1. Introduction

Consider a bar rigidly clamped at one end and twisted at the other end by a torque (twisting moment) $M_t = F.d$ applied in a plane perpendicular to the axis of the bar, as shown in Fig. 1. Such a bar is in ***torsion***. An alternative representation of the torque is the curved arrow shown in the figure.

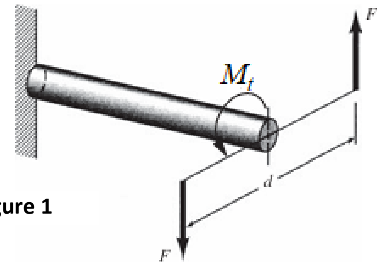

Figure 1

The twisting moment is then : $M_t = F.d$

Basic assumptions:

- Assumptions of plane sections, straightness of radii, and conservation of distances between sections of the torsionally oriented shaft;
- the nature of the deformation is pure shear;

Note : the present study concerns the case of circular shafts

2. Deformation study

Let us derive an expression relating the applied twisting moment acting on a shaft of circular cross section and the shearing stress at any point in the shaft. In Fig. 2, the shaft is shown loaded by the two torques $\boldsymbol{M_t}$ in static equilibrium.

A generator O_1A, deforms into the configuration O_1B. The angle between these configurations is denoted by α.

By definition, the shearing strain γ on the surface of the shaft is :

$$\gamma = \tan \alpha = \alpha$$

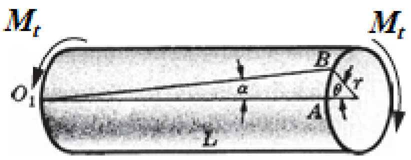

Figure 2- Torque acting on a section of a shaft.

Where the angle α is assumed to be small. From the geometry of the figure,

$$\alpha = \frac{AB}{L} = \frac{r\theta}{L}$$

Angle θ is defined as the angle of twist.

Hence $$\gamma = \frac{r\theta}{L} \quad (1)$$

But since a diameter of the shaft remain the same after torsion, the shearing strain at a distance ρ is $\gamma_\rho = \rho\theta / L$.

Consequently the shearing strains of the longitudinal fibers ***vary linearly as the distances from the center of the shaft***.

3. Twisting moment and shearing stress relation

The distribution of shearing stresses is symmetric around the geometric axis of the shaft. They have the appearance shown in Fig. 3. For equilibrium, the sum of the moments of these distributed shearing forces over the entire circular cross section is equal to the torque M_t. Thus we have

$$M_t = \int_0^r \tau\rho \, dA \quad (2)$$

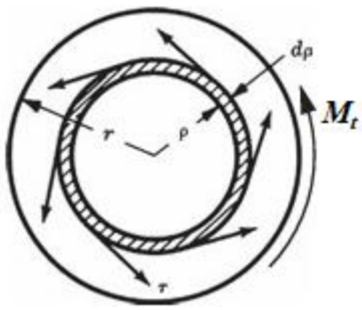

Figure 3- Shearing stress act ing on a differential area element.

The shearing stress varies as the distance from the axis; hence

$$\frac{\tau_\rho}{\rho} = \frac{\tau_r}{r} = \text{constant}$$

Consequently we may write

$$M_t = \int_0^r \frac{\tau_\rho}{\rho}\left(\rho^2\right)\, dA = \frac{\tau_\rho}{\rho}\int_0^r \rho^2\, dA$$

The expression $\int_0^r \rho^2\, dA$ is by definition the polar mement of inertia of the cross-sectional area. Hence the desired relationship is

$$M_t = \frac{\tau_\rho I_p}{\rho} \quad \text{or} \quad \tau_\rho = \frac{M_t \rho}{I_p} \qquad (3)$$

3.1 Torsional Shearing Stress

So, for either a solid or a hollow circular shaft subject to a twisting moment $\boldsymbol{M_t}$ the torsional shearing stress τ at a distance $\boldsymbol{r}$ from the center of the shaft is written as

$$\tau = \frac{M_t}{I_p}\rho \qquad (4)$$

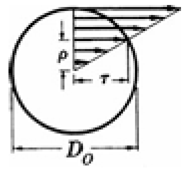

Figure 4 -Shearing stress distribution in a solid shaft.

3.2 Shearing Strain

The ratio of the shear stress t to the shear strain γ is called the *shear modulus* and is given by

$$G = \frac{\tau}{\gamma} \qquad (5)$$

Again the units of $\boldsymbol{G}$ are the same as those of shear stress, since the shear strain is dimensionless.

Using Eq. (4) for τ and Eq. (1) for γ along with Eq. (5), we find the expression for θ to be

$$\theta = \frac{M_t L}{I_p G} \qquad (6)$$

where at the outermost fiber, ρ=r.