}} (t - x)^{n+1}$$

We have

$$g(x) = 0 ; g(x_0) = f(x) - \sum_{j=0}^n \frac{f^{(j)}(x_0)}{j!} (x - x_0)^j \text{ and } g'(t) = -\frac{f^{(n+1)}(t)}{n!} (x - t)^n.$$

We have too

$$\varphi(x) = \varphi(x_0) = 0$$

By applying Rolle's theorem to the function  $\varphi$  in the interval  $[x_0, x]$  (or  $]x, x_0[$ , if  $x < x_0$ ) we get:

there is a real number  $c$  in  $]x_0, x[$  (or  $]x, x_0[$ , if  $x < x_0$ ) where

$$\varphi'(c) = 0$$

or

$$g'(c) - (n+1) \frac{g(x_0)}{(x_0 - x)^{n+1}} (c - x)^n = 0$$

or

$$-\frac{f^{(n+1)}(c)}{n!} (x - c)^n - (n+1) \frac{g(x_0)}{(x_0 - x)^{n+1}} (c - x)^n = 0$$

or

$$g(x_0) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}.$$

So

$$f(x) - \sum_{j=0}^n \frac{f^{(j)}(x_0)}{j!} (x - x_0)^j = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}.$$

Finally we obtain

$$f(x) = \sum_{j=0}^n \frac{f^{(j)}(x_0)}{j!} (x - x_0)^j + \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$$

### 3.1.2 Taylor's formula with Young's remainder

#### Definition 3.1

Let  $f$  and  $g$  be functions defined in the neighborhood  $v$  of the point  $x_0$  where

$\forall x \in v - \{x_0\}: g(x) \neq 0$ , we say that  $f$  is negligible before  $g$  when  $x \rightarrow x_0$  and we write  $f = o(g)$ , if the following is true:  $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} = 0$ .

#### Definition 3.2

By putting  $\varepsilon(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)$ . The Taylor's- Lagrange formula is rewritten in the form:

$$f(x) = \sum_{j=0}^n \frac{f^{(j)}(x_0)}{j!} (x - x_0)^j + (x - x_0)^n \varepsilon(x) \quad \text{where} \quad \lim_{x \rightarrow x_0} \varepsilon(x) = 0.$$

The remainder  $R_n = (x - x_0)^n \varepsilon(x)$ , is called Young remainder.

And by putting  $R_n = (x - x_0)^n \varepsilon(x) = o((x - x_0)^n)$ , the previous formula is rewritten into the form:

$$f(x) = \sum_{j=0}^n \frac{f^{(j)}(x_0)}{j!} (x - x_0)^j + o((x - x_0)^n).$$

**Note:** We mention that  $R_n = o((x - x_0)^n)$  means that  $\lim_{x \rightarrow x_0} \frac{R_n}{(x - x_0)^n} = 0$ .

### 3.1.3 McLaurin's formula with Young's remainder

By taking  $x_0 = 0$  in Taylor's-Young's formula, we get:

called McLaurin's- Young's formula.

#### Examples 3.1

1) Let  $f(x) = e^x$ ;  $x_0 = 0$  we have  $f \in C^\infty(\mathbb{R})$ ;  $\forall k \in \mathbb{N}^*$ :  $f^{(k)}(x) = e^x$  and

$$\forall k \in \mathbb{N}^*: f^{(k)}(0) = 1.$$

So

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k + R_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \dots + \frac{x^n}{n!} + R_n(x),$$

where

\*  $R_n(x) = \frac{x^{n+1}}{(n+1)!} e^c$  with  $c \in ]0, x[$  or  $c \in ]x, 0[$  In Taylor's- Lagrange formula.

\*  $R_n(x) = x^n \varepsilon(x) = o(x^n)$  with  $\lim_{x \rightarrow x_0} \varepsilon(x) = 0$ . In Taylor's - Young's formula.

2) Let  $f(x) = \frac{1}{1-x}$ ;  $x, x_0 \in ]1, +\infty[$  we have  $f \in C^\infty(]1, +\infty[)$  and

$$\forall k \in \mathbb{N}: f^{(k)}(x_0) = \frac{k!}{(1-x_0)^{k+1}}.$$

So

$$\begin{aligned} f(x) &= \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} x^k + R_n(x, x_0) = \sum_{k=0}^n \frac{1}{(1-x_0)^{k+1}} x^k + R_n(x, x_0) \\ &= \frac{1}{1-x_0} + \frac{x-x_0}{(1-x_0)^2} + \frac{(x-x_0)^2}{(1-x_0)^3} + \frac{(x-x_0)^3}{(1-x_0)^4} + \dots + \frac{(x-x_0)^n}{(1-x_0)^{n+1}} + R_n(x, x_0). \end{aligned}$$

Where

\*  $R_n(x, x_0) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1} = \frac{(x-x_0)^{n+1}}{(1-c)^{n+2}}$  with  $c \in ]x_0, x[$  or  $c \in ]x, x_0[$  In Taylor's-Lagrange formula.

\*  $R_n(x, x_0) = (x - x_0)^n \varepsilon(x, x_0) = 0((x - x_0)^n)$  with  $\lim_{x \rightarrow x_0} \varepsilon(x, x_0) = 0$ . In Taylor's-Young's formula.

**For example if  $x_0 = 3$ , then**

$$f(x) = -\frac{1}{2} + \frac{1}{4}(x - 3) - \frac{1}{8}(x - 3)^2 + \frac{1}{16}(x - 3)^3 + \dots + \frac{(-1)^n}{2^n}(x - 3)^n + R_n(x).$$

$$R_n(x) = \frac{(x - 3)^{n+1}}{(1 - c)^{n+2}} \text{ with } c \in ]3, x[ \text{ or } c \in ]x, 3[ \text{ (Or } |c - 3| < |x - 3|),$$

or

$$R_n(x) = (x - 3)^n \varepsilon(x) = 0((x - 3)^n) \text{ with } \lim_{x \rightarrow 3} \varepsilon(x) = 0$$

### 3.2 Apply Tyler's formulas to find local extrema

#### Theorem 3.3

Let  $f$  be a function of class  $C^n$  in the neighborhood of the point  $x_0$  such that:

$$f'(x_0) = f''(x_0) = f^{(3)}(x_0) = \dots = f^{(n-1)}(x_0) = 0 \text{ and } f^{(n)}(x_0) \neq 0. \text{ Then}$$

$f$  accepts a local maximum (local minimum, respectively) at  $x_0$  if and only if  $n$  is even and  $f^{(n)}(x_0) < 0$  ( $f^{(n)}(x_0) > 0$ , respectively).

#### Proof

Applying the Tyler-Lagrange formula and taking into account the hypothesis on the derivatives we get:

$$f(x) = f(x_0) + \frac{f^{(n)}(c)}{n!} (x - x_0)^n \text{ where } c \text{ is a real number between } x_0 \text{ and } x.$$

Assume that  $f^{(n)}(x_0) < 0$ .

Since  $f^{(n)}$  is continuous at  $x_0$ , there exists a real number  $\delta > 0$  such that:

$$\forall x \in ]x_0 - \delta, x_0 + \delta[: f^{(n)}(x) < 0 \implies f^{(n)}(c) < 0 \text{ (Because } c \text{ confined between } x_0, x).$$

If  $n$  is even, then  $\frac{f^{(n)}(c)}{n!} (x - x_0)^n < 0$ . So  $\forall x \in ]x_0 - \delta, x_0 + \delta[: f(x) < f(x_0)$ .

So  $f(x_0)$  is local maximum.

In the same way the proof is performed if  $f^{(n)}(x_0) > 0$ .

If  $n$  is odd, then the sign of the difference  $f(x) - f(x_0)$  changes at  $x_0$ .

### Example 3.2

$$\text{Let } f(x) = \frac{1}{5}x^5 - \frac{1}{4}x^4.$$

**Necessary condition:**

$$f'(x) = 0 \Leftrightarrow x = 0, x = 1$$

**Sufficient condition:**

$f'(0) = f''(0) = f^{(3)}(0) = 0$  and  $f^{(4)}(0) = -6 < 0 \Rightarrow f(0) = 0$  is local maximum value of  $f$ .

Similarly we have

$f'(1) = 0$  and  $f''(1) = 1 > 0 \Rightarrow f(1) = -\frac{1}{20}$  is local minimum value of  $f$ .

## 3.2 Limited Development

### 3.2.1 Limited Development of order $n$ in a neighborhood of 0

#### Definition 3.3

Let  $f$  be a function defined in a neighborhood of 0 - with the possible exception of 0 - we say that  $f$  admits a limited Development of order  $n$  in a neighborhood of 0 if and only if there exists a neighborhood  $v$  of 0 and constant numbers  $a_0, a_1, a_2, \dots, a_n$  where

$$\forall x \in v; x \neq 0: f(x) = a_0 + a_1x + a_2x^2 \dots + a_nx^n + x^n\varepsilon(x) \text{ with } \lim_{x \rightarrow 0} \varepsilon(x) = 0.$$

The polynomial  $a_0 + a_1x + a_2x^2 \dots + a_nx^n$ , is called the regular part and we denote it by  $P_n(x)$ .

The term  $x^n\varepsilon(x)$  ( or  $o(x^n)$  ) is called the remainder or the complementary term, and it is symbolized by  $R_n(x)$ .

#### Theorem 3.4 (uniqueness)

If a function  $f$  admits a limited Development of order  $n$  in the neighborhood of 0, then this Development is unique.

#### Proof

Assume that for all  $x \in v - \{0\}$ :

$$f(x) = a_0 + a_1x + a_2x^2 \dots + a_nx^n + x^n\varepsilon_1(x) \text{ with } \lim_{x \rightarrow 0} \varepsilon_1(x) = 0,$$

and

$$f(x) = b_0 + b_1x + b_2x^2 \dots \dots \dots + b_nx^n + x^n\varepsilon_2(x) \text{ with } \lim_{x \rightarrow 0} \varepsilon_2(x) = 0.$$

So we have for all  $x \in v - \{0\}$ :

$$a_0 - b_0 + (a_1 - b_1)x + (a_2 - b_2)x^2 \dots \dots \dots + (a_n - b_n)x^n + x^n(\varepsilon_1(x) - \varepsilon_2(x)) = 0.$$

Taking the limit at 0 we obtain

$$a_0 - b_0 = 0.$$

So we have for all  $x \in v - \{0\}$ :

$$(a_1 - b_1)x + (a_2 - b_2)x^2 \dots \dots \dots + (a_n - b_n)x^n + x^n(\varepsilon_2(x) - \varepsilon_1(x)) = 0,$$

or

$$(a_1 - b_1) + (a_2 - b_2)x \dots \dots \dots + (a_n - b_n)x^{n-1} + x^{n-1}(\varepsilon_2(x) - \varepsilon_1(x)) = 0.$$

Taking the limit again at 0 we obtain

$$(a_1 - b_1) = 0.$$

By continuing the operation, we obtain for all  $x \in v - \{0\}$

$$(a_n - b_n) + (\varepsilon_2(x) - \varepsilon_1(x)) = 0,$$

from where

$$(a_n - b_n) = \lim_{x \rightarrow 0} (\varepsilon_1(x) - \varepsilon_2(x)) = 0$$

So

$$a_n = b_n \text{ and } \varepsilon_1(x) = \varepsilon_2(x).$$

### Theorem 3.5

If a function  $f$  admits a limited Development of order  $n$  ( $n \geq 1$ ) in the neighborhood of 0 and if  $f(0) = a_0$  then  $f$  is differentiable at 0 and we have  $f'(0) = a_1$ .

#### Proof

If we have  $f(0) = a_0$  we can write

$$\frac{f(x) - f(0)}{x - 0} = a_1 + a_2x \dots \dots \dots + a_nx^{n-1} + x^{n-1}\varepsilon_1(x) \text{ with } \lim_{x \rightarrow 0} \varepsilon_1(x) = 0,$$

so

$$\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0} (a_1 + a_2x \dots \dots \dots + a_nx^{n-1} + x^{n-1}\varepsilon_1(x)) = a_1.$$

### Theorem 3.6

If  $f$  is of class  $C^n$  in the neighborhood of 0, then  $f$  admits a limited development to the neighborhood of 0, which is obtained in the McLaurin's- Young's formula, i.e.:

$$f(x) = \sum_{j=0}^n \frac{f^{(j)}(0)}{j!} x^j + o(x^n).$$

### Proof

The theorem results from the application of the Maclaurin-Young formula and the uniqueness of limited development.

### Theorem 3.7

If an even (respectively odd) function admits an limited development to the neighborhood of 0, then its regular part is even (respectively odd). The limited development of  $f$  is therefore written:

$$f(x) = a_0 + a_2x^2 + \dots + a_{2n}x^{2n} + o(x^{2n}) \text{ If } f \text{ is even.}$$

$$f(x) = a_1 + a_3x^2 + \dots + a_{2n+1}x^{2n+1} + o(x^{2n+1}) \text{ If } f \text{ is odd.}$$

### Remark 3.1

The order of the limited development is determined by the degree of the remainder  $o(x^n)$  and not the degree of the regular part.

### Example 3.3

$$f(x) = 1 - \frac{1}{2}x + 3x^2 + \frac{3}{5}x^3 + o(x^5) \text{ Is a limited development of order 5.}$$

$$g(x) = 1 - \frac{1}{2}x + 3x^2 + \frac{3}{5}x^3 - \frac{7}{13}x^4 + 2x^5 + o(x^3) \text{ Is a limited development of order 3.}$$

$$\text{Since } -\frac{7}{13}x^4 + 2x^5 = o(x^3) \text{ we get } g(x) = 1 - \frac{1}{2}x + 3x^2 + \frac{3}{5}x^3 + o(x^3)$$

## 3.2.2 Operation on Limited Development

### Theorem 3.8

Let  $f, g$  be two functions admitting limited developments to the same order  $n$  in the neighborhood of 0. We denote their regulars parts as  $P_n(x), Q_n(x)$ , respectively. That is

$$f(x) = P_n(x) + x^n \varepsilon_1(x) \quad ; \quad g(x) = Q_n(x) + x^n \varepsilon_2(x).$$

Then, the functions  $f + g, fg, \frac{f}{g}$  (if  $\lim_{x \rightarrow 0} g(x) \neq 0$ ),  $fog$  (if  $\lim_{x \rightarrow 0} g(x) = 0$ ), admitting limited developments of order  $n$  in the neighborhood of 0 and we have:

$$1) f(x) + g(x) = P_n(x) + Q_n(x) + x^n \varepsilon_3(x) \text{ with } \lim_{x \rightarrow 0} \varepsilon_3(x) = 0.$$

$$2) f(x)g(x) = A_n(x) + x^n \varepsilon_4(x) \text{ with } \lim_{x \rightarrow 0} \varepsilon_4(x) = 0.$$

Where  $A_n(x)$  is the polynomial we obtain by retaining in the multiplication  $P_n(x)Q_n(x)$  only the terms with degrees less than or equal to  $n$ .

$$3) \frac{f(x)}{g(x)} = B_n(x) + x^n \varepsilon_5(x) \text{ with } \lim_{x \rightarrow 0} \varepsilon_5(x) = 0.$$

Where  $B_n(x)$  is the polynomial we obtain by Euclidean division of  $P_n(x)$  by  $Q_n(x)$  according to increasing powers of  $x$  keeping only terms with degrees less than or equal to  $n$ .

$$4) f \circ g(x) = C(x) + x^n \varepsilon_6(x) \text{ with } \lim_{x \rightarrow 0} \varepsilon_6(x) = 0.$$

Where  $C_n(x)$  is the polynomial we obtain by retaining in the composite  $P_n \circ Q_n(x)$  only the terms with degrees less than or equal to  $n$ .

### Proof

Let us prove the third case, for example.

We first recall with the following proposition

#### Proposition 3.1 (DIVISION BY INCREASING POWER ORDER)

Let  $n \in \mathbb{N}^*$  and  $A, B$  two polynomials. We write them

$$A(x) = a_0 + a_1x + a_2x^2 \dots \dots \dots + a_px^p \text{ and } B(x) = b_0 + b_1x + b_2x^2 \dots \dots \dots + b_mx^m.$$

We assume that  $b_0 \neq 0$ . Then there exist a unique pair  $(Q, R)$  of polynomials such that

$$\begin{cases} A(x) = B(x)Q(x) + x^{n+1}R(x) \\ \deg(Q) \leq n \end{cases}.$$

Let us now return to proving the third case in Theorem 3.6. we've got.

$$f(x) = P_n(x) + x^n \varepsilon_1(x),$$

$$g(x) = Q_n(x) + x^n \varepsilon_2(x).$$

The division according to the increasing powers to the order  $n$  of  $P_n(x)$  by  $Q_n(x)$  gives

$$\begin{cases} P_n(x) = Q_n(x)Q(x) + x^{n+1}R(x) \\ \deg(Q) \leq n \end{cases}.$$

From where

$$\begin{aligned} f(x) - x^n \varepsilon_1(x) &= P_n(x) \\ &= Q_n(x)Q(x) + x^{n+1}R(x) \\ &= (g(x) - x^n \varepsilon_2(x))Q(x) + x^{n+1}R(x). \end{aligned}$$

So

$$f(x) = g(x)Q(x) + x^n(\varepsilon_1(x) - \varepsilon_2(x)Q(x) + xR(x)),$$

$$\frac{f(x)}{g(x)} = Q(x) + x^n \frac{(\varepsilon_1(x) - \varepsilon_2(x)Q(x) + xR(x))}{g(x)}$$

Or

$$\frac{f(x)}{g(x)} = Q(x) + x^n \varepsilon(x) \quad \text{with} \quad \varepsilon(x) = \frac{(\varepsilon_1(x) - \varepsilon_2(x)Q(x) + xR(x))}{g(x)}.$$

Since  $\lim_{x \rightarrow 0} g(x) \neq 0$  we have  $\lim_{x \rightarrow 0} \varepsilon(x) = 0$ .

### Examples 3.4

$$1) f(x) = \ln(1+x) = \underbrace{x - \frac{1}{2}x^2 + \frac{1}{3}x^3}_{P_n(x)} + x^3 \varepsilon_1(x) \quad \text{where} \quad \lim_{x \rightarrow 0} \varepsilon_1(x) = 0$$

$$g(x) = e^x = \underbrace{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3}_{Q_n(x)} + x^3 \varepsilon_2(x) \quad \text{where} \quad \lim_{x \rightarrow 0} \varepsilon_2(x) = 0$$

$$\begin{aligned} * f(x) + g(x) &= P_n(x) + Q_n(x) = \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right) + \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3\right) \\ &= 1 + 2x + \frac{1}{2}x^3 + x^3 \varepsilon_3(x). \quad \text{where} \quad \lim_{x \rightarrow 0} \varepsilon_3(x) = 0 \end{aligned}$$

$$\begin{aligned} * f(x)g(x) &= P_n(x)Q_n(x) = \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3\right) \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3\right) \\ &= x \left(1 + x + \frac{1}{2}x^2\right) - \frac{1}{2}x^2(1+x) + \frac{1}{3}x^3(1) \\ &= x + x^2 + \frac{1}{2}x^3 - \frac{1}{2}x^2 - \frac{1}{2}x^3 + \frac{1}{3}x^3 \\ &= x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + x^3 \varepsilon_4(x). \quad \text{where} \quad \lim_{x \rightarrow 0} \varepsilon_4(x) = 0 \end{aligned}$$

\* We have  $\lim_{x \rightarrow 0} g(x) \neq 0$ , so  $\frac{f}{g}$  admits a limited development.

$$\frac{f(x)}{g(x)} = \frac{P_n(x)}{Q_n(x)} = \frac{x - \frac{1}{2}x^2 + \frac{1}{3}x^3}{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3}. \quad \text{By Euclidean division of } P_n(x) \text{ by } Q_n(x) \text{ according to}$$

increasing powers of  $x$  we obtain

$$\frac{f(x)}{g(x)} = x - \frac{3}{2}x^2 + \frac{4}{3}x^3 + x^3 \varepsilon_5(x) \quad \text{where} \quad \lim_{x \rightarrow 0} \varepsilon_5(x) = 0$$

$$2) f(x) = e^x = \underbrace{1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3}_{P(x)} + x^3 \varepsilon_1(x) \quad \text{where} \quad \lim_{x \rightarrow 0} \varepsilon_1(x) = 0$$

$$g(x) = \sin x = \underbrace{x - \frac{1}{6}x^3}_{Q(x)} + x^3 \varepsilon_2(x) \quad \text{where} \quad \lim_{x \rightarrow 0} \varepsilon_2(x) = 0$$

\* We have  $\lim_{x \rightarrow 0} g(x) = 0$ , so  $f \circ g$  admits a limited development.

$$\begin{aligned} f \circ g(x) &= e^{\sin x} = P \circ Q(x) = 1 + \left(x - \frac{1}{6}x^3\right) + \frac{1}{2}\left(x - \frac{1}{6}x^3\right)^2 + \frac{1}{6}\left(x - \frac{1}{6}x^3\right)^3 \\ &= 1 + \left(x - \frac{1}{6}x^3\right) + \frac{1}{2}(x)^2 + \frac{1}{6}(x)^3 \\ &= 1 + x + \frac{1}{2}x^2 + x^3 \varepsilon_3(x) \text{ where } \lim_{x \rightarrow 0} \varepsilon_1(x) = 0. \end{aligned}$$

### 3.2.3 Limited Development of order $n$ in a neighborhood of $x_0$

#### Definition 3.4

Let  $f$  be a function defined in a neighborhood of  $x_0$  - with the possible exception of  $x_0$  - we say that  $f$  admits a limited Development of order  $n$  in a neighborhood of  $x_0$  if and only if the function  $F: h \rightarrow F(h) = f(h + x_0)$  admits a limited Development of order  $n$  in a neighborhood of 0. And if

$$F(h) = a_0 + a_1h + a_2h^2 \dots \dots \dots a_n h^n + h^n \varepsilon_1(h) \text{ with } \lim_{h \rightarrow 0} \varepsilon_1(h) = 0.$$

Then for all  $x \in v - \{x_0\}$ :

$$\begin{aligned} f(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 \dots \dots \dots + a_n(x - x_0)^n + (x - x_0)^n \varepsilon(x, x_0) \text{ with} \\ \lim_{x \rightarrow x_0} \varepsilon(x, x_0) &= 0. \end{aligned}$$

### 3.2.4 Limited Development of order $n$ in a neighborhood of $\infty$

#### Definition 3.5

Let  $f$  be a function defined in a neighborhood of  $+\infty$  ( $-\infty$ , respectively). We say that  $f$  admits a limited Development of order  $n$  in a neighborhood of  $+\infty$  ( $-\infty$ , respectively) if and only if the function  $F: h \rightarrow F(h) = f\left(\frac{1}{h}\right)$  admits a limited Development of order  $n$  in a neighborhood of 0. And if

$$F(h) = a_0 + a_1h + a_2h^2 \dots \dots \dots a_n h^n + h^n \varepsilon_1(h) \text{ with } \lim_{h \rightarrow 0} \varepsilon_1(h) = 0 \left( \lim_{x \rightarrow -\infty} \varepsilon(x) = 0, \text{ resp} \right)$$

Then

$$f(x) = a_0 + a_1 \frac{1}{x} + a_2 \frac{1}{x^2} \dots \dots \dots + a_n \frac{1}{x^n} + \frac{1}{x^n} \varepsilon(x) \text{ with } \lim_{x \rightarrow +\infty} \varepsilon(x) = 0 \left( \lim_{x \rightarrow -\infty} \varepsilon(x) = 0, \text{ resp} \right),$$

where  $\varepsilon(x) = \varepsilon_1\left(\frac{1}{x}\right)$ .

#### Remark 3.2

If the function  $F$  admits a limited Development of order  $n$  in a neighborhood of 0, then the two limited Developments of  $f$  in the neighborhood of  $+\infty$  and in the neighborhood

of  $-\infty$  are identical. In this case, we say that  $f$  admits a limited Development of order  $n$  in a neighborhood of infinity.

### 3.3 Applications of limited development

Let  $x_0$  be a real number or  $-\infty$  or  $+\infty$ , and  $f, g$  are non-zero functions that accepts limited developments in the neighborhood of  $x_0$ . We denote by  $(a_i)_{i \in \mathbb{N}}$  and  $(b_j)_{j \in \mathbb{N}}$  for the coefficients of their regular parts respectively.

#### 3.3.1 Calculation of limits

When calculating the limit  $\lim_{x \rightarrow x_0} \frac{f(x)}{g(x)}$  and if we obtain one of the indeterminate form  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ , to raise the indeterminacy we develop the functions  $f$  and  $g$  in the neighborhood of  $x_0$  to the smallest orders  $m$  and  $n$ , respectively, where  $b_n \neq 0$  and  $a_m \neq 0$ .

#### Example 3.5

Calculate the limit

$$\lim_{x \rightarrow 0} \frac{\ln(1+x) + \frac{1}{2}\sin^2 x - \tan x}{(1 - \cos x) \sinh^2 x}.$$

We have

$$\ln(1+x)$$

$$\sinh^2 x ($$

$$= \frac{1}{2}x^4 + o(x^4) = \frac{1}{2}x^4 + x^4\varepsilon_2(x) \text{ with } \lim_{x \rightarrow 0} \varepsilon_2(x) = 0.$$

So

$$\lim_{x \rightarrow 0} \frac{\ln(1+x) + \frac{1}{2}\sin^2 x - \tan x}{(1 - \cos x) \sinh^2 x} = \lim_{x \rightarrow 0} \frac{-\frac{5}{12}x^4 + x^4\varepsilon_1(x)}{\frac{1}{2}x^4 + x^4\varepsilon_2(x)} = \lim_{x \rightarrow 0} \frac{-\frac{5}{12} + \varepsilon_1(x)}{\frac{1}{2} + \varepsilon_2(x)} = -\frac{5}{6}.$$

#### Example 3.6

Calculate the limit

We have

$$\frac{x^2 - x}{x + 1} = -x + 2x^2 + o(x^2), \text{ so } \sin \frac{x^2 - x}{x + 1} = -x + 2x^2 + o(x^2)$$

and

$$\sin \frac{x^2 - x}{x + 1}$$

$$\arcsin x - \tan x = x + \frac{1}{3}x^3 - \left(x + \frac{1}{6}x^3\right) = -\frac{1}{6}x^3 + x^3\varepsilon_2(x) \text{ with } \lim_{x \rightarrow 0} \varepsilon_2(x) = 0.$$

So

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$$\lim_{x \rightarrow 0^-} \frac{\sin \frac{x^2 - x}{x + 1} + \ln(1 + x)}{\arcsin x - \tan x} = +\infty; \quad \lim_{x \rightarrow 0^+} \frac{\sin \frac{x^2 - x}{x + 1} + \ln(1 + x)}{\arcsin x - \tan x} = -\infty.$$

### 3.3.2 Study the relative position of the graph and the tangent line

To determine the relative position of the graph of a function  $f$  and its tangent line at the point  $x_0$ , we develop the function  $f$  in the neighborhood of  $x_0$  to the smallest order  $n$  such that  $a_n \neq 0$  and  $n \geq 2$ .

#### Exercise 3.1

Let the function  $f$  be defined by  $f(x) = \sqrt{x} - \ln\left(\cos\left(\frac{1}{2}x - \frac{1}{2}\right)\right)$ , and we denote by  $(C_f)$  the graph representing the function  $f$ .

- 1) Find a limited Development of order 3 in a neighborhood of 1 for the functions  $f$ .
- 2) Deduce the equation of the tangent  $(T)$  to the curve  $(C_f)$  at the abscissa point  $x = 1$ .
- 3) Determine the relative positions of  $(C_f)$  and  $(T)$ . What do you conclude?

#### Solution

$$1) F(h) = f(h + x_0) = f(h + 1) = \sqrt{1 + h} - \ln\left(\cos\frac{h}{2}\right).$$

We have

$$\ln\left(\cos\frac{h}{2}\right) = \ln\left(1 + \cos\frac{h}{2} - 1\right) = UoV(h) \text{ where } U(h) = \ln(1 + h) \text{ and } V(h) = \cos\frac{h}{2} - 1.$$

Since  $\ln(1 + x) = h - \frac{1}{2}h^2 + \frac{1}{3}h^3 + o(h^3)$  and  $\cos\frac{h}{2} - 1 = -\frac{1}{8}h^2 + o(h^3)$ , we get

$$\ln\left(\cos\frac{h}{2}\right) = UoV(h) = \left(-\frac{1}{8}h^2\right) - \frac{1}{2}\left(-\frac{1}{8}h^2\right)^2 + \frac{1}{3}\left(-\frac{1}{8}h^2\right)^3 = -\frac{1}{8}h^2 + o(h^3)$$

And we have  $\sqrt{1+h} = 1 + \frac{1}{2}h - \frac{1}{8}h^2 + \frac{1}{16}h^3 + o(h^3)$ .

So

$$\sqrt{1+h} - \ln\left(\cos\frac{h}{2}\right) = 1 + \frac{1}{2}h - \frac{1}{8}h^2 + \frac{1}{16}h^3 - \left(-\frac{1}{8}h^2\right) = 1 + \frac{1}{2}h + \frac{1}{16}h^3 + o(h^3).$$

Substituting  $h = x - 1$  we get:

$$f(x) = 1 + \frac{1}{2}(x-1) + \frac{1}{16}(x-1)^3 + o((x-1)^3) = \frac{1}{2}x - \frac{1}{2} + \frac{1}{16}(x-1)^3 + (x-1)^3 \varepsilon(x).$$

2) We have  $f(1) = -\frac{1}{2} = a_0$  so  $f$  is differentiable at  $x_0 = 1$ , and therefore  $(C_f)$  accepts tangent line which we denote by  $(T)$ . And the equation of  $(T)$  is  $y = \frac{1}{2}x - \frac{1}{2}$ .

3) We have  $f(x) - \left(\frac{1}{2}x - \frac{1}{2}\right) = (x-1)^3 \left(\frac{1}{16} + \varepsilon(x)\right)$ .

If  $x$  is sufficiently close to 1, the sign of the difference  $f(x) - \left(\frac{1}{2}x - \frac{1}{2}\right)$  is the same sign of  $(x-1)^3$ , hence the following result:

For  $x < 1$ ,  $(C_f)$  is located under the tangent.

For  $x > 1$ ,  $(C_f)$  is located above the tangent.

We conclude that  $(C_f)$  accepts an inflection point  $A_0(1,1)$ .

### 3.3.3 The study of infinite branches of curves.

To study the infinite branches and determine the asymptotic lines of the graph  $(C_f)$  of function  $f$  in the neighborhood of  $+\infty$  ( $-\infty$ , respectively), we develop the function  $f$  in the neighborhood of  $+\infty$  ( $-\infty$ , respectively) to the smallest order  $n$ , where  $a_n \neq 0$  and  $n \in \mathbb{N}^*$ .

#### Exercise 3.2

Let the function  $g$  be defined by  $g(x) = xe^{\frac{x-1}{2x^2-3x}}$ , and we denote by  $(C_g)$  the graph representing the function  $g$ .

1) Find a limited Development of order 2 in a neighborhood of  $\infty$  for the functions  $f$ .

2) Deduce that the curve  $(C_g)$  accepts an asymptote  $(\Delta)$  and write an equation for it. Study the relative position of  $(C_g)$  and  $(\Delta)$  in a neighborhood of  $\infty$ .

#### Solution

1) We put  $G(h) = g\left(\frac{1}{h}\right) = \frac{1}{h} e^{\frac{h-h}{3h-2}}$ .

We have

$$\frac{h^2 - h}{3h - 2} = \frac{1}{2}h + \frac{1}{4}h^2 + o(h^2) \quad \text{and} \quad e^h = 1 + h + \frac{1}{2}h^2 + o(h^2).$$

So

$$\begin{aligned} G(h) &= \frac{1}{h} \left( 1 + \left( \frac{1}{2}t + \frac{1}{4}t^2 \right) + \frac{1}{2} \left( \frac{1}{2}t + \frac{1}{4}t^2 \right)^2 \right) \\ &= \frac{1}{h} \left( 1 + \frac{1}{2}t + \frac{3}{8}t^2 + o(h^2) \right). \end{aligned}$$

Substituting  $h = \frac{1}{x}$  we get:

$g($

2) Since  $\lim_{x \rightarrow \infty} \left( \frac{3}{8x} + \frac{1}{x} \varepsilon(x) \right) = 0$ , then

a) The line with the equation:  $y = x + \frac{1}{2}$  is asymptotic to the curve  $(C_g)$  in the neighborhood of  $\infty$ .

If  $x$  is sufficiently close to 1, the sign of the difference  $f(x) - \left( \frac{1}{2}x - \frac{1}{2} \right)$  is the same sign of  $(x - 1)^3$ , hence the following result:

b) We also have:  $g(x) - \left( x + \frac{1}{2} \right) = \frac{1}{x} \left( \frac{3}{8} + \varepsilon(x) \right)$ , for  $|x|$  Big enough, the sign of the difference  $g(x) - \left( x + \frac{1}{2} \right)$  is the same sign of  $\frac{3}{8x}$ . Hence the following result:

In a neighborhood of  $-\infty$ ,  $(C_f)$  is located under the asymptotic.

In a neighborhood of  $+\infty$ ,  $(C_f)$  is located above the asymptotic.

### 3.4 generalised limited development

Let  $f$  be a function defined in the neighborhood of a point 0, - with the possible exception of 0. We suppose that  $f$  does not admit a limited development to the neighborhood of 0 but the function  $x \rightarrow x^\alpha f(x)$  ( $\alpha \in \mathbb{R}_+^*$ ) admits a limited development of order  $n$  in the neighborhood of 0. We can then write in the neighborhood of 0 and for  $x \neq 0$ :

$$x^\alpha f(x) = a_0 + a_1x + a_2x^2 \dots \dots \dots + a_nx^n + x^n\varepsilon(x) \quad \text{with} \quad \lim_{x \rightarrow 0} \varepsilon(x) = 0.$$

From where, the generalized limited development of  $f$  in the neighborhood of 0 is:

$$f(x) = \frac{1}{x^\alpha} \left( a_0 + a_1x + a_2x^2 \dots \dots \dots + a_nx^n + x^n\varepsilon(x) \right) \quad \text{with} \quad \lim_{x \rightarrow 0} \varepsilon(x) = 0.$$

**Limited Development of usual Functions ( in the neighborhood of 0).**

$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots + \frac{1}{n!}x^n + o(x^n)$
$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 \dots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + o(x^{2n+2})$
$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 \dots + \frac{(-1)^n}{(2n)!}x^{2n} + o(x^{2n+1})$
$\sinh x = x + \frac{1}{6}x^3 + \frac{1}{120}x^5 \dots + \frac{1}{(2n+1)!}x^{2n+1} + o(x^{2n+2})$
$\cosh x = 1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \dots + \frac{1}{(2n)!}x^{2n} + o(x^{2n+1})$
$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \dots + \frac{\alpha(\alpha-1)\dots(\alpha-n+1)}{n!}x^n + o(x^n) \quad (\alpha \in \mathbb{R}_+^*)$
$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n + o(x^n)$
$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + o(x^n)$
$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \dots - \frac{1}{n}x^n + o(x^n)$
$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots + \frac{(-1)^{n-1}}{n}x^n + o(x^n)$
$\arctan x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 \dots + \frac{(-1)^n}{2n+1}x^{2n+1} + o(x^{2n+2})$
$\arcsin x = x + \frac{1}{2} \frac{x^3}{3} + \frac{1 \times 3}{2 \times 4} \frac{x^5}{5} + \frac{1 \times 3 \times 5}{2 \times 4 \times 6} \frac{x^7}{7} \dots + \frac{1 \times 3 \times (2n-1)}{2 \times 4 \times (2n)} \frac{x^{2n+1}}{(2n+1)} + o(x^{2n+2})$
$\operatorname{argtanh} x = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 \dots + \frac{1}{2n+1}x^{2n+1} + o(x^{2n+2})$
$\operatorname{argsinh} x = x - \frac{1}{2} \frac{x^3}{3} + \frac{1 \times 3}{2 \times 4} \frac{x^5}{5} - \frac{1 \times 3 \times 5}{2 \times 4 \times 6} \frac{x^7}{7} \dots + (-1)^n \frac{1 \times 3 \times (2n-1)}{2 \times 4 \times (2n)} \frac{x^{2n+1}}{(2n+1)} + o(x^{2n+2})$
$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \frac{1382}{155925}x^{11} + o(x^{12})$
$\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \frac{62}{2835}x^9 - \frac{1382}{155925}x^{11} + o(x^{12})$
$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 - \frac{21}{1024}x^6 + \frac{33}{2048}x^7 - \frac{429}{32768}x^8 + o(x^8)$