MATH20142

Complex Analysis

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The University of Manchester

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0. Preliminaries

§0.1 Contact details

The lecturer is Dr. Charles Walkden, Room 2.241, Tel: 0161 275 5805, Email: charles.walkden@manchester.ac.uk.

My office hour is: Monday 10am–11am. If you want to see me at another time then please email me first to arrange a mutually convenient time.

§0.2 Course structure

§0.2.1 Learning outcomes

At the end of the course you will be able to

- prove the Cauchy-Riemann Theorem and its converse and use them to decide whether a given function is holomorphic;
- use power series to define a holomorphic function and calculate its radius of convergence;
- define and perform computations with elementary holomorphic functions such as sin, cos, sinh, cosh, exp, log, and functions defined by power series;
- define the complex integral and use a variety of methods (the Fundamental Theorem of Contour Integration, Cauchy's Theorem, the Generalised Cauchy Theorem and the Cauchy Residue Theorem) to calculate the complex integral of a given function;
- use Taylor's Theorem and Laurent's Theorem to expand a holomorphic function in terms of power series on a disc and Laurent series on an annulus, respectively;
- identify the location and nature of a singularity of a function and, in the case of poles, calculate the order and the residue;
- apply techniques from complex analysis to deduce results in other areas of mathematics, including proving the Fundamental Theorem of Algebra and calculating infinite real integrals, trigonometric integrals, and the summation of series.

§0.2.2 Lectures

There will be approximately 21 lectures in total.

The lecture notes are available on the course webpage. The course webpage is available via Blackboard or directly at
`personalpages.manchester.ac.uk/staff/charles.p.walkden/complex-analysis`.
Please let me know of any mistakes or typos that you find in the notes.

I will use the visualiser for the majority of the lectures. Each week, I will upload scanned copies of what I write on the visualiser onto the course webpage.

The lectures will be recorded via the University's 'Lecture Capture' (podcast) system. Remember that Lecture Capture is a useful revision tool but it is not a substitute for attending lectures. The support classes are not podcasted.

§0.2.3 Exercises

The lecture notes also contain the exercises (at the end of each section). The exercises are an integral part of the course and you should make a serious attempt at them.

The lecture notes also contain the solutions to the exercises. I will trust you to have serious attempts at solving the exercises without looking at the solutions.

§0.2.4 Tutorials and support classes

The tutorial classes start in Week 2. There are 6 classes for this course but you only need go to one each week. You will be assigned to a class. Attendance at tutorial classes is recorded and monitored by the Teaching and Learning Office. If you go to a class other than the one you've been assigned to then you will normally be recorded as being absent.

I try to run the tutorial classes so that the majority of people get some benefit from them. Each week I will prepare a worksheet. The worksheets will normally contain exercises from the lecture notes or from past exam questions. I will often break the exercises down into easier, more manageable, subquestions; the idea is that then everyone in the class can make progress on them within the class. (If you find the material in the examples classes too easy then great!—it means that you are progressing well with the course.) You still need to work on the remaining exercises (and try past exams) in your own time!

There will be regular Kahoot quizzes in the tutorials. The Kahoot quizzes are designed so that you can get instant feedback on your understanding of important parts of the course.

I will not normally put the worksheets (or their solutions) on the course webpage. There is nothing on the worksheets that isn't already contained in either the exercises, lecture notes or past exam papers that are already on the course webpage. The worksheets tell you which exercises are being covered. When the worksheets contain material that you do not already have access to then I will make these available on the course webpage.

§0.2.5 Coursework

The coursework for this year will be a closed-book test taking place during Week 7. You can see the time and location on your personalised timetable, available via my.manchester.ac.uk. You will need to know §§2,3,4 from the course for the test (this is the material that we will cover in weeks 1–5).

Your coursework script, with feedback, will be returned to you within 15 working days of the test. You will be able to collect your script from the reception desk in the Alan Turing Building.

§0.2.6 The exam

In terms of what is examinable:

- Anything that I cover (including proofs) in the lectures can be regarded as being examinable (unless I explicitly say otherwise in the lectures).
- There may be a small amount of material in these lecture notes that I do not cover in the lectures; this will not be examinable.

- For the avoidance of doubt, the proofs of the following theorems will be discussed in the lectures but are **not** examinable: Proposition 2.5.2, Theorem 3.3.2, Lemma 3.3.2, Lemma 4.4.2, Proposition 4.5.1, Theorem 4.5.5, Theorem 5.1.1, Theorem 5.2.1, Theorem 6.2.1, Theorem 7.3.1. However, understanding the ideas in the proofs may help you gain a wider understanding of the subject and how different parts of the course relate to each other.

- The exercises are at a similar level (in terms of style/difficulty) to the (non-bookwork) parts of the exam.

§0.3 Recommended texts

The lecture notes cover everything that is in the course and you probably do not need to buy, or refer to, a book.

If you do want a text to refer to then the most suitable is

I.N. Stewart and D.O. Tall, Complex Analysis, Cambridge University Press, 1983.

(This is also an excellent source of additional exercises.)

The best book (in my opinion) on complex analysis is

L.V. Ahlfors, Complex Analysis, McGraw-Hill, 1979

although it is perhaps too advanced to be used as a substitute for the lectures/lecture notes for this course. There are many other books on complex analysis available either in the library, on Amazon, or online; many will be suitable for this course, although I should also warn you that some are not very good...

1. Introduction

§1.1 Where we are going

You are already familiar with how to differentiate and integrate real-valued functions defined on the real line. For example, if $f : \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = 3x^2 + 2x$ then you already know that $f'(x) = 6x + 2$ and that $\int f(x)\,dx = x^3 + x^2 + c$. You saw how to formally define differentiation and integration for functions that map the reals to the reals in the Real Analysis course. In this course, we will look at what it means for functions defined on the complex plane to be differentiable or integrable and look at ways in which one can integrate complex-valued functions. Surprisingly, the theory turns out to be considerably easier than the real-valued case! Thus the word 'complex' in the title refers to the presence of complex numbers, and not that the analysis is harder!

One of the highlights towards the end of the course is Cauchy's Residue Theorem. This theorem gives a new method for calculating real integrals that would be difficult or impossible just using techniques that you know from real analysis. For example, let $0 < a < b$ and consider

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2 + a^2)(x^2 + b^2)}\,dx. \tag{1.1.1}$$

If you try calculating this using techniques that you know (integration by substitution, integration by parts, etc) then you will quickly hit an impasse. However, using complex analysis one can evaluate (1.1.1) in about five lines of work![1]

§1.2 Recap on complex numbers

A complex number is an expression of the form $x + iy$ where $x, y \in \mathbb{R}$. (Here i denotes $\sqrt{-1}$ so that $i^2 = -1$.) We denote the set of complex numbers by $\mathbb{C}$. We can represent $\mathbb{C}$ as the Argand diagram or complex plane by drawing the point $x + iy \in \mathbb{C}$ as the point with co-ordinates (x, y) in the plane $\mathbb{R}^2$ (see Figure 1.2.1).

If $a + ib, c + id \in \mathbb{C}$ then we can add and multiply them as follows

$$(a + ib) + (c + id) = (a + c) + i(b + d)$$

$$(a + ib)(c + id) = ac + iad + ibc + i^2bd = (ac - bd) + i(ad + bc).$$

To divide complex numbers we use the following trick (often referred to as 'realising the denominator')

$$\frac{1}{a+ib} = \frac{1}{a+ib}\frac{a-ib}{a-ib} = \frac{a-ib}{a^2 - i^2b^2} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}.$$

We shall often denote a complex number by the letters z or w. Suppose that $z = x + iy$ where $x, y \in \mathbb{R}$. We call x the *real part* of z and write $x = \mathrm{Re}(z)$. We call y the *imaginary part* of z and write $y = \mathrm{Im}(z)$. (Note: the imaginary part of $x + iy$ is y, and not iy.)

[1] The answer is $\frac{\pi}{b^2-a^2}(e^{-a} - e^{-b})$.

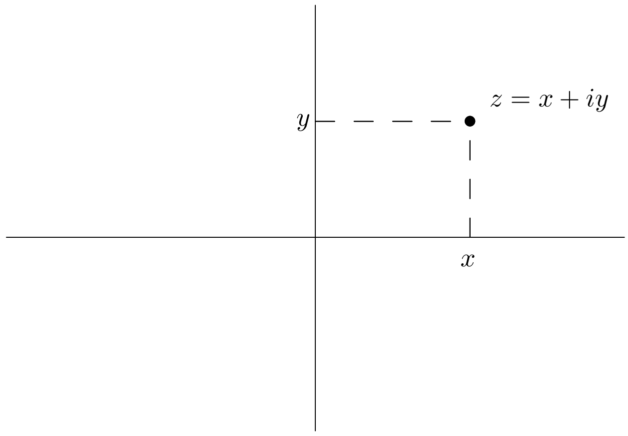

Figure 1.2.1: The Argand diagram or the complex plane. Here $z = x + iy$.

We say that $z \in \mathbb{C}$ is *real* if $\mathrm{Im}(z) = 0$ and we say that $z \in \mathbb{C}$ is *imaginary* if $\mathrm{Re}(z) = 0$. In the complex plane, the set of real numbers corresponds to the x-axis (which we will often call the *real axis*) and the set of imaginary numbers corresponds to the y-axis (which we will often call the *imaginary axis*).

If $z = x + iy$, $x, y \in \mathbb{R}$ then we define $\bar{z} = x - iy$ to be the *complex conjugate* of z.

Let $z = x + iy$, $x, y \in \mathbb{R}$. The *modulus* (or *absolute value*) of z is

$$|z| = \sqrt{x^2 + y^2} \geq 0.$$

(If z is real then this is just the usual absolute value.) It is straightforward to check that $|\bar{z}| = |z|$ and that

$$z\bar{z} = (x + iy)(x - iy) = x^2 + y^2 = |z|^2.$$

Here are some basic properties of $|z|$:

Proposition 1.2.1
Let $z, w \in \mathbb{C}$. Then

(i) $|z| = 0$ *if and only if* $z = 0$*;*

(ii) $|zw| = |z|\,|w|$*;*

(iii) $\left|\frac{1}{z}\right| = \frac{1}{|z|}$ *if* $z \neq 0$*;*

(iv) $|z + w| \leq |z| + |w|$*;*

(v) $||z| - |w|| \leq |z - w|$.

Remark. The inequality $|z + w| \leq |z| + |w|$ is often called the *triangle inequality*. The inequality $||z| - |w|| \leq |z - w|$ is often called the *reverse triangle inequality*.

Proof. Parts (i), (ii) and (iii) follow easily from the definition of $|z|$. We leave (v) as an exercise (see Exercise 1.6). To see (iv), first note that if $z = x + iy$ then $\mathrm{Re}(z) = x \leq$

$\sqrt{x^2 + y^2} = |z|$. Then

$$\begin{aligned}
|z+w|^2 &= (z+w)\overline{(z+w)} \\
&= (z+w)(\bar{z}+\bar{w}) \\
&= z\bar{z} + w\bar{w} + z\bar{w} + \bar{z}w \\
&= |z|^2 + |w|^2 + z\bar{w} + \overline{z\bar{w}} \\
&= |z|^2 + |w|^2 + 2\,\mathrm{Re}(z\bar{w}) \text{ using Exercise 1.5(iv)} \\
&\leq |z|^2 + |w|^2 + 2|z\bar{w}| \\
&= |z|^2 + |w|^2 + 2|z||\bar{w}| \\
&= |z|^2 + |w|^2 + 2|z||w| \\
&= (|z| + |w|)^2.
\end{aligned}$$

□

Let $z \neq 0$. If we plot the point z in the complex plane then $|z|$ denotes the length of the vector joining the origin 0 to the point z. See Figure 1.2.2. The angle θ in Figure 1.2.2 is

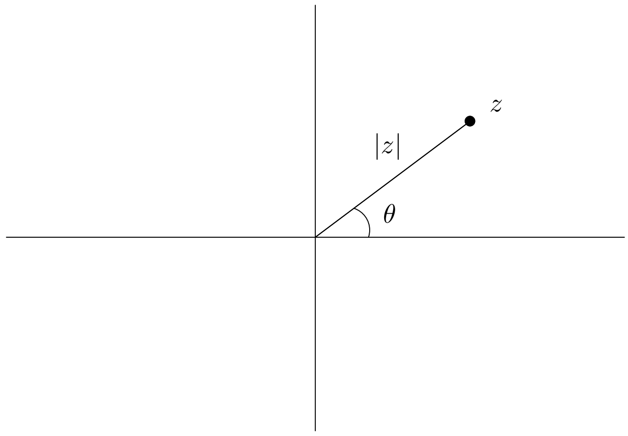

Figure 1.2.2: The modulus $|z|$ and argument $\arg z$ of z.

called the argument of z and we write $\theta = \arg z$. We have that $\tan\theta = y/x$. Note that θ is not uniquely determined: if we replace θ by $\theta + 2n\pi$, $n \in \mathbb{Z}$, then we get the same point. However, there is a unique value of θ such that $-\pi < \theta \leq \pi$; this is called the *principal value* of $\arg z$. We write $\mathrm{Arg}(z)$ for the principal value of the argument of z.

Let $z \in \mathbb{C}$. We can represent z in polar co-ordinates as follows. First write $z = x + iy$ and draw z in the complex plane; see Figure 1.2.3. Then $x = r\cos\theta$ and $y = r\sin\theta$ where θ is the argument of z and $r = \sqrt{x^2 + y^2} = |z|$. We call (r, θ) the *polar co-ordinates* of z and write $z = r(\cos\theta + i\sin\theta)$.

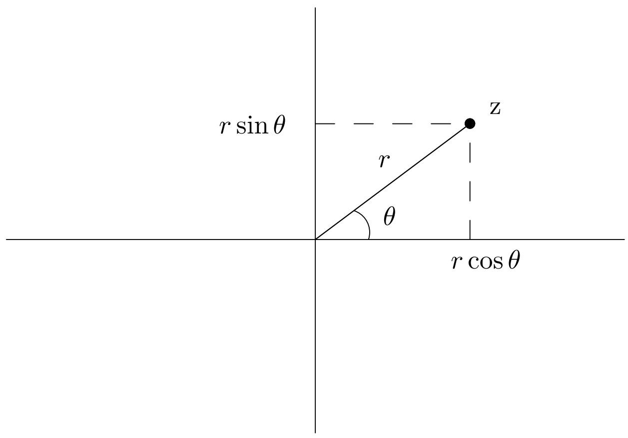

Figure 1.2.3: If z has polar co-ordinates (r, θ) then the real part of z is $r\cos\theta$ and the imaginary part of z is $r\sin\theta$, and conversely.

Exercises for Part 1

The following exercises are provided for you to revise complex numbers.

Exercise 1.1
Write the following expressions in the form $x + iy$, $x, y \in \mathbb{R}$:

$$\text{(i) } (3+4i)^2; \text{ (ii) } \frac{2+3i}{3-4i}; \text{ (iii) } \frac{1-5i}{3i-1}; \text{ (iv) } \frac{1-i}{1+i} - i + 2; \text{ (v) } \frac{1}{i}.$$

Exercise 1.2
Find the modulus, the argument and the principal value of the argument for the following complex numbers:

$$\text{(i) } 2i; \text{ (ii) } -1 - i\sqrt{3}; \text{ (iii) } -4.$$

Exercise 1.3
By writing $z = x + iy$ find all solutions of the following equations:

$$\text{(i) } z^2 = -5 + 12i; \text{ (ii) } z^2 + 4z + 12 - 6i = 0.$$

Exercise 1.4
Let $z, w \in \mathbb{C}$. Show that (i) $\mathrm{Re}(z \pm w) = \mathrm{Re}(z) \pm \mathrm{Re}(w)$, (ii) $\mathrm{Im}(z \pm w) = \mathrm{Im}(z) \pm \mathrm{Im}(w)$. Give examples to show that neither $\mathrm{Re}(zw) = \mathrm{Re}(z)\,\mathrm{Re}(w)$ nor $\mathrm{Im}(zw) = \mathrm{Im}(z)\,\mathrm{Im}(w)$ hold in general.

Exercise 1.5
Let $z, w \in \mathbb{C}$. Show that (i) $\overline{z \pm w} = \bar{z} \pm \bar{w}$, (ii) $\overline{zw} = \bar{z}\bar{w}$, (iii) $\overline{\left(\frac{1}{z}\right)} = \frac{1}{(\bar{z})}$ if $z \neq 0$, (iv) $z + \bar{z} = 2\,\mathrm{Re}(z)$, (v) $z - \bar{z} = 2i\,\mathrm{Im}(z)$.

Exercise 1.6
Let $z, w \in \mathbb{C}$. Show, using the triangle inequality, that the reverse triangle inequality holds:

$$||z| - |w|| \leq |z - w|.$$

Exercise 1.7
Draw the set of all $z \in \mathbb{C}$ satisfying the following conditions

$$\text{(i) } \mathrm{Re}(z) > 2; \text{ (ii) } 1 < \mathrm{Im}(z) < 2; \text{ (iii) } |z| < 3; \text{ (iv) } |z-1| < |z+1|.$$

Exercise 1.8
(i) Let $z, w \in \mathbb{C}$ and write them in polar form as $z = r(\cos\theta + i\sin\theta)$, $w = s(\cos\phi + i\sin\phi)$ where $r, s > 0$ and $\theta, \phi \in \mathbb{R}$. Compute the product zw. Hence, using formulæ for $\cos(\theta + \phi)$ and $\sin(\theta + \phi)$, show that $\arg zw = \arg z + \arg w$ (we write $\arg z_1 = \arg z_2$ if the principal argument of z_1 differs from that of z_2 by $2k\pi$ with $k \in \mathbb{Z}$).

(ii) By induction on n, derive De Moivre's Theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

(iii) Use De Moivre's Theorem to derive formulæ for $\cos 3\theta$, $\sin 3\theta$, $\cos 4\theta$, $\sin 4\theta$ in terms of $\cos\theta$ and $\sin\theta$.

Exercise 1.9
Let $w_0 \neq 0$ be a complex number such that $|w_0| = r$ and $\arg w_0 = \theta$. Find the polar forms of all the solutions z to $z^n = w_0$, where $n \geq 1$ is a positive integer.

Exercise 1.10
Let $\mathrm{Arg}(z)$ denote the principal value of the argument of z. Give an example to show that, in general, $\mathrm{Arg}(z_1 z_2) \neq \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$ (c.f. Exercise 1.8(i)).

Exercise 1.11
Try evaluating the integral in (1.1.1), i.e.

$$\int_{-\infty}^{\infty} \frac{x\sin x}{(x^2+a^2)(x^2+b^2)}\,dx$$

using the methods that you already know (substitution, partial fractions, integration by parts, etc). (There will be a prize for anyone who can do this integral by hand in under 2 pages using such methods!)

2. Limits and differentiation in the complex plane and the Cauchy-Riemann equations

§2.1 Open sets and domains

In real analysis, one normally studies functions $f : (a, b) \to \mathbb{R}$, $f : [a, b] \to \mathbb{R}$ or $f : \mathbb{R} to \mathbb{R}$ where $(a, b) \subset \mathbb{R}$ is an open interval or $[a, b] \subset \mathbb{R}$ is a closed interval. Usually, when talking about continuity or differentiability in real analysis, one studies functions defined on open intervals or on the entire real line. This is often because, when studying whether a function $f : (a, b) \to \mathbb{R}$ is continuous or differentiable at a point $x_0 \in (a, b)$, we want to be able to look at points x near to x_0 (either to the left or to the right of x_0) and then take the limit at $x \to x_0$. We will need the appropriate generalisation to complex analysis of the notion of 'defined on a open interval' that captures similar behaviour.

A major difference between real and complex analysis is that the geometry of the complex plane is far richer than that of the real line. For example, the only connected subsets of $\mathbb{R}$ are intervals, whereas there are far more complicated connected subsets in $\mathbb{C}$ ('connected' has a rigorous meaning, but for now you can assume that a subset is connected if it 'looks' connected: i.e. any two points in the subset can be joined by a line that does not leave the subset). We need to make precise what we mean by convergence, open sets (generalising open intervals), etc, in $\mathbb{C}$.

Remark. Throughout, we use $\subset$ (rather than $\subseteq$) to denote 'is a subset of'. Thus $A \subset B$ means that A is a subset (indeed, possibly equal to) B.

Definition. Let $z_0 \in \mathbb{C}$ and let $\varepsilon > 0$. We write

$$B_\varepsilon(z_0) = \{z \in \mathbb{C} \mid |z - z_0| < \varepsilon\}$$

to denote the open disc in $\mathbb{C}$ of complex numbers that are distance at most ε from z_0. We call $B_\varepsilon(z_0)$ the *ε-neighbourhood* of z_0.

Definition. Let $D \subset \mathbb{C}$. We say that D is an *open set* if for every $z_0 \in D$ there exists $\varepsilon > 0$ such that $B_\varepsilon(z_0) \subset D$.

Definition. We call a set D *closed* if its complement $\mathbb{C} \setminus D$ is open

Remark. Note that a set is closed precisely when the complement is open. A very common mistake is to think that 'closed' means 'not open': this is not the case, and it is easy to write down examples of sets that are neither open nor closed (can you think of any?).

In our setting, one can often decide whether a set is open or not by looking at it and thinking carefully. (A more rigorous treatment of open sets is given in the MATH20122

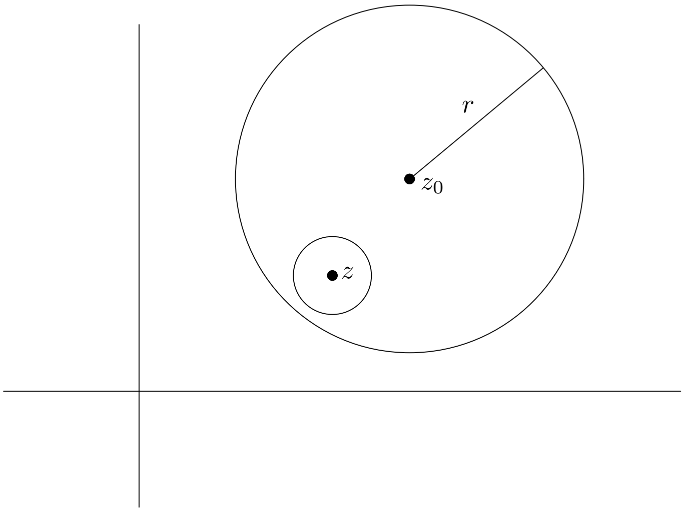

Figure 2.1.1: The open disc $B_r(z_0)$ with centre z_0 and radius $r > 0$. This is an open set as, given any point $z \in \mathbb{C}$, we can find another open disc centred at z that is contained in $B_r(z_0)$.

Metric Spaces course.) For example, any open disc $\{z \in \mathbb{C} \mid |z - z_0| < r\}$ is an open set; see Figure 2.1.1

We will also need the notion of a polygonal arc in $\mathbb{C}$. Let $z_0, z_1 \in \mathbb{C}$. We denote the straight line from z_0 to z_1 by $[z_0, z_1]$. Now let $z_0, z_1, \ldots, z_r \in \mathbb{C}$. We call the union of the straight lines $[z_0, z_1], [z_1, z_2], \ldots, [z_{r-1}, z_r]$ a *polygonal arc* joining z_0 to z_r.

Open subsets of $\mathbb{C}$ may be very complicated. We will only be interested in 'nice' open sets called domains.

Definition. Let $D \subset \mathbb{C}$ be a non-empty set. Then we say that D is a *domain* if

(i) D is open;

(ii) given any two point $z_1, z_2 \in D$, there exists a polygonal arc contained in D that joins z_1 to z_2.

Examples.

(i) The open disc $\{z \in \mathbb{C} \mid |z - z_0| < r\}$ centred at $z_0 \in \mathbb{C}$ and of radius r is a domain.

(ii) An annulus $\{z \in \mathbb{C} \mid r_1 < |z - z_0| < r_2\}$ is a domain.

(iii) A half-plane such as $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > a\}$ is a domain.

(iv) A closed disc $\{z \in \mathbb{C} \mid |z - z_0| \leq r\}$ or a closed half-plane $\{z \in \mathbb{C} \mid \mathrm{Re}(z) \geq a\}$ are not domains as they are not open sets.

(v) The set $D = \{z \in \mathbb{C} \mid \mathrm{Im}(z) \neq 0\}$, corresponding to the complex plane with the real axis deleted, is not a domain. Although it is an open set, there are points (such as i, $-i$) that cannot be connected by a polygonal arc lying entirely in D.

See Figure 2.1.2 for examples of domains.

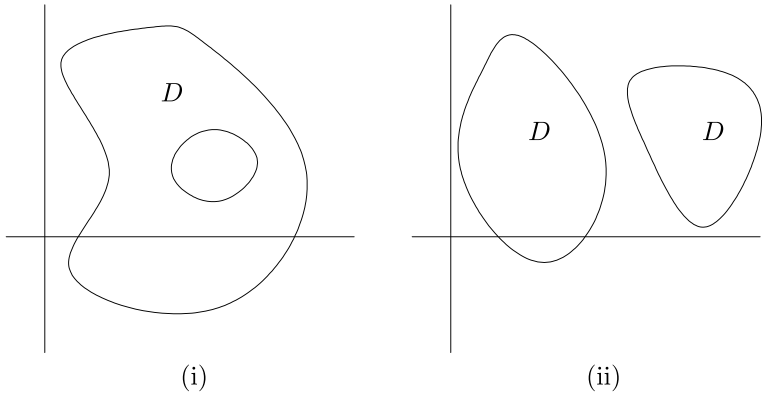

Figure 2.1.2: In (i), D is a domain. In (ii) D is not a domain as it is not connected.

§2.2 Limits of complex sequences

Let $z_n \in \mathbb{C}$ be a sequence of complex numbers. We say that $z_n \to z$ as $n \to \infty$ if: for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $n \geq N$ then $|z_n - z| < \varepsilon$. (Equivalently, we may say that $z_n \to z$ if $|z_n - z|$ is a null sequence.)

Lemma 2.2.1
Let $z_n \in \mathbb{C}$ and write $z_n = x_n + iy_n$, $x_n, y_n \in \mathbb{R}$. Then z_n converges if and only if x_n and y_n converge.

Proof. Suppose that $z_n \to z$ and write $z = x + iy$. Then

$$|x_n - x| \leq \sqrt{|x_n - x|^2 + |y_n - y|^2} = |z_n - z| \to 0$$

as $n \to \infty$. Hence $x_n \to x$. A similar argument show that $y_n \to y$.

Conversely, suppose that $x_n \to x$ and $y_n \to y$. Then

$$|z_n - z| = \sqrt{|x_n - x|^2 + |y_n - y|^2} \to 0$$

so that $z_n \to z$. □

§2.3 Complex functions and continuity

Let $D \subset \mathbb{C}$, $D \neq \emptyset$. A *function* $f : D \to \mathbb{C}$ is a rule that assigns to each point $z \in D$ an image $f(z) \in \mathbb{C}$.

Write $z = x + iy$. Then saying that f is a function is equivalent to saying that there are two real-valued functions $u(x, y)$ and $v(x, y)$ of the two real variables x, y such that

$$f(z) = u(x, y) + iv(x, y).$$

Example. Let $f(z) = z^2$. Then $f(x + iy) = (x + iy)^2 = x^2 - y^2 + 2ixy$. Here $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$.

Example. Let $f(z) = \bar{z}$. Then $f(x + iy) = x - iy$. Here $u(x, y) = x$, $v(x, y) = -y$.

Let D be a domain and let $f : D \to \mathbb{C}$. Let $z_0 \in D$. We say that $\lim_{z \to z_0} f(z) = \ell$ (or, equivalently, $f(z)$ tends to ℓ as z tends to z_0) if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that if $z \in D$ and $0 < |z - z_0| < \delta$ then $|f(z) - \ell| < \varepsilon$.

That is, $f(z) \to \ell$ as $z \to z_0$ means that if z is very close (but not equal to) z_0 then $f(z)$ is very close to ℓ. Note that in this definition we do not need to know the value of $f(z_0)$.

Example. Let $f : \mathbb{C} \to \mathbb{C}$ be defined by $f(z) = 1$ if $z \neq 0$ and $f(0) = 0$. Then $\lim_{z \to 0} f(z) = 1$. Here $\lim_{z \to 0} f(z) \neq f(0)$.

We will be interested in functions which do behave nicely when taking limits.

Definition. Let D be a domain and let $f : D \to \mathbb{C}$ be a function. We say that f is *continuous at* $z_0 \in D$ if

$$\lim_{z \to z_0} f(z) = f(z_0).$$

We say that f is *continuous on* D if it is continuous at z_0 for all $z_0 \in D$.

Continuity obeys the same rules as in Real Analysis. In particular, suppose that $f, g : D \to \mathbb{C}$ are complex functions which are continuous at z_0. Then

$$f(z) + g(z),\ f(z)g(z),\ cf(z)\ (c \in \mathbb{C})$$

are all continuous at z_0, as is $f(z)/g(z)$ provided that $g(z_0) \neq 0$.

§2.4 Differentiable functions

Let us first consider how one differentiates real valued functions defined on $\mathbb{R}$. You will cover this properly in the Real Analysis course, and some of you will have seen 'differentiation from first principles' at A-level or high school. Let $(a, b) \subset \mathbb{R}$ be an open interval and let$f : (a, b) \to \mathbb{R}$ be a function. Let $x_0 \in (a, b)$. The idea is that $f'(x_0)$ is the slope of the graph of f at the point x_0. Heuristically, one takes a point x that is near x_0 and looks at the gradient of the straight line drawn between the points $(x_0, f(x_0))$ and $(x, f(x))$ on the graph of f; this is an approximation to the slope at x_0, and becomes more accurate as x approaches x_0. We then say that f is differentiable at x_0 if this limit exists, and define the derivative of f at x_0 to be the value of this limit.

Definition. Let $(a, b) \subset \mathbb{R}$ be an interval and let $x_0 \in (a, b)$. A function $f : (a, b) \to \mathbb{R}$ is *differentiable at* x_0 if

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{2.4.1}$$

exists. We call $f'(x_0)$ the *derivative of* f *at* x_0. We say that f is *differentiable* if it is differentiable at all points $x_0 \in (a, b)$.

Remark. Notice that there are two ways that x can approach x_0: x can either approach x_0 from the left or from the right. The definition of the derivative in (2.4.1) requires the limit to exist from both the left and the right and for the value of these limits to be the same.

(As an aside, one could instead look at left-handed and right-handed derivatives. For example, consider $f(x) = |x|$ defined on $\mathbb{R}$. The left-handed derivative at 0 is

$$\lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{-x}{x} = -1$$

and the right-handed derivative at 0 is

$$\lim_{x\to 0+} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0-} \frac{x}{x} = 1.$$

(Here $x \to 0-$ ($x \to 0+$) means x tends to 0 from the left-hand side (right-hand side, respectively).) Thus the left-handed and right-handed derivatives are not equal, so f is not differentiable at the origin. This corresponds to our intuition, as the graph of the function $f(x) = |x|$ has a corner at the origin and so there is no well-defined tangent.)

Remark. The above remark illustrates why we are interested in functions defined on open sets: we want to approach the point x_0 from either side. If f was defined on the closed interval $[a, b]$ then we could only consider right-handed derivatives at a (and left-handed derivatives at b).

The generalisation to complex functions is as one would expect.

Definition. Let $D \subset \mathbb{C}$ be an open set and let $f : D \to \mathbb{C}$ be a function. Let $z_0 \in D$. We say that f is *differentiable at* z_0 if

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0} \tag{2.4.2}$$

exists. (Note that in (2.4.2) we are allowing z to converge to z_0 from any direction.) We call $f'(z_0)$ the *derivative of* f *at* z_0. If f is differentiable at every point $z_0 \in D$ then we say that f is *differentiable on* D.

Remark. Sometimes we use the notation

$$\frac{df}{dz}(z_0)$$

to denote the derivative of f at z_0.

Remark. Although the definition of differentiability of a function in complex analysis is, essentially, the same as the definition in real analysis, we lose many of the geometrical interpretations of the derivative. For example, one cannot easily interpret $f'(z_0)$ as the gradient or slope of f at z_0. As another example, in real analysis one can normally interpret points x_0 for which $f'(x_0) = 0$ as turning points or local maxima/minima of f. The notion of a local maximum or local minimum does not exist in complex analysis; this is because there is no natural ordering on the set of complex numbers.

Differentiability is a very strong property for a complex function to possess; it is much stronger than the real case. For example (as we shall see) there are many functions that are differentiable when restricted to the real axis but that are not differentiable as a function defined on $\mathbb{C}$. For this reason, we shall often use the following alternative terminology.

Definition. Suppose that $f : D \to \mathbb{C}$ is differentiable on a domain D. Then we say that f is *holomorphic on* D. If f is defined on a domain D and is holomorphic on that domain then we say that f is *holomorphic*.

The higher derivatives are defined similarly, and we denote them by

$$f''(z_0),\ f'''(z_0), \ldots, f^{(n)}(z_0).$$

Example. Let $f(z) = z^2$, defined on $\mathbb{C}$. Let $z_0 \in \mathbb{C}$ be any point. Then

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \frac{z^2 - z_0^2}{z - z_0} = \lim_{z \to z_0} \frac{(z + z_0)(z - z_0)}{z - z_0} = \lim_{z \to z_0} z + z_0 = 2z_0.$$

Hence $f'(z_0) = 2z_0$ for all $z_0 \in \mathbb{C}$. Thus f is differentiable at every point in $\mathbb{C}$ and so is a holomorphic function on $\mathbb{C}$.

All of the standard rules of differentiable functions continue to hold in the complex case:

Proposition 2.4.1
Let f, g be holomorphic on D. Let $c \in \mathbb{C}$. Then the following hold:

(i) sum rule: $(f + g)' = f' + g'$,

(ii) scalar rule: $(cf)' = cf'$,

(iii) product rule: $(fg)' = f'g + fg'$,

(iv) quotient rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$,

(v) chain rule: $(f \circ g)' = f' \circ g \cdot g'$.

Proof. The proofs are all very simple modifications of the corresponding arguments in the real-valued case. (Usually the only modification needed is to replace the absolute value $|\cdot|$ defined on $\mathbb{R}$ with the modulus $|\cdot|$ defined on $\mathbb{C}$.) □

We will need the following fact.

Proposition 2.4.2
Suppose that f is differentiable at z_0. Then f is continuous at z_0.

Proof. To show that f is continuous, we need to show that $\lim_{z \to z_0} f(z) = f(z_0)$, i.e. $\lim_{z \to z_0} f(z) - f(z_0) = 0$. Note that

$$\lim_{z \to z_0} f(z) - f(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}(z - z_0) = f'(z_0) \times 0 = 0,$$

as required. □

§2.5 The Cauchy-Riemann equations

Throughout, let D be a domain. Let $z = x + iy \in D$. Let $f : D \to \mathbb{C}$ be a complex valued function. We write f as the sum of its real part and imaginary part by setting

$$f(z) = u(x, y) + iv(x, y)$$

where $u, v : D \to \mathbb{R}$ are real-valued functions.

Example. Let $f(z) = z^3$. Then

$$f(z) = z^3 = (x + iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3) = u(x, y) + iv(x, y)$$

where $u(x, y) = x^3 - 3xy^2$ and $v(x, y) = 3x^2y - y^3$.

If f is differentiable, then the Cauchy-Riemann equations give two relationships between u and v. To state them, we need to recall the notion of a partial derivative.

Definition. Suppose that $g(x, y)$ is a real-valued function depending on two co-ordinates x, y. Define

$$\frac{\partial g}{\partial x}(x, y) = \lim_{h \to 0} \frac{g(x+h, y) - g(x, y)}{h}, \quad \frac{\partial g}{\partial y}(x, y) = \lim_{k \to 0} \frac{g(x, y+k) - g(x, y)}{k}$$

(if these limits exist). For brevity (and provided there is no confusion), we leave out the (x, y) and write

$$\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}.$$

Thus, to calculate $\partial g/\partial x$ we treat y as a constant and differentiate with respect to x, and to calculate $\partial g/\partial y$ we treat x as a constant and differentiate with respect to y.

Theorem 2.5.1 (The Cauchy-Riemann Theorem)
Let $f : D \to \mathbb{C}$ and write $f(x + iy) = u(x, y) + iv(x, y)$. Suppose that f is differentiable at $z_0 = x_0 + iy_0$. Then

(i) the partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

exist at (x_0, y_0) and

(ii) the following relations hold

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \ \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0). \tag{2.5.1}$$

Remark. The relationships in (2.5.1) are called the *Cauchy-Riemann equations*.

Proof. Recall from (2.4.2) that to calculate $f'(z_0)$ we look at points that are close to z_0 and then let these points tend to z. The trick is to calculate $f'(z_0)$ in two different ways: by looking at points that converge to z_0 horizontally, and by looking at points that converge to z_0 vertically.

Let h be real and consider $z_0 + h = (x_0 + h) + iy_0$. Then as $h \to 0$ we have $z_0 + h \to z_0$. Hence

$$\begin{aligned} f'(z_0) &= \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \\ &= \lim_{h \to 0} \frac{u(x_0 + h, y_0) + iv(x_0 + h, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{h} \\ &= \lim_{h \to 0} \frac{u(x_0 + h, y_0) - u(x_0, y_0)}{h} + i\frac{v(x_0 + h, y_0) - v(x_0, y_0)}{h} \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0). \end{aligned} \tag{2.5.2}$$

Now take k to be real and consider $z_0 + ik = x_0 + i(y_0 + k)$. Then as $k \to 0$ we have $z_0 + ik \to z_0$. Hence

$$\begin{aligned} f'(z_0) &= \lim_{k \to 0} \frac{f(z_0 + ik) - f(z_0)}{ik} \\ &= \lim_{k \to 0} \frac{u(x_0, y_0 + k) + iv(x_0, y_0 + k) - u(x_0, y_0) - iv(x_0, y_0)}{ik} \\ &= \lim_{k \to 0} \frac{u(x_0, y_0 + k) - u(x_0, y_0)}{ik} + i\frac{v(x_0, y_0 + k) - v(x_0, y_0)}{ik} \\ &= -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0), \end{aligned} \tag{2.5.3}$$

recalling that $1/i = -i$. Comparing the real and imaginary parts of (2.5.2) and (2.5.3) gives the result. $\square$

Example. We can use the Cauchy-Riemann equations to examine whether the function $f(z) = \bar{z}$ might be differentiable on $\mathbb{C}$. Note that writing $z = x + iy$ allows us to write $f(z) = \bar{z} = x - iy$. Hence $f(z) = u(x, y) + iv(x, y)$ with $u(x, y) = x$ and $v(x, y) = -y$. Now

$$\frac{\partial u}{\partial x} = 1, \ \frac{\partial u}{\partial y} = 0, \ \frac{\partial v}{\partial x} = 0, \ \frac{\partial v}{\partial y} = -1.$$

Hence there are no points at which

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

so that $f(z) = \bar{z}$ is not differentiable at any point in $\mathbb{C}$.

Remark. Notice however that $f(z) = \bar{z}$ is continuous at every point in $\mathbb{C}$. Hence $f(z) = \bar{z}$ is an example of an everywhere continuous but nowhere differentiable function. Such functions also exist in real analysis, but they are much harder to write down and considerably harder to study (one of the simplest is known as Weierstrass' function $w(x) = \sum_{n=0}^{\infty} 2^{-n\alpha} \cos 2\pi b^n x$ where $\alpha \in (0, 1)$, $b \geq 2$; such functions are still of interest in current research).

We have seen that if f is differentiable at z_0 then the partial derivatives of u and v exist at z_0 and the Cauchy-Riemann equations are satisfied. One could ask whether the converse is true: if the Cauchy-Riemann equations are satisfied at the point z_0 then is f differentiable at z_0? The answer is no, as the following example shows. Define

$$f(x + iy) = \begin{cases} 0 \text{ if } (x, y) \text{ lies on either the } x \text{ or } y \text{ axes,} \\ 1 \text{ otherwise.} \end{cases}$$

Note that if we write $f(x+iy) = u(x, y)+iv(x, y)$ then $u(x, y) = f(x+iy)$ and $v(x, y) = 0$. Then at the origin

$$\frac{\partial u}{\partial x}(0, 0) = \lim_{h \to 0} \frac{u(h, 0) - u(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = \lim_{h \to 0} 0 = 0$$

and similarly $\partial u/\partial y(0, 0)$. Clearly, $\partial v/\partial x$, $\partial v/\partial y$ are all equal to zero at the origin. Hence the partial derivatives exist at the origin and the Cauchy-Riemann equations hold at the origin, so that the conclusions of the Cauchy-Riemann Theorem hold at the origin. However,

f is not continuous at the origin; this is because $h + ih \to 0$ as $h \to 0$ but $1 = f(h + ih) \not\to f(0) = 0$ as $h \to 0$. As f is not continuous at the origin, it cannot be differentiable at the origin.

The problem with the above example is that in the definition of differentiability (2.4.2) we need to let z tend to z_0 in an arbitrary way. In calculating the partial derivatives we only know what happens at z tends to z_0 either horizontally or vertically. Hence we need some extra hypotheses on u, v at z_0; the correct hypotheses are to assume the continuity of the partial derivatives.

Proposition 2.5.2 (Converse to the Cauchy-Riemann Theorem)
Let $f : D \to \mathbb{C}$ be a continuous function and write $f(x + iy) = u(x, y) + iv(x, y)$. Let $z_0 = x_0 + iy_0 \in D$. Suppose that

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$$

exist and are continuous at z_0, and further suppose that the Cauchy-Riemann equations hold at z_0. Then f is differentiable at z_0.

Proof. The proof is based on the following lemma; we omit the proof.

Lemma 2.5.3
Suppose that $\partial w/\partial x$, $\partial w/\partial y$ exist at (x_0, y_0) and $\partial w/\partial x$ is continuous at (x_0, y_0). Then there exist functions $\varepsilon(h, k)$ and $\eta(h, k)$ such that

$$w(x_0 + h, y_0 + k) - w(x_0, y_0) = h\left(\frac{\partial w}{\partial x}(x_0, y_0) + \varepsilon(h, k)\right) + k\left(\frac{\partial w}{\partial y}(x_0, y_0) + \eta(h, k)\right)$$

and $\varepsilon(h, k), \eta(h, k) \to 0$ as $h, k \to 0$.

Now consider $z = z_0 + h + ik$. Applying the above lemma to both u and v we can write

$$\begin{aligned}
&f(z) - f(z_0) \\
&= u(x_0 + h, y_0 + k) + iv(x_0 + h, y_0 + k) - u(x_0, y_0) - iv(x_0, y_0) \\
&= h\left(\frac{\partial u}{\partial x} + \varepsilon_1\right) + k\left(\frac{\partial u}{\partial y} + \eta_1\right) + ih\left(\frac{\partial v}{\partial x} + \varepsilon_2\right) + ik\left(\frac{\partial v}{\partial y} + \eta_2\right)
\end{aligned}$$

where $\varepsilon_1, \varepsilon_2, \eta_1, \eta_2 \to 0$ as $h, k \to 0$.

Using the Cauchy-Riemann equations we can write the above expression as

$$\begin{aligned}
f(z) - f(z_0) &= (h + ik)\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + h\varepsilon_1 + k\eta_1 + ih\varepsilon_2 + ik\eta_2 \\
&= (z - z_0)\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \rho
\end{aligned}$$

where $\rho = h\varepsilon_1 + k\eta_1 + ih\varepsilon_2 + ik\eta_2$. Hence

$$\frac{f(z) - f(z_0)}{z - z_0} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \frac{\rho}{z - z_0}$$

and so it remains to show that $\rho/(z - z_0) \to 0$ as $z \to z_0$. To see this, note that

$$\left|\frac{\rho}{z - z_0}\right| = \frac{|\rho|}{\sqrt{h^2 + k^2}} \leq \frac{|h||\varepsilon_1| + |k||\eta_1| + |h||\varepsilon_2| + |k||\eta_2|}{\sqrt{h^2 + k^2}} \leq |\varepsilon_1| + |\eta_1| + |\varepsilon_2| + |\eta_2|$$

which tends to zero as $h, k \to 0$. □

Exercises for Part 2

Exercise 2.1
Which of the following sets are domains?

(i) $\{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$,

(ii) $\{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0,\ |z| < 2\}$,

(iii) $\{z \in \mathbb{C} \mid |z| \leq 6\}$,

(iv) $\{z \in \mathbb{C} \mid |z| < 2\} \cup \{z \in \mathbb{C} \mid |z| > 4\}$.

Exercise 2.2
Using the definition in (2.4.2), differentiate the following complex functions from first principles:

$$\text{(i) } f(z) = z^2 + z; \text{ (ii) } f(z) = 1/z \ (z \neq 0); \text{ (iii) } f(z) = z^3 - z^2.$$

Exercise 2.3

(i) In each of the following cases, write $f(z)$ in the form $u(x, y)+iv(x, y)$ where $z = x+iy$ and u, v are real-valued functions.

$$(a)\ f(z) = z^2;\ (b)\ f(z) = \frac{1}{z}\ (z \neq 0).$$

(ii) Show that u and v satisfy the Cauchy-Riemann equations everywhere for (a), and for all $z \neq 0$ in (b).

(iii) Write the function $f(z) = |z|$ in the form $u(x, y)+iv(x, y)$. Using the Cauchy-Riemann equations, decide whether there are any points in $\mathbb{C}$ at which f is differentiable.

Exercise 2.4

(i) Show that the Cauchy-Riemann equations hold for the functions u, v given by $u(x, y) = x^3 - 3xy^2,\ v(x, y) = 3x^2y - y^3$. Show that u, v are the real and imaginary parts of a holomorphic function $f : \mathbb{C} \to \mathbb{C}$.

(ii) Show that the Cauchy-Riemann equations hold for the functions u, v given by

$$u(x, y) = \frac{x^4 - 6x^2y^2 + y^4}{(x^2 + y^2)^4},\ v(x, y) = \frac{4xy^3 - 4x^3y}{(x^2 + y^2)^4}$$

where $(x, y) \neq (0, 0)$.

Show that u, v are the real and imaginary parts of a holomorphic function $f : \mathbb{C}\backslash\{0\} \to \mathbb{C}$.

Exercise 2.5
Let $f(z) = \sqrt{|xy|}$ where $z = x + iy$.

(i) Show from the definition (2.4.2) that f is not differentiable at the origin.

(ii) Show however that the Cauchy-Riemann equations are satisfied at the origin. Why does this not contradict Proposition 2.5.2?

Exercise 2.6
Suppose that $f(z) = u(x, y) + iv(x, y)$ is holomorphic. Use the Cauchy-Riemann equations to show that both u and v satisfy Laplace's equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

(you may assume that the second partial derivatives exist and are continuous). (Functions which satisfy Laplace's equation are called *harmonic functions*.)

Exercise 2.7
Let $f(z) = z^3$, $f : \mathbb{C} \to \mathbb{C}$. Determine real-valued functions u, v so that $f(z) = u(x, y) + iv(x, y)$ (where $z = x + iy$). Verify that both u and v satisfy Laplace's equation.

Exercise 2.8
Suppose $f(z) = u(x, y) + iv(x, y)$ is holomorphic on $\mathbb{C}$. Suppose we know that $u(x, y) = x^5 - 10x^3y^2 + 5xy^4$. By using the Cauchy-Riemann equations, find all the possible forms of $v(x, y)$.

(The Cauchy Riemann equations have the following remarkable implication: suppose $f(z) = u(x, y) + iv(x, y)$ is holomorphic and that we know a formula for u, then we can recover v (up to a constant); similarly, if we know v then we can recover u (up to a constant). Hence for complex differentiable functions, the real part of a function determines the imaginary part (up to constants), and vice versa.)

Exercise 2.9
Suppose that

$$u(x, y) = x^3 - kxy^2 + 12xy - 12x$$

for some constant $k \in \mathbb{C}$. Find all values of k for which u is the real part of a holomorphic function $f : \mathbb{C} \to \mathbb{C}$.

Exercise 2.10
Show that if $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and f has a constant real part then f is constant.

Exercise 2.11
Show that the only holomorphic function $f : \mathbb{C} \to \mathbb{C}$ of the form $f(x + iy) = u(x) + iv(y)$ is given by $f(z) = \lambda z + a$ for some $\lambda \in \mathbb{R}$ and $a \in \mathbb{C}$.

Exercise 2.12
Suppose that $f(z) = u(x, y) + iv(x, y)$, $f : \mathbb{C} \to \mathbb{C}$, is a holomorphic function and that

$$2u(x, y) + v(x, y) = 5 \text{ for all } z = x + iy \in \mathbb{C}.$$

Show that f is constant.

3. Power series and elementary analytic functions

§3.1 Recap on convergence and absolute convergence of series

Recall that we have already discussed what it means for an infinite sequence of complex numbers to converge. Recall that if $s_n \in \mathbb{C}$ then we say that s_n converges to $s \in \mathbb{C}$ if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|s - s_n| < \varepsilon$ for all $n \geq N$.

Let $z_k \in \mathbb{C}$. We say that the series $\sum_{k=0}^{\infty} z_k$ converges if the sequence of partial sums $s_n = \sum_{k=0}^{n} z_k$ converges. The limit of this sequence of partial sums is called the *sum* of the series. A series which does not converge is called *divergent*.

Remark. One can show (see Exercise 3.1) that $\sum_{n=0}^{\infty} z_n$ is convergent if, and only if, both $\sum_{n=0}^{\infty} \mathrm{Re}(z_n)$ and $\sum_{n=0}^{\infty} \mathrm{Im}(z_n)$ are convergent.

We will need a stronger property than just convergence.

Definition. Let $z_n \in \mathbb{C}$. We say that $\sum_{n=0}^{\infty} z_n$ is *absolutely convergent* if the real series $\sum_{n=0}^{\infty} |z_n|$ is convergent.

Lemma 3.1.1
Suppose that $\sum_{n=0}^{\infty} z_n$ is absolutely convergent. Then $\sum_{n=0}^{\infty} z_n$ is convergent.

Proof. Suppose that $\sum_{n=0}^{\infty} z_n$ is absolutely convergent. Let $z_n = x_n + iy_n$. Then $|x_n|, |y_n| \leq |z_n|$. Hence by the comparison test, the real series $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ are absolutely convergent. As x_n, y_n are real, we know that

$$\left|\sum_{n=0}^{\infty} x_n\right| \leq \sum_{n=0}^{\infty} |x_n|, \quad \left|\sum_{n=0}^{\infty} y_n\right| \leq \sum_{n=0}^{\infty} |y_n|,$$

so that $\sum_{n=0}^{\infty} x_n$ and $\sum_{n=0}^{\infty} y_n$ are convergent. By the above remark, $\sum_{n=0}^{\infty} z_n$ is convergent. □

Remark. It is easy to give an example of a series which is convergent but not absolutely convergent. In fact, we can give an example using real series. Recall that $\sum_{n=0}^{\infty}(-1)^n/n$ is convergent but $\sum_{n=0}^{\infty} |(-1)^n/n| = \sum_{n=0}^{\infty} 1/n$ is divergent.

The reason for working with absolutely convergent series is that they behave well when multiplied together. Indeed, two series which converge absolutely may be multiplied in a similar way to two finite sums. First note that if we have two finite sums then we can multiply them together systematically as follows:

$$\begin{aligned}&(a_0 + a_1 + a_2 + a_3 + \cdots + a_n)(b_0 + b_1 + b_2 + b_3 + \cdots + b_n) \\ &\quad = (a_0b_0) + (a_0b_1 + a_1b_0) + (a_0b_2 + a_1b_1 + a_2b_0) + (a_0b_3 + a_1b_2 + a_2b_1 + a_3b_0) + \cdots\end{aligned}$$

For absolutely convergent series the following proposition holds. (We remark that Proposition 3.1.2 is not true in general if one of the infinite series converges but is not absolutely convergent.)

Proposition 3.1.2
Let $a_n, b_n \in \mathbb{C}$. Suppose that $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent. Then

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} c_n$$

where $c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_n b_0$ and $\sum_{n=0}^{\infty} c_n$ is absolutely convergent.

Proof. Omitted. □

From real analysis or sequences and series, you know some tests to see whether a real series converges. The same tests continue to hold for complex series and we state them below as propositions.

Proposition 3.1.3 (The ratio test)
Let $z_n \in \mathbb{C}$. Suppose that

$$\lim_{n \to \infty} \frac{|z_{n+1}|}{|z_n|} = \ell. \tag{3.1.1}$$

If $\ell < 1$ then $\sum_{n=0}^{\infty} z_n$ is absolutely convergent. If $\ell > 1$ then $\sum_{n=0}^{\infty} z_n$ diverges.

Remark. If $\ell = 1$ in (3.1.1) then we can say nothing: the series may converge absolutely, it may converge but not absolutely converge, or it may diverge.

Proposition 3.1.4 (The root test)
Let $z_n \in \mathbb{C}$. Suppose that

$$\lim_{n \to \infty} |z_n|^{1/n} = \ell. \tag{3.1.2}$$

If $\ell < 1$ then $\sum_{n=0}^{\infty} z_n$ is absolutely convergent. If $\ell > 1$ then $\sum_{n=0}^{\infty} z_n$ diverges.

Remark. Again, if $\ell = 1$ in (3.1.2) then we can say nothing: the series may converge absolutely, it may converge but not absolutely converge, or it may diverge.

Example. Consider the series

$$\sum_{n=0}^{\infty} \frac{i^n}{2^n}.$$

Here $z_n = i^n/2^n$. We can use the ratio test to show that this series converges absolutely. Indeed, note that

$$\left|\frac{z_{n+1}}{z_n}\right| = \left|\frac{i^{n+1}}{2^{n+1}} \frac{2^n}{i^n}\right| = \left|\frac{i}{2}\right| = \frac{1}{2}.$$

Hence $\lim_{n \to \infty} |z_{n+1}/z_n| = 1/2 < 1$ and so by the ratio test the series converges absolutely.

We could also have used the root test to show that this series converges absolutely. To see this, note that

$$|z_n|^{1/n} = \left|\frac{i^n}{2^n}\right|^{1/n} = \left(\frac{1}{2^n}\right)^{1/n} = 1/2.$$

Hence $\lim_{n \to \infty} |z_n|^{1/n} = 1/2 < 1$ and so by the root test the series converges absolutely.

§3.2 Power series and the radius of convergence

Definition. A series of the form $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ where $a_n \in \mathbb{C}$, $z \in \mathbb{C}$ is called a *power series at* z_0.

By changing variables and replacing $z - z_0$ by z we need only consider power series at 0, i.e. power series of the form

$$\sum_{n=0}^{\infty} a_n z^n.$$

When does a power series converge? Let

$$R = \sup \left\{ r \geq 0 \mid \text{there exists } z \in \mathbb{C} \text{ such that } |z| = r \text{ and} \sum_{n=0}^{\infty} a_n z^n \text{ converges} \right\}.$$

(We allow $R = \infty$ if no finite supremum exists.)

Theorem 3.2.1
Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series and let R be defined as above. Then

(i) $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely for $|z| < R$;

(ii) $\sum_{n=0}^{\infty} a_n z^n$ diverges for $|z| > R$.

Remark. We cannot say what happens in the case when $|z| = R$: the power series may converge, it may converge but not absolutely converge, or it may diverge.

Proof. Let $z \in \mathbb{C}$ be such that $|z| < R$. Choose $z_1 \in \mathbb{C}$ such that $|z| < |z_1| \leq R$ and such that $\sum_{n=0}^{\infty} a_n z_1^n$ converges. As $\sum_{n=0}^{\infty} a_n z_1^n$ converges, it follows that $a_n z_1^n \to 0$ as $n \to \infty$. Hence $|a_n z_1^n| \to 0$ as $n \to \infty$. It follows that $|a_n z_1^n|$ is a bounded sequence; that is, there exists $K > 0$ such that $|a_n z_1^n| < K$ for all n. Let $q = |z|/|z_1|$. As $|z| < |z_1|$, we have that $q < 1$. Now

$$|a_n z^n| = |a_n z_1^n| \left| \frac{z}{z_1} \right|^n < K q^n.$$

Hence by the comparison test, $\sum_{n=0}^{\infty} |a_n z^n|$ converges (noting that $\sum_{n=0}^{\infty} K q^n = K/(1-q)$). Hence $\sum_{n=0}^{\infty} a_n z^n$ converges absolutely, and so converges.

Now suppose that $\sum_{n=0}^{\infty} a_n z_2^n$ diverges. If $|z| > |z_2|$ and $\sum_{n=0}^{\infty} a_n z^n$ converges then the above paragraph shows that $\sum_{n=0}^{\infty} a_n z_2^n$ must also converge, a contradiction. Hence $\sum_{n=0}^{\infty} a_n z^n$ diverges.

These two facts show that R must exist. □

Definition. The number R given in Theorem 3.2.1 is called the *radius of convergence* of the power series $\sum_{n=0}^{\infty} a_n z^n$. We call the set $\{z \in \mathbb{C} \mid |z| < R\}$ the *disc of convergence*.

We would like some ways of computing the radius of convergence of a power series.

Proposition 3.2.2
Let $\sum_{n=0}^{\infty} a_n z^n$ be a power series.

(i) If $\lim_{n\to\infty} |a_{n+1}|/|a_n|$ exists then

$$\frac{1}{R} = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|}.$$

(ii) If $\lim_{n\to\infty} |a_n|^{1/n}$ *exists then*

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}.$$

(Here we interpret $1/0$ *as* ∞ *and* $1/\infty$ *as* 0.*)*

Remark. If the limit in (i) exists then the limit in (ii) exists and they give the same answer for the radius of convergence. It is straightforward to find examples of sequences a_n for which the limit in (ii) exists but the limit in (i) does not.

Remark. You may wonder why we state the above formulæ in terms of $1/R$ rather than R, given that this introduces the extra notational difficulty of how to interpret $1/0$ and $1/\infty$. The reason is to make the formulæ in Proposition 3.2.2 resemble the ratio test and the root test (Propositions 3.1.3 and 3.1.4, respectively) for the convergence of infinite series.

Example. Consider

$$\sum_{n=0}^{\infty} \frac{z^n}{n}.$$

Here $a_n = 1/n$. In this case

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1} \to 1 = \frac{1}{R}$$

as $n \to \infty$. Hence the radius of convergence is equal to 1.

Example. Consider

$$\sum_{n=0}^{\infty} \frac{z^n}{2^n}.$$

Here $a_n = 1/2^n$. Using Proposition 3.2.2(i) we can calculate the radius of convergence as

$$\frac{1}{R} = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} = \lim_{n\to\infty} \frac{1}{2} = \frac{1}{2}$$

so that $R = 2$. Alternatively, we could use Proposition 3.2.2(ii) and see that

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{1}{2^n}\right)^{1/n} = \frac{1}{2}$$

so that again $R = 2$.

Proof of Proposition 3.2.2. We prove (i). Suppose that $|a_{n+1}/a_n|$ converges to a limit, say ℓ, as $n \to \infty$, i.e.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \ell.$$

Then

$$\lim_{n\to\infty} \frac{|a_{n+1} z^{n+1}|}{|a_n z^n|} \to \ell |z|.$$

By the ratio test, the power series $\sum_{n=0}^{\infty} a_n z^n$ converges for $\ell|z| < 1$ and diverges for $\ell|z| > 1$. Hence the radius of convergence $R = 1/\ell$.

We prove (ii). Suppose that $|a_n|^{1/n} \to \ell$ as $n \to \infty$. By the root test, the power series $\sum_{n=0}^{\infty} a_n z^n$ converges if $\lim_{n\to\infty} |a_n z^n|^{1/n} = \lim_{n\to\infty} |a_n|^{1/n}|z| = \ell|z| < 1$ and diverges if $\lim_{n\to\infty} |a_n z^n|^{1/n} = \lim_{n\to\infty} |a_n|^{1/n}|z| = \ell|z| > 1$. Hence the radius of convergence $R = 1/\ell$. $\square$

Remark. It may happen that neither of the limits in (i) nor (ii) of Proposition 3.2.2 exist. However, there is a formula for the radius of convergence R that works for any power series $\sum_{n=0}^{\infty} a_n z^n$.

Let x_n be a sequence of real numbers. For each n, consider $\sup_{k\geq n} x_k$. As n increases, this sequence decreases. Recall that any decreasing sequence of non-negative reals converges. Hence

$$\lim_{n\to\infty} \left\{ \sup_{k\geq n} x_k \right\}$$

exists (although it may be equal to ∞). We denote the limit by $\limsup_{n\to\infty} x_n$. Thus $\limsup x_n$ exists for any sequence x_n. (One can show that if $\lim_{n\to\infty} x_n = \ell$ then $\limsup_{n\to\infty} x_n = \ell$.)

With this definition, it is always the case that

$$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{1/n}.$$

§3.3 Differentiation of power series

We know that for a polynomial

$$p(z) = a_0 + a_1 z + \cdots + a_n z^n$$

the derivative is given by

$$p'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1}.$$

This suggests that a power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{3.3.1}$$

can be differentiated term by term to give

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}. \tag{3.3.2}$$

However, because we are dealing with infinite sums, this needs to be proved. There are two steps to this: (i) we have to show that if (3.3.1) converges for $|z| < R$ then (3.3.2) converges for $|z| < R$, and (ii) that $f(z)$ is differentiable for $|z| < R$ and the derivative is given by (3.3.2).

Lemma 3.3.1
Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R. Then $g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ converges for $|z| < R$.

Proof. Let $|z| < R$ and choose r such that $|z| < r < R$. Then $\sum_{n=1}^{\infty} a_n r^n$ converges absolutely. Hence the summands must be bounded, so there exists $K > 0$ such that $|a_n r^n| < K$ for all $n \geq 0$.

Let $q = |z|/r$ and note that $0 < q < 1$. Then

$$|n a_n z^{n-1}| = n|a_n| \left|\frac{z}{r}\right|^{n-1} r^{n-1} < n\frac{K}{r} q^{n-1}.$$

But $\sum_{n=1}^{\infty} nq^{n-1}$ converges to $(1-q)^{-2}$. Hence by the comparison test, $\sum_{n=0}^{\infty} |na_n z^{n-1}|$ converges. Hence $\sum_{n=0}^{\infty} na_n z^{n-1}$ converges absolutely and so converges. □

Theorem 3.3.2
Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R. Then $f(z)$ is holomorphic on the disc of convergence $\{z \in \mathbb{C} \mid |z| < R\}$ and $f'(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$.

Proof. Let $g(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$. By Lemma 3.3.1 we know that this converges for $|z| < R$.

We have to show that if $|z_0| < R$ then $f(z)$ is differentiable at z_0 and, moreover, the derivative is equal to $g(z_0)$, i.e. we have to show that if $|z_0| < R$ then

$$f'(z_0) := \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = g(z_0)$$

or equivalently

$$\lim_{z \to z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} - g(z_0) \right) = 0.$$

For any $N \geq 1$ we have the following

$$\begin{aligned}
&\frac{f(z) - f(z_0)}{z - z_0} - g(z_0) \\
&= \sum_{n=1}^{\infty} \left(a_n \frac{z^n - z_0^n}{z - z_0} - na_n z_0^{n-1} \right) \\
&= \sum_{n=1}^{\infty} \left(a_n (z^{n-1} + z_0 z^{n-2} + \cdots + z_0^{n-2} z + z_0^{n-1}) - na_n z_0^{n-1} \right) \\
&= \sum_{n=1}^{\infty} a_n (z^{n-1} + z_0 z^{n-2} + \cdots + z_0^{n-2} z + z_0^{n-1} - nz_0^{n-1}) \\
&= \sum_{n=1}^{N} a_n (z^{n-1} + z_0 z^{n-2} + \cdots + z_0^{n-2} z + z_0^{n-1} - nz_0^{n-1}) \\
&\quad + \sum_{n=N+1}^{\infty} a_n (z^{n-1} + z_0 z^{n-2} + \cdots + z_0^{n-2} z + z_0^{n-1} - nz_0^{n-1}) \\
&= \Sigma_{1,N}(z) + \Sigma_{2,N}(z), \text{say.}
\end{aligned}$$

Let $\varepsilon > 0$. Choose r such that $|z_0| < r < R$. Then, as in the proof of Lemma 3.3.1, $\sum_{n=1}^{\infty} na_n r^{n-1}$ is absolutely convergent. Hence we can choose $N = N(\varepsilon)$ such that

$$\sum_{n=N+1}^{\infty} |na_n r^{n-1}| < \frac{\varepsilon}{4}.$$

Since $|z_0| < r$, provided z is close enough to z_0 so that $|z| < r$ then we have that

$$|\Sigma_{2,N}(z)| \leq \sum_{n=N+1}^{\infty} 2n|a_n| r^{n-1} < \frac{\varepsilon}{2}. \tag{3.3.3}$$

Now consider $\Sigma_{1,N}(z)$. This is a polynomial in z and so is a continuous function. Note that $\Sigma_{1,N}(z_0) = 0$. Hence, as $z \to z_0$, we have that $\Sigma_{1,N}(z) \to 0$. Hence, provided z is close enough to z_0 we have that

$$|\Sigma_{1,N}(z)| < \frac{\varepsilon}{2}. \tag{3.3.4}$$

Finally, if z is close enough to z_0 so that both (3.3.3) and (3.3.4) hold then

$$\begin{aligned} \left| \frac{f(z) - f(z_0)}{z - z_0} - g(z_0) \right| &= |\Sigma_{1,N}(z) + \Sigma_{2,N}(z)| \\ &\leq |\Sigma_{1,N}(z)| + |\Sigma_{2,N}(z)| \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

As ε is arbitrary, it follows that $f'(z_0) = g(z)$. □

The above two results have a very important consequence. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < R$ then we can differentiate it as many times as we like within the disc of convergence.

Proposition 3.3.3
Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence R. Then all of the higher derivatives $f', f'', f''', \ldots, f^{(k)}, \ldots$ of f exist for z within the disc of convergence. Moreover,

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n z^{n-k} = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n z^{n-k}.$$

Proof. This is a simple induction on k. □

Instead of using a power series at the origin, by replacing z by $z - z_0$ we can consider a power series at the point z_0. (This will be useful later on when we look at Taylor series.) Suppose that $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has disc of convergence $|z| < R$. Then, replacing z by $z - z_0$, we have that the power series $g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$ has disc of convergence $\{z \in \mathbb{C} \mid |z - z_0| < R\}$. That is, the power series $g(z)$ converges for all z inside the disc with centre z_0 and radius R. Moreover, inside this disc of convergence all the higher derivatives of g exist and

$$g^{(k)}(z) = \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} a_n (z - z_0)^{n-k}.$$

§3.4 Special functions

§3.4.1 The exponential function

You have probably already met the exponential function $e^x = \sum_{n=0}^{\infty} x^n/n!$, certainly in the case when x is real. Here we study the (complex) exponential function.

Definition. The exponential function is defined to be the power series

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

By Proposition 3.2.2(i) we see that the radius of convergence R for $\exp z$ is given by

$$\frac{1}{R} = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n+1} = 0$$

so that $R = \infty$. Hence this series has radius convergence ∞, and so converges absolutely for all $z \in \mathbb{C}$.

By Theorem 3.3.2 we may differentiate term-by-term to obtain

$$\frac{d}{dz} \exp z = \sum_{n=1}^{\infty} n \frac{z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!} = \exp z,$$

which we already knew to be true in the real-valued case.

In the real case we know that if $x, y \in \mathbb{R}$ then $e^{x+y} = e^x e^y$. This is also true in the complex-valued case, and the proof involves a neat trick. First we need the following fact:

Lemma 3.4.1
Suppose that f is holomorphic on a domain D and $f'(z) = 0$ for all $z \in D$. Then f is constant on D.

Remark. This is well-known in the real case: a function with zero derivative must be constant. The proof in the complex case is somewhat more involved and we omit it. (See Stewart and Tall, p.71.)

Proposition 3.4.2
Let $z_1, z_2 \in \mathbb{C}$. Then $\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$.

Proof. Let $c \in \mathbb{C}$ and define the function $f(z) = \exp(z)\exp(c - z)$. Then

$$f'(z) = \exp(z)\exp(c - z) - \exp(z)\exp(c - z) = 0$$

by the product rule. Hence by Lemma 3.4.1 we must have that $f(z)$ is constant; in particular this constant must be $f(0) = \exp c$. Hence $\exp(z)\exp(c - z) = \exp(c)$. Putting $c = z_1 + z_2$ and $z = z_1$ gives the result. $\square$

Remark. In particular, if we take $z_1 = z$ and $z_2 = -z$ in Proposition 3.4.2 then we have that

$$1 = \exp 0 = \exp(z - z) = \exp(z)\exp(-z).$$

Hence $\exp z \neq 0$ for any $z \in \mathbb{C}$. (We already knew that $e^x = 0$ has no real solutions; now we know that it has no complex solutions either.)

Finally, we want to connect the real number e to the complex exponential function. We define e to be the real number $e = \exp 1$. Then, iterating Proposition 3.4.2 inductively, we obtain

$$e^n = \exp(1)^n = \exp(1 + \cdots + 1) = \exp n.$$

For a rational number m/n ($n > 0$) we have that

$$(\exp(m/n))^n = \exp(nm/n) = \exp(m) = e^m$$

so that $\exp(m/n) = e^{m/n}$. Thus the notation $e^z = \exp z$ does not conflict with the usual definition of e^x when z is real. Hence we shall normally write e^z for $\exp z$. In particular, if we write $z = x + iy$ then Proposition 3.4.2 tells us that

$$e^{x+iy} = e^x e^{iy}.$$

We already understand real exponentials e^x. Hence to understand complex exponentials we need to understand expressions of the form e^{iy}.

§3.4.2 Trigonometric functions

Define

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}, \ \sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.$$

By Proposition 3.2.2(i) it is straightforward to check that these converge absolutely for all $z \in \mathbb{C}$.

Substituting $z = -z$ we see that cos is an even function and that sin is an odd function, i.e.

$$\cos(-z) = \cos z, \ \sin(-z) = -\sin z.$$

Moreover, $\cos(0) = 1$, $\sin(0) = 0$.

By Theorem 3.3.2 we can differentiate term-by-term to see that

$$\frac{d}{dz}\cos z = -\sin z, \ \frac{d}{dz}\sin z = \cos z.$$

Term-by-term addition of the power series for $\cos z$ and $\sin z$ shows that

$$\exp iz = \cos z + i \sin z.$$

Replacing z by $-z$ we see that $e^{-iz} = \cos z - i \sin z$. Hence

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz}), \ \sin z = \frac{1}{2i}(e^{iz} - e^{-iz}).$$

Squaring the above expressions and adding them gives $\cos^2 z + \sin^2 z = 1$. These are all expressions that we already knew in the case when z is a real number; now we know that they continue to hold when z is any complex number. Carrying on in the same way, one can prove the addition formulæ $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$, etc, for complex z_1, z_2, and all the other usual trigonometric identities.

§3.4.3 Hyperbolic functions

Define

$$\cosh z = \frac{1}{2}(e^z + e^{-z}), \ \sinh z = \frac{1}{2}(e^z - e^{-z}).$$

Differentiating these we see that

$$\frac{d}{dz}\cosh z = \sinh z, \ \frac{d}{dz}\sinh z = \cosh z.$$

One can also prove addition formulæ for the hyperbolic trigonometric functions, and other identities including (for example)

$$\cosh^2 z - \sinh^2 z = 1 \text{ for all } z \in \mathbb{C}$$

(again, we knew this already when $z \in \mathbb{R}$).

We also have the relations

$$\cos iz = \cosh z,\ \sin iz = i \sinh z;$$

these follow from Exercise 3.6.

§3.4.4 Periods of the exponential and trigonometric functions

Definition. Let $f : \mathbb{C} \to \mathbb{C}$. We say that a number $p \in \mathbb{C}$ is a *period* for f if $f(z+p) = f(z)$ for all $z \in \mathbb{C}$.

Clearly if $p \in \mathbb{C}$ is a period and $n \in \mathbb{Z}$ is any integer then np is also a period.

For the exponential function, we have that

$$e^{2\pi i} = \cos 2\pi + i \sin 2\pi = 1$$

so that

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z.$$

Hence $2\pi i$ is a period for exp, as is $2n\pi i$ for any integer n. In Exercise 3.11 we shall see that these are the only periods for exp.

We shall also see in the exercises that the only complex periods for sin and cos are $2n\pi$.

§3.4.5 The logarithmic function

In real analysis, the (natural) logarithm is the inverse function to the exponential function. That is, if $e^x = y$ then $x = \ln y$. (Throughout we will write ln to denote the (real) natural logarithm.) Here we consider the complex analogue of this.

Let $z \in \mathbb{C}$, $z \neq 0$, and consider the equation

$$\exp w = z. \tag{3.4.1}$$

By §3.4.4, if w_1 is a solution to (3.4.1) then so is $w_1 + 2n\pi i$. Each of these values is called a logarithm of z, and we denote any of these values by $\log z$. Thus, unlike in the real case, the complex logarithm is a multi-valued function.

We want to find a formula for $\log z$. In (3.4.1) write $w = x + iy$. Then

$$z = \exp w = \exp(x + iy) = e^x(\cos y + i \sin y). \tag{3.4.2}$$

By taking the modulus of both sides of (3.4.2) we see that $e^x = |z|$. Note that both x and $|z|$ are real numbers. Hence $x = \ln |z|$. By taking the argument of both sides of (3.4.2) we see that $y = \arg z$. Hence we can make the following definition.

Definition. Let $z \in \mathbb{C}$, $z \neq 0$. Then *a complex logarithm of z* is

$$\log z = \ln |z| + i \arg z$$

where $\arg z$ is any argument of z.

The *principal value* of $\log z$ is the value of $\log z$ when $\arg z$ has its principal value $\operatorname{Arg} z$, i.e. the unique value of the argument in $(-\pi, \pi]$. We denote the principal logarithm by $\operatorname{Log} z$:

$$\operatorname{Log} z = \ln |z| + i \operatorname{Arg} z.$$

Note that we say *a* complex logarithm (rather than *the* complex logarithm) to emphasise the fact that the complex logarithm is multi-valued.

Dealing with multivalued functions is tricky. One way is to only consider the logarithm function on a subset of $\mathbb{C}$.

Definition. The complex plane with the negative real-axis (including 0) removed is called the *cut plane*. See Figure 3.4.1.

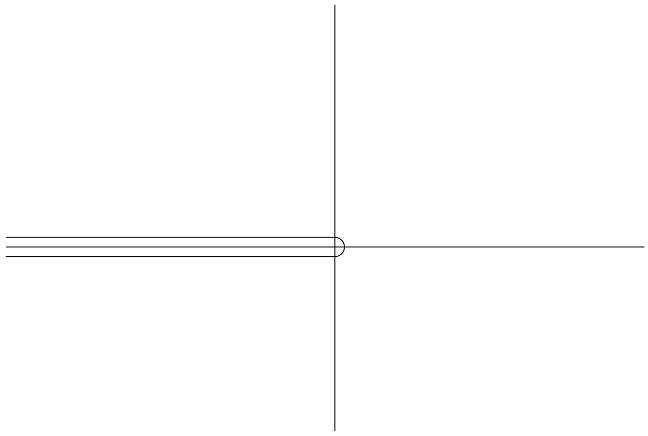

Figure 3.4.1: The cut plane: this is the complex plane with the negative real axis removed.

Proposition 3.4.3
The principal logarithm $\operatorname{Log} z$ *is continuous on the cut plane.*

Proof. This follows from the fact (which we shall not prove, although the proof is easy) that the principal value of the argument $\operatorname{Arg} z$ is continuous on the cut-plane. □

Having seen that the principal logarithm is continuous, we can go on to show that it is differentiable.

Proposition 3.4.4
The principal logarithm $\operatorname{Log} z$ *is holomorphic on the cut plane and*

$$\frac{d}{dz}\operatorname{Log} z = \frac{1}{z}.$$

Proof. Let $w = \operatorname{Log} z$. Then $z = \exp w$. Let $\operatorname{Log}(z+h) = w+k$. Then by Proposition 3.4.3 Log is continuous on the cut plane so we have that $k \to 0$ as $h \to 0$. Then

$$\begin{aligned}
\frac{d}{dz}\operatorname{Log} z &= \lim_{h\to 0}\frac{\operatorname{Log}(z+h)-\operatorname{Log}(z)}{h} \\
&= \lim_{k\to 0}\frac{(w+k)-w}{\exp(w+k)-\exp(w)} \\
&= \lim_{k\to 0}\left(\frac{\exp(w+k)-\exp(w)}{k}\right)^{-1} \\
&= \left(\frac{d}{dw}\exp(w)\right)^{-1} \\
&= \frac{1}{z}.
\end{aligned}$$

$\square$

Having defined the complex logarithm we can go on to define complex powers. For $b, z \in \mathbb{C}$ with $b \neq 0$ we define the *principal value of* b^z to be

$$b^z = \exp(z \operatorname{Log} b)$$

and the *subsidiary values* to be $\exp(z \log b)$.

Exercises for Part 3

Exercise 3.1
Let $z_n \in \mathbb{C}$. Show that $\sum_{n=0}^{\infty} z_n$ is convergent if, and only if, both $\sum_{n=0}^{\infty} \mathrm{Re}(z_n)$ and $\sum_{n=0}^{\infty} \mathrm{Im}(z_n)$ are convergent.

Exercise 3.2
Find the radius of convergence of each of the following power series:

$$\text{(i) } \sum_{n=1}^{\infty} \frac{2^n z^n}{n}, \text{ (ii) } \sum_{n=1}^{\infty} \frac{z^n}{n!}, \text{ (iii) } \sum_{n=1}^{\infty} n! z^n, \text{ (iv) } \sum_{n=1}^{\infty} n^p z^n \ (p \in \mathbb{N}).$$

Exercise 3.3
Consider the power series

$$\sum_{n=0}^{\infty} a_n z^n$$

where $a_n = 1/2^n$ if n is even and $a_n = 1/3^n$ if n is odd. Show that neither of the formulæ for the radius of convergence for this power series given in Proposition 3.2.2 converge. Show by using the comparison test that this power series converges for $|z| < 2$.

Exercise 3.4

(i) By multiplying two series together, show using Proposition 3.1.2 that for $|z| < 1$, we have

$$\sum_{n=1}^{\infty} n z^{n-1} = \frac{1}{(1-z)^2}.$$

(ii) By multiplying two series together, show using Proposition 3.1.2 that for $z, w \in \mathbb{C}$ we have

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \sum_{n=0}^{\infty} \frac{w^n}{n!} = \sum_{n=0}^{\infty} \frac{(z+w)^n}{n!}.$$

Exercise 3.5
Recall that if $|z| < 1$ then we can sum the geometric progression with common ratio z and initial term 1 as follows:

$$1 + z + z^2 + z^3 + \cdots + z^n + \cdots = \frac{1}{1-z}.$$

Use Theorem 3.3.2 to show that for each $k \geq 1$

$$\frac{1}{(1-z)^k} = \sum_{n=k-1}^{\infty} \left(\begin{array}{c} n \\ k-1 \end{array} \right) z^{n-(k-1)}$$

for $|z| < 1$. (When $k = 2$ this gives an alternative proof of the result in Exercise 3.4 (i).)

Exercise 3.6
Show that for $z, w \in \mathbb{C}$ we have

$$\text{(i)}\ \cos z = \frac{e^{iz} + e^{-iz}}{2}, \ \text{(ii)}\ \sin z = \frac{e^{iz} - e^{-iz}}{2i}.$$

Show also that

(iii) $\sin(z + w) = \sin z \cos w + \cos z \sin w$,

(iv) $\cos(z + w) = \cos z \cos w - \sin z \sin w$.

Exercise 3.7
Derive formulæ for the real and imaginary parts of the following complex functions and check that they satisfy the Cauchy-Riemann equations:

$$\text{(i)}\ \sin z, \ \text{(ii)}\ \cos z, \ \text{(iii)}\ \sinh z, \ \text{(iv)}\ \cosh z.$$

Exercise 3.8
For each of the complex functions $\exp, \cos, \sin, \cosh, \sinh$ find the set of points on which it assumes (i) real values, and (ii) purely imaginary values.

Exercise 3.9
We know that the only real numbers $x \in \mathbb{R}$ for which $\sin x = 0$ are $x = n\pi$, $n \in \mathbb{Z}$. Show that there are no further complex zeros for sin, i.e., if $\sin z = 0$, $z \in \mathbb{C}$, then $z = n\pi$ for some $n \in \mathbb{Z}$. Also show that if $\cos z = 0$, $z \in \mathbb{C}$ then $z = (n + 1/2)\pi$, $n \in \mathbb{Z}$.

Exercise 3.10
Find the zeros of the following functions

$$\text{(i)}\ 1 + e^z, \ \text{(ii)}\ 1 + i - e^z.$$

Exercise 3.11

(i) Recall that a complex number $p \in \mathbb{C}$ is called a period of $f : \mathbb{C} \to \mathbb{C}$ if $f(z+p) = f(z)$ for all $z \in \mathbb{C}$. Calculate the set of periods of $\sin z$.

(ii) We know that $p = 2n\pi i$, $n \in \mathbb{Z}$, are periods of $\exp z$. Show that there are no other periods.

Exercise 3.12
(So far, there has been little difference between the real and the complex versions of elementary functions. Here is one instance of where they can differ.)

Let $z_1, z_2 \in \mathbb{C} \setminus \{0\}$. Show that

$$\operatorname{Log} z_1 z_2 = \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2n\pi i.$$

where $n = n(z_1, z_2)$ is an integer which need not be zero. Give an explicit example of two complex numbers z_1, z_2 for which $\operatorname{Log} z_1 z_2 \neq \operatorname{Log} z_1 + \operatorname{Log} z_2$.

Exercise 3.13
Calculate the principal value of i^i and the subsidiary values. (Do you find it surprising that these turn out to be real?)

Exercise 3.14

(i) Let $\alpha \in \mathbb{C}$ and suppose that α is not a non-negative integer. Define the power series

$$\begin{aligned} f(z) &= 1 + \alpha z + \frac{\alpha(\alpha-1)}{2!} z^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!} z^3 + \cdots \\ &= 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} z^n. \end{aligned}$$

(Note that, as α is not a non-negative integer, this is an infinite series.)

Show that the this power series has radius of convergence 1.

(ii) Show that, for $|z| < 1$, we have $f'(z) = \dfrac{\alpha f(z)}{1+z}$.

(iii) By considering the derivative of the function $g(z) = \dfrac{f(z)}{(1+z)^{\alpha}}$, show that $f(z) = (1+z)^{\alpha}$ for $|z| < 1$.

4. Complex integration and Cauchy's Theorem

§4.1 Introduction

Consider the real integral

$$\int_a^b f(x)\,dx.$$

We often read this as 'the integral of f from a to b'. That is, we think of starting at the point a and moving along the real axis to b, integrating f as we go.

Now let $z_0, z_1 \in \mathbb{C}$. How might we define

$$\int_{z_0}^{z_1} f(z)\,dz?$$

We want to start at z_0, move through the complex plane to z_1, integrating f as we go. But in the complex plane there are lots of ways of moving from z_0 to z_1. Suppose γ is a path from z_0 to z_1 (we shall make precise what we mean by a path below, but intuitively just think of it as a continuous curve starting at z_0 and ending at z_1). Then, using similar ideas to those from MATH10121/10131 Calculus and Vectors, we can define

$$\int_\gamma f(z)\,dz.$$

A priori this looks like it will depend on the path γ. However, as we shall see, in complex analysis in many cases this quantity is independent of the path chosen.

§4.2 Paths and contours

First we need to make precise what we mean by a path.

Definition. A *path* is a continuous function $\gamma : [a, b] \to \mathbb{C}$ where $[a, b]$ is a real interval.

Remark. So, for each $a \leq t \leq b$, $\gamma(t)$ is a point on the path. We say that the path γ *starts* at $\gamma(a)$ and *ends* at $\gamma(b)$.

Remark. Note that a path is a function. Sometimes, it is convenient to regard a path as a set of points in $\mathbb{C}$, i.e. we identify the function γ with its image. However, we should regard this set of points as having an orientation: a path starts at one end-point and ends at the other. If we think of the path γ in this way then we sometimes call the function $\gamma(t)$ a *parametrisation* of the path γ. Note that the same path can have different parametrisations. For example

$$\gamma_1(t) = t + it,\ \gamma_2(t) = t^2 + it^2,\ 0 \leq t \leq 1$$

are both parametrisations of the straight line that starts at 0 and ends at $1 + i$. We shall see later (Proposition 4.3.1) that when we calculate an integral along a path then it is independent of the choice of parametrisation.

As an example of a path, let $z_0, z_1 \in \mathbb{C}$. Define

$$\gamma(t) = (1-t)z_0 + tz_1,\ 0 \leq t \leq 1. \tag{4.2.1}$$

Then $\gamma(0) = z_0, \gamma(1) = z_1$ and the image of γ is the straight line joining z_0 to z_1. We sometimes denote this path by $[z_0, z_1]$. See Figure 4.2.1.

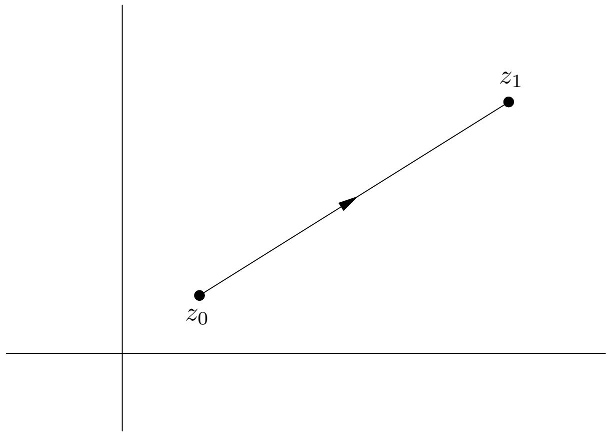

Figure 4.2.1: The path $\gamma(t) = (1-t)z_0 + tz_1$, $0 \leq t \leq 1$, describes the straight line joining z_0 to $z-1$. We sometimes denote this path by $[z_0, z_1]$.

Definition. Let $\gamma : [a, b] \to \mathbb{C}$ be a path. If $\gamma(a) = \gamma(b)$ (i.e. if γ starts and ends at the same point) then we say that γ is a *closed path* or a *closed loop*.

Example. An important example of a closed path is given by

$$\gamma(t) = e^{it} = \cos t + i \sin t,\ 0 \leq t \leq 2\pi. \tag{4.2.2}$$

This is the path that describes the circle in $\mathbb{C}$ with centre 0 and radius 1, starting and ending at the point 1, travelling around the circle in an anticlockwise direction. See Figure 4.2.2.

Definition. A path γ is said to be *smooth* if $\gamma : [a, b] \to \mathbb{C}$ is differentiable and γ' is continuous. (By differentiable at a we mean that the one-sided derivative exists, similarly at b.)

All of the examples of paths above are smooth.

We can use integrals to define the lengths of paths:

Definition. Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth path. Then the *length of* γ is defined to be

$$\text{length}(\gamma) = \int_a^b |\gamma'(t)|\, dt.$$

Example. It is straightforward to check from (4.2.1) that

$$\text{length}([z_0, z_1]) = |z_1 - z_0|.$$

If $\gamma(t)$ is the path given in (4.2.2) then

$$\text{length}(\gamma) = 2\pi.$$

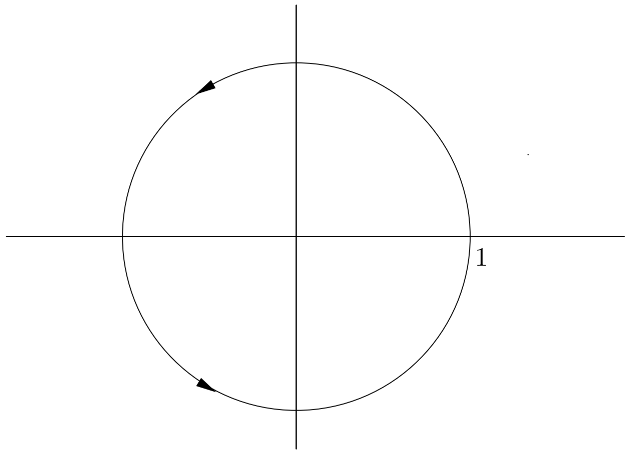

Figure 4.2.2: The circular path $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$. Note that it starts at 1 and travels anticlockwise around the unit circle.

Often we will want to integrate over a number of paths joined together. One could make the latter a path by constructing a suitable reparametrisation, but in practice this makes things complicated; in particular the joins may not be smooth. It is simpler to give a name to several smooth paths joined together.

Definition. A *contour* γ is a collection of smooth paths $\gamma_1, \ldots, \gamma_n$ where the end-point of γ_r coincides with the start point of γ_{r+1}, $1 \leq r \leq n-1$. We write

$$\gamma = \gamma_1 + \cdots + \gamma_n.$$

If the end-point of γ_n coincides with the start point of γ_1 then we call γ a *closed contour*.

Thus a contour is a path that is smooth except at finitely many places. A contour looks like a smooth path but with finitely many corners.

Example. Let $0 < \varepsilon < R$. Define

$$\begin{aligned} \gamma_1 : [\varepsilon, R] \to \mathbb{C} &\qquad \gamma_1(t) = t, \\ \gamma_2 : [0, \pi] \to \mathbb{C} &\qquad \gamma_2(t) = Re^{it}, \\ \gamma_3 : [-R, -\varepsilon] \to \mathbb{C} &\qquad \gamma_3(t) = t, \\ \gamma_4 : [-\pi, 0] \to \mathbb{C} &\qquad \gamma_4(t) = \varepsilon e^{-it}. \end{aligned}$$

Then $\gamma = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ is a closed contour (see Figure 4.2.3).

Definition. The length of a contour $\gamma = \gamma_1 + \cdots + \gamma_n$ is defined to be

$$\text{length}(\gamma) = \text{length}(\gamma_1) + \cdots + \text{length}(\gamma_n).$$

Suppose that $\gamma : [a, b] \to \mathbb{C}$ is a path that starts at $\gamma(a)$ and ends at $\gamma(b)$. Then we can consider the reverse of this path, where we start at $\gamma(b)$ and, travelling backwards along γ, end at $\gamma(a)$. More formally, we make the following definition.

Definition. Let $\gamma : [a, b] \to \mathbb{C}$ be a path. Define $-\gamma : [a, b] \to \mathbb{C}$ to be the path

$$-\gamma(t) = \gamma(a + b - t).$$

We call $-\gamma$ the *reversed path* of γ.

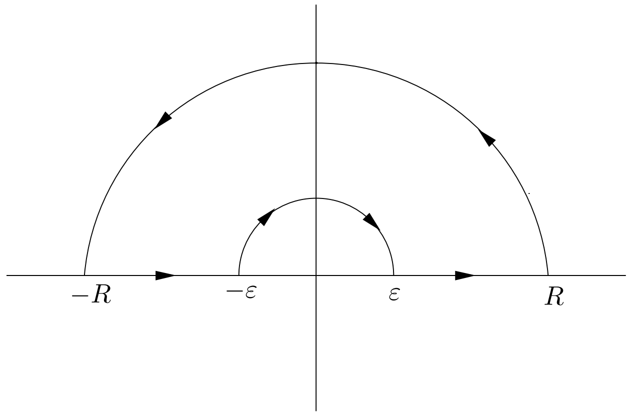

Figure 4.2.3: The contour $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$.

§4.3 Contour integration

Let $f : D \to \mathbb{C}$ be a complex functions defined on a domain D. Let $\gamma : [a, b] \to D$ be a smooth path in D.

Definition. The integral of f along γ is defined to be

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\gamma'(t)\,dt. \tag{4.3.1}$$

We will often write $\int_\gamma f$ for $\int_\gamma f(z)\,dz$.

Remark. Strictly speaking we should write $f(\gamma(t))\gamma'(t) = u(t)+iv(t)$ where $u, v : [a, b] \to \mathbb{R}$ and define $\int_\gamma f$ to be $\int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$.

Example. Take $f(z) = z^2$ and $\gamma(t) = t^2 + it$, $0 \le t \le 1$. Then $f(\gamma(t)) = (t^2 + it)^2 = t^4 - t^2 + 2it^3$ and $\gamma'(t) = 2t + i$. Hence

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_0^1 f(\gamma(t))\gamma'(t)\,dt = \int_0^1 (t^4 - t^2 + 2it^3)(2t + i)\,dt \\ &= \int_0^1 2t^5 - 4t^3\,dt + i\int_0^1 5t^4 - t^2\,dt \\ &= \left[\frac{1}{3}t^6 - t^4\right]_0^1 + i\left[t^5 - \frac{1}{3}t^3\right]_0^1 \\ &= \frac{-2}{3} + i\frac{2}{3}.\end{aligned}$$

The following proposition shows that the definition (4.3.1) is independent of the choice of parametrisation of the path.

Proposition 4.3.1
Let $\gamma : [a, b] \to \mathbb{C}$ be a smooth path. Let $\phi : [c, d] \to [a, b]$ be an increasing smooth bijection. Then $\gamma \circ \phi : [c, d] \to \mathbb{C}$ is a path that has the same image as γ. Moreover,

$$\int_{\gamma\circ\phi} f = \int_\gamma f$$

for any continuous function f.

Proof. It is clear that both γ and $\gamma \circ \phi$ have the same image. Thus γ and $\gamma \circ \phi$ are different parametrisations of the same path. Note that

$$\begin{aligned}
\int_{\gamma \circ \phi} f &= \int_c^d f(\gamma(\phi(t)))(\gamma\phi)'(t)\,dt \\
&= \int_c^d f(\gamma(\phi(t)))\gamma'(\phi(t))\phi'(t)\,dt \text{ by the chain rule} \\
&= \int_a^b f(\gamma(t))\gamma'(t)\,dt \text{ by the change of variables formula.}
\end{aligned}$$

□

Remark. If ϕ in Proposition 4.3.1 is a decreasing smooth bijection then $\gamma\phi$ has the same image as ϕ but the path traverses this in the opposite direction, i.e. $\gamma\phi$ is a parametrisation of $-\gamma$. Following the above calculation we see that $\int_{\gamma\phi} f = -\int_\gamma f$, corresponding to the fact stated below that $\int_{-\gamma} f = -\int_\gamma f$.

Now suppose that $\gamma = \gamma_1 + \cdots + \gamma_n$ is a contour in D. We define

$$\int_\gamma f = \int_{\gamma_1} f + \cdots + \int_{\gamma_n} f.$$

The following basic properties of contour integration follow easily from this definition.

Proposition 4.3.2
Let $f, g : D \to \mathbb{C}$ be continuous and let $c \in \mathbb{C}$. Suppose that $\gamma, \gamma_1, \gamma_2$ are contours in D. Suppose that the end point of γ_1 is the start point of γ_2. Then

(i)

$$\int_{\gamma_1 + \gamma_2} f = \int_{\gamma_1} f + \int_{\gamma_2} f;$$

(ii)

$$\int_\gamma (f + g) = \int_\gamma f + \int_\gamma g;$$

(iii)

$$\int_\gamma cf = c \int_\gamma f;$$

(iv)

$$\int_{-\gamma} f = -\int_\gamma f.$$

Recall from real calculus (or, indeed, from A-level or high school) that one way to calculate the integral of f is to find an anti-derivative, i.e. find a function F such that $F' = f$. The Fundamental Theorem of Calculus then says that $\int_a^b f(x)\,dx = F(b) - F(a)$. We have an analogue of this in for the complex integral. We first need the following definition.

Definition. Let $f : D \to \mathbb{C}$ be a continuous function. We say that a function $F : D \to \mathbb{C}$ is an *anti-derivative of f on D* if $F' = f$.

Theorem 4.3.3 (The Fundamental Theorem of Contour Integration)
Suppose that $f : D \to \mathbb{C}$ is continuous, $F : D \to \mathbb{C}$ is an antiderivative of f on D, and γ is a contour from z_0 to z_1. Then

$$\int_\gamma f = F(z_1) - F(z_0). \tag{4.3.2}$$

Proof. It is sufficient to prove the theorem for smooth paths. Let $\gamma : [a, b] \to D$, $\gamma(a) = z_0$, $\gamma(b) = z_1$, be a smooth path.

Let $w(t) = f(\gamma(t))\gamma'(t)$ and let $W(t) = F(\gamma(t))$. Then by the chain rule

$$W'(t) = F'(\gamma(t))\gamma'(t) = f(\gamma(t))\gamma'(t) = w(t).$$

Write $w(t) = u(t) + iv(t)$ and $W(t) = U(t) + iV(t)$ so that $U' = u$, $V' = v$. Hence

$$\begin{aligned}
\int_\gamma f &= \int_a^b f(\gamma(t))\gamma'(t)\,dt \\
&= \int_a^b w(t)\,dt \\
&= \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt \\
&= U(t)|_a^b + i\,V(t)|_a^b \quad \text{by the Fundamental Theorem of Calculus} \\
&= W(t)|_a^b \\
&= F(z_1) - F(z_0).
\end{aligned}$$

□

Remark. Notice that (4.3.2) does not depend on the choice of path γ from z_0 to z_1; all we need to know is that there exists an anti-derivative for f on a domain that contains z_0, z_1.

Example. Let $f(z) = z^2$ and let γ be any contour from $z_0 = 0$ to $z_1 = 1 + i$. Then $F(z) = z^3/3$ is an anti-derivative for f and

$$\int_\gamma z^2\,dz = \frac{1}{3}z_1^3 - \frac{1}{3}z_0^3 = \frac{(1+i)^3}{3} = -\frac{2}{3} + \frac{2}{3}i.$$

Remark. If γ is a closed path (i.e. γ starts and end at the same point) and f has an anti-derivative on a domain that contains γ then $\int_\gamma f = 0$. However, possessing an anti-derivative is a very strong hypothesis on f (see the following remarks).

Remark. In real analysis, any sufficiently nice function f has an anti-derivative: we define

$$F(x) = \int_0^x f(t)\,dt.$$

then $F' = f$. In complex analysis, however, the existence of an anti-derivative in on domain D is a very strong hypothesis. Consider for example $f(z) = 1/z$ defined on $D = \mathbb{C} \setminus \{0\}$. Does this function have an anti-derivative on D? The natural candidate would be $\operatorname{Log} z$. However, $\operatorname{Log} z$ is only continuous on the cut-plane; $\operatorname{Log} z$ is not continuous on D and so cannot be differentiable. So $\operatorname{Log} z$ is not an anti-derivative of $f(z) = 1/z$.

Remark. Let $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$ denote the unit circle in $\mathbb{C}$ described anticlockwise. note that γ is a closed path. Let $f(z) = 1/z$. The above remark suggests that f does not have an anti-derivative on any domain that contains γ. Thus to evaluate $\int_\gamma f$ we need to use the definition of the contour integral given in (4.3.1). We have

$$\int_\gamma f = \int_0^{2\pi} f(\gamma(t))\gamma'(t)\,dt = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it}\,dt = 2\pi i.$$

If f had an anti-derivative on a domain that contains γ then, by the Fundamental Theorem of Contour Integration, we would have that $\int_\gamma f = 0$. Hence $f(z) = 1/z$ does not have an anti-derivative on any domain that contains γ.

In general, looking for an anti-derivative is not the best way of calculating complex integrals. There are much more powerful techniques that allows us to calculate many complex integrals without having to worry about anti-derivatives. One such technique that applies in the case when γ is a closed contour is Cauchy's Theorem. Before discussing Cauchy's Theorem, we need a technical result about integration known as the Estimation Lemma.

§4.4 The Estimation Lemma

There are two results about real integration that are obvious from considering the integral of $f(x)$ over $[a, b]$ as the area underneath the graph of f. Firstly

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx \tag{4.4.1}$$

and secondly, if $|f(x)| \leq M$ then

$$\left| \int_a^b f(x)\,dx \right| \leq M(b-a). \tag{4.4.2}$$

See Figures 4.4.1 and 4.4.2.

Both of these results have analogies in the context of complex analysis. However, the proofs are surprisingly intricate.

Here is the complex analogue of (4.4.1).

Lemma 4.4.1
Let $u, v : [a, b] \to \mathbb{R}$ be continuous functions. Then

$$\left| \int_a^b u(t) + iv(t)\,dt \right| \leq \int_a^b |u(t) + iv(t)|\,dt. \tag{4.4.3}$$

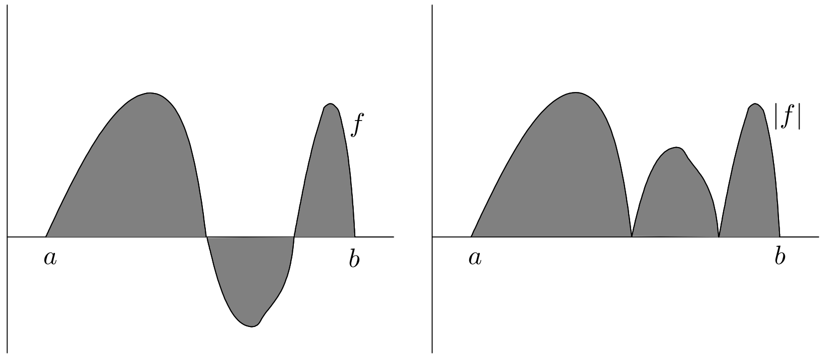

Figure 4.4.1: If $f[a,b] \to \mathbb{R}$ is negative on some subset of $[a,b]$ then the area underneath that part of the graph is negative. When f is replaced by $|f|$, this area becomes positive.

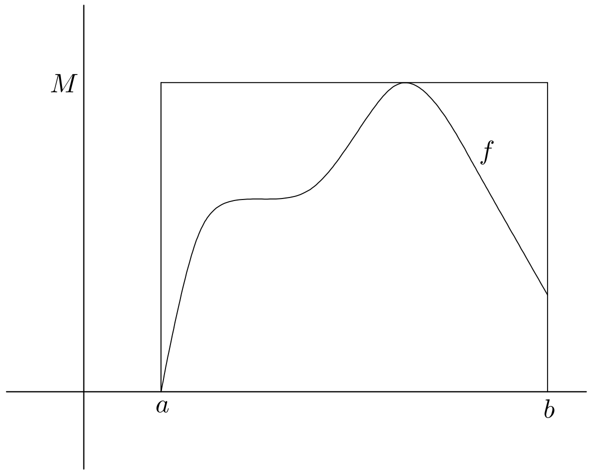

Figure 4.4.2: The graph of f is contained inside the rectangle of width $b-a$ and height M. Hence the area underneath the graph is at most $M(b-a)$.

Proof. Write

$$\int_a^b u(t) + iv(t)\,dt = X + iY.$$

Then

$$\begin{aligned} X^2 + Y^2 &= (X - iY)(X + iY) \\ &= \int_a^b (X - iY)(u(t) + iv(t))\,dt \\ &= \int_a^b Xu(t) + Yv(t)\,dt + i\int_a^b Xv(t) - Yu(t)\,dt. \end{aligned}$$

However, $X^2 + Y^2$ is real, hence the imaginary part of the above expression must be zero, i.e.

$$\int_a^b Xv(t) - Yu(t)\,dt = 0$$

so that

$$X^2 + Y^2 = \int_a^b Xu(t) + Yv(t)\,dt. \tag{4.4.4}$$

Notice that the integrand in (4.4.4) is the real part of $(X - iY)(u(t) + iv(t))$. Recalling that for any complex number z we have that $\mathrm{Re}(z) \leq |z|$, we have that

$$\begin{aligned} Xu(t) + Yv(t) &\leq |(X - iY)(u(t) + iv(t))| \\ &= |X - iY||u(t) + iv(t)| \\ &= \sqrt{X^2 + Y^2}|u(t) + iv(t)|. \end{aligned}$$

Hence

$$\begin{aligned} X^2 + Y^2 &= \int_a^b Xu(t) + Yv(t)\,dt \\ &\leq \sqrt{X^2 + Y^2} \int_a^b |u(t) + iv(t)|\,dt \end{aligned}$$

and cancelling the term $\sqrt{X^2 + Y^2}$ gives

$$\left| \int_a^b u(t) + iv(t)\,dt \right| = |X + iY| = \sqrt{X^2 + Y^2} \leq \int_a^b |u(t) + iv(t)|\,dt$$

as claimed. □

We can now prove the following important result—the complex analogue of (4.4.2)—which we will use many times in the remainder of the course.

Lemma 4.4.2 (The Estimation Lemma)
Let $f : D \to \mathbb{C}$ be continuous and let γ be a contour in D. Suppose that $|f(z)| \leq M$ for all z on γ. Then

$$\left| \int_\gamma f \right| \leq M \operatorname{length}(\gamma).$$

Remark. We shall use the Estimation Lemma in two different ways: (i) suppose f is a function which takes small (in modulus) values along a contour γ, then $\int_\gamma f$ is small; (ii) if f is any continuous function and γ is a contour with small length, then $\int_\gamma f$ is small.

Proof. Simply note that by Lemma 4.4.1 we have that

$$\begin{aligned} \left| \int_\gamma f \right| &= \left| \int_a^b f(\gamma(t))\gamma'(t)\,dt \right| \\ &\leq \int_a^b |f(\gamma(t))||\gamma'(t)|\,dt \\ &\leq M \int_a^b |\gamma'(t)|\,dt \\ &= M \operatorname{length}(\gamma). \end{aligned}$$

□

Example. Let $f(z) = 1/(z^2 + z + 1)$ and let $\gamma(t) = 5e^{it}$, $0 \leq t \leq 2\pi$, be the circle of radius 5 centred at 0. We use the Estimation Lemma to bound $\int_\gamma f(z)\,dz$.

First note that if z is a point on γ then $|z| = 5$. Hence

$$\begin{aligned} |z^2 + z + 1| &\geq |z|^2 - |z+1| \text{ by the reverse triangle inequality} \\ &\geq |z|^2 - |z| - 1 \text{ by the triangle inequality} \\ &= 25 - 5 - 1 = 19. \end{aligned}$$

Thus for z on γ we have that

$$|f(z)| = \left| \frac{1}{z^2 + z + 1} \right| \leq \frac{1}{19}.$$

Next we note that length$(\gamma) = 2\pi \times 5 = 10\pi$.

Thus, by the Estimation Lemma,

$$\left| \int_\gamma f(z)\,dz \right| \leq \frac{10\pi}{19}.$$

§4.5 Winding numbers and Cauchy's Theorem

Suppose that $f : D \to \mathbb{C}$. The Fundamental Theorem of Contour Integration (Theorem 4.3.3) tells us that if f has an anti-derivative F in D and γ is any path in D from z_0 to z_1 then

$$\int_\gamma f = F(z_1) - F(z_0).$$

We say that $\gamma : [a,b] \to \mathbb{D}$ is *closed* if it begins and ends at the same point, i.e. if $z_0 = \gamma(a) = \gamma(b) = z_1$.

In particular, it follows from Theorem 4.3.3 that if f has an anti-derivative F on D then

$$\int_\gamma f = 0 \tag{4.5.1}$$

for all closed paths γ in D. What happens if we do not know if f has an anti-derivative? In this case, Cauchy's Theorem gives conditions under which (4.5.1) continues to hold. (Actually, there are many different theorems of this kind, most of which are either due to, or were known to, Cauchy and are often referred to as 'Cauchy's Theorem'. We will give one version expressed in terms of winding numbers.)

Let γ be a closed path and let z_0 be a point that is not on γ. Imagine you have a piece of string. Tie one end to (say) a pencil and place the tip of the pencil on the point z_0. Now trace around the closed path γ with the other end of the piece of string. When you get back to where you started, the string will be wrapped around the pencil some number of times. This number (counted positively for anti-clockwise turns and negatively for clockwise turns) is the winding number of γ at z_0. See Figure 4.5.1 for examples of winding numbers.

In examples, it is easy to calculate winding numbers by eye and this is how we shall always do it. However, in order to use winding numbers to develop the theory of integration, we shall need an analytic expression for the winding number $w(\gamma, z)$ of a closed path γ around a point z. Let us first consider the case when the closed path γ does not pass through the origin 0. We need the following result, which we state without proof.

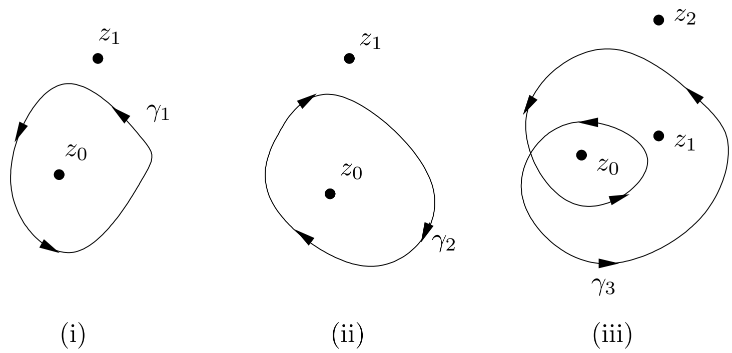

Figure 4.5.1: In (i), $w(\gamma_1, z_0) = 1$ and $w(\gamma_1, z_1) = 0$. In (ii), $w(\gamma_2, z_0) = -1$ and $w(\gamma_2, z_1) = 0$ as γ_2 winds clockwise around z_0. In (iii), $w(\gamma_3, z_0) = 2, w(\gamma_3, z_1) = 1, w(\gamma_3, z_2) = 0$ as γ_3 winds anticlockwise twice around z_0, anticlockwise once around z_1 and does not wind at all around z_2.

Proposition 4.5.1
Let γ be a path in $\mathbb{C} \setminus \{0\}$. Then there exists a parametrisation $\gamma : [a, b] \to \mathbb{C} \setminus \{0\}$ of γ for which $t \mapsto \arg \gamma(t)$ is a continuous function. Any other choice of parametrisation with a continuous choice of argument differs from this argument by a constant integer multiple of 2π.

Example. For example, consider

$$\gamma(t) = \begin{cases} e^{it}, \ 0 \leq t \leq \pi \\ e^{i(t+2\pi)}, \ \pi < t \leq 2\pi. \end{cases}$$

Then γ describes the unit circle with centre 0 and radius 1. Here

$$\arg \gamma(t) = \begin{cases} t, \ 0 \leq t \leq \pi \\ t + 2\pi, \ \pi < t \leq 2\pi. \end{cases}$$

and this is not continuous. However, we can find a parametrisation of γ for which the argument is continuous, for example

$$\gamma(t) = e^{it}, \ 0 \leq t \leq 2\pi$$

and note that $\arg \gamma(t) = t$, $0 \leq t \leq 2\pi$, is continuous.

Now consider the closed path γ. We can reinterpret the winding number $w(\gamma, 0)$ of γ around 0 as the multiple of 2π giving the total change in argument along γ.

Proposition 4.5.2
Let γ be a closed path that does not pass through the origin. Then

$$w(\gamma, 0) = \frac{1}{2\pi i} \int_\gamma \frac{1}{z} \, dz.$$

Example. Let $\gamma(t) = e^{4\pi i t}$, $0 \leq t \leq 1$. Intuitively, this winds around the origin twice anticlockwise, and so should have winding number $w(\gamma, 0) = 2$. We can check this using

Proposition 4.5.2 as follows:

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,dz &= \frac{1}{2\pi i}\int_0^1 \frac{1}{e^{4\pi it}}4\pi i e^{4\pi it}\,dt \\ &= \int_0^1 2\,dt = 2.\end{aligned}$$

Example. Let $\gamma(t) = e^{-it}$, $0 \leq t \leq 2\pi$. In this case, γ winds around the origin once, but clockwise. Thus $w(\gamma, 0) = -1$. Again, we can check this using Proposition 4.5.2 as follows:

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{1}{z}\,dz &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{e^{-it}}(-i)e^{-it}\,dt \\ &= \int_0^{2\pi} \frac{-1}{2\pi}\,dt = -1.\end{aligned}$$

Proof of Proposition 4.5.2. Intuitively this is clear: let $\gamma : [a,b] \to \mathbb{C} \setminus \{0\}$ be a closed path that does not pass through the origin. Note that $\gamma(a) = \gamma(b)$. Then (and we put quotes around the following to indicate that this is not a valid proof)

$$\begin{aligned}\text{“}\int_\gamma \frac{1}{z}\,dz &= \int_a^b \frac{1}{\gamma(t)}\gamma'(t) \\ &= [\log(\gamma(t))]_a^b \\ &= (\ln|\gamma(b)| + i\arg\gamma(b)) - (\ln|\gamma(a)| + i\arg\gamma(a)) \\ &= i\,(\arg\gamma(b) - \arg\gamma(a)) \\ &= 2\pi i w(\gamma, 0)\text{”}.\end{aligned}$$

The reason that the above computation does not work is that $1/z$ does not have $\log(z)$ (or, indeed, the principal logarithm $\mathrm{Log}(z)$) as an antiderivative on $\mathbb{C} \setminus \{0\}$. This is because $\mathrm{Log}(z)$ is not continuous on $\mathbb{C} \setminus \{0\}$ and so cannot be differentiable. However, $\mathrm{Log}(z)$ is continuous and is an anti-derivative for $1/z$ on the cut plane, where we remove the negative real axis from $\mathbb{C}$. More generally, one can define a logarithm continuously on a cut plane where one removes any ray from $\mathbb{C}$. (A *ray* is an infinite straight line starting at 0; for example, the negative real axis is a ray.)

For each $\alpha \in [-\pi, \pi)$ define the cut plane at angle α to be

$$\mathbb{C}_\alpha = \mathbb{C} \setminus \{re^{i\alpha} \mid r > 0\},$$

i.e. the complex plane with the ray inclined at angle α from the positive x-axis removed. On $\mathbb{C}_\alpha$ we can define $\arg z$ to be $\arg_\alpha z = \theta$ where

$$z = re^{i\theta},\ r > 0,\ \alpha - 2(m+1)\pi < \theta \leq \alpha - 2m\pi$$

where we have the freedom to choose any $m \in \mathbb{Z}$. (The case $\alpha = \pi, m = 0$ corresponds to the usual principal value of the argument.)

Let γ be a closed path that does not pass through the origin. In general, γ will not lie in one cut plane. Split γ up into pieces $\gamma_1, \ldots, \gamma_n$ defined on $[t_0, t_1], \ldots, [t_{n-1}, t_n]$ so that each γ_r lies in a single cut plane, $\mathbb{C}_{\alpha_r}$, say. Along each γ_r we will choose a value of the argument $\arg_{\alpha_r}$ which is continuous on $\mathbb{C}_{\alpha_r}$ and such that $\arg_{\alpha_r} \gamma_r(t_r) = \arg_{\alpha_{r+1}} \gamma_{r+1}(t_r)$, $0 \leq r \leq n-1$. Hence

$$\begin{aligned}
\int_{\gamma_r} \frac{1}{z}\,dz &= \log\gamma(t_r) - \log(\gamma(t_{r-1})) \\
&= \log|\gamma(t_r)| - \log|\gamma(t_{r-1})| + i\left(\arg_{\alpha_r}(\gamma(t_r)) - \arg_{\alpha_r}(\gamma(t_{r-1}))\right).
\end{aligned}$$

Now

$$\begin{aligned}
\int_{\gamma} \frac{1}{z}\,dz &= \sum_{r=1}^{n} \int_{\gamma_r} \frac{1}{z}\,dz \\
&= \sum_{r=1}^{n} (\log|\gamma(t_r)| - \log|\gamma(t_{r-1})|) + i\sum_{r=1}^{n}\left(\arg_{\alpha_r}(\gamma(t_r)) - \arg_{\alpha_r}(\gamma(t_{r-1}))\right).
\end{aligned}$$

The real parts cancel. The imaginary parts sum to

$$\arg_{\alpha_n}(\gamma(t_n)) - \arg_{\alpha_0}(\gamma(t_0)),$$

the total change in argument around γ, i.e. $2\pi w(\gamma, 0)$. □

More generally, we have the following formula for the winding number around z_0 for a closed path that does not pass through z_0.

Proposition 4.5.3
Let γ be a closed path that does not pass through z_0. Then

$$w(\gamma, z_0) = \frac{1}{2\pi i}\int_{\gamma} \frac{1}{z - z_0}\,dz.$$

Proof. This is just a change-of-origin argument. Let $\gamma : [a, b] \to \mathbb{C}$ be a closed path that does not pass through z_0. Consider the path $\gamma_1(t) = \gamma(t) - z_0$; this is γ translated by z_0. Then $w(\gamma, z_0) = w(\gamma_1, 0)$. Now

$$\begin{aligned}
\frac{1}{2\pi i}\int_{\gamma} \frac{1}{z - z_0}\,dz &= \frac{1}{2\pi i}\int_a^b \frac{1}{\gamma(t) - z_0}\gamma'(t)\,dt \\
&= \frac{1}{2\pi i}\int_a^b \frac{1}{\gamma_1(t)}\gamma_1'(t)\,dt \text{ as } \gamma'(t) = \gamma_1'(t) \\
&= \frac{1}{2\pi i}\int_{\gamma_1} \frac{1}{z}\,dz \\
&= w(\gamma_1, 0).
\end{aligned}$$

□

The following results are obvious in terms of the geometric meaning of winding number.

Proposition 4.5.4
(i) Let γ_1, γ_2 be closed paths that do not pass through z_0. Then

$$w(\gamma_1 + \gamma_2, z_0) = w(\gamma_1, z_0) + w(\gamma_2, z_0).$$

(ii) Let γ be a closed path that does not pass through z_0. Then

$$w(-\gamma, z_0) = -w(\gamma, z_0).$$

Proof. We prove (i). To see this, note that by Proposition 4.3.2(i) we have that

$$\begin{aligned} w(\gamma_1 + \gamma_2, z_0) &= \frac{1}{2\pi i}\int_{\gamma_1+\gamma_2} \frac{1}{z - z_0}\,dz \\ &= \frac{1}{2\pi i}\int_{\gamma_1} \frac{1}{z - z_0}\,dz + \frac{1}{2\pi i}\int_{\gamma_2} \frac{1}{z - z_0}\,dz \\ &= w(\gamma_1, z_0) + w(\gamma_2, z_0). \end{aligned}$$

We prove (ii). By Proposition 4.3.2(i) we have that

$$\begin{aligned} w(-\gamma, z_0) &= \frac{1}{2\pi i}\int_{-\gamma} \frac{1}{z - z_0}\,dz \\ &= -\frac{1}{2\pi i}\int_{\gamma} \frac{1}{z - z_0}\,dz \\ &= w(\gamma, z_0). \end{aligned}$$

□

We can now state Cauchy's Theorem.

Theorem 4.5.5 (Cauchy's Theorem)
Let f be holomorphic on a domain D and let γ be a closed contour in D that does not wind around any point outside D (i.e. $w(\gamma, z) = 0$ for all $z \notin D$). Then $\int_\gamma f = 0$.

Remark. The strength of Cauchy's Theorem is that we do not need to know if f has an anti-derivative on D. (If f did have an antiderivative on D then $\int_\gamma f = 0$ follows immediately from the Fundamental Theorem of Contour Integration; however, possessing an antiderivative is an extremely strong assumption on f. See Theorem 4.3.3 and the remarks following it.)

Remark. See Figure 4.5.2 for examples of the hypotheses of Cauchy's Theorem.

Proof. There are many proofs of Cauchy's Theorem; here we give one based on Green's Theorem (see MATH10121 Calculus and Vectors). We assume (in addition to the hypotheses stated) that f has continuous partial derivatives.

Green's theorem states the following: suppose that γ is a piecewise smooth closed contour bounding a region Γ, g, h are C^1 functions defined on an open set containing Γ, then

$$\int_\gamma g(x,y)\,dx + h(x,y)\,dy = \int\int_\Gamma \left(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y}\right) dx\,dy. \tag{4.5.2}$$

Let f be as in the hypotheses and write $f(z) = f(x+iy) = u(x,y) + iv(x,y)$. Note that $dz = dx + i\,dy$. Then

$$\int_\gamma f\,dz = \int_\gamma (u + iv)(dx + i\,dy)$$

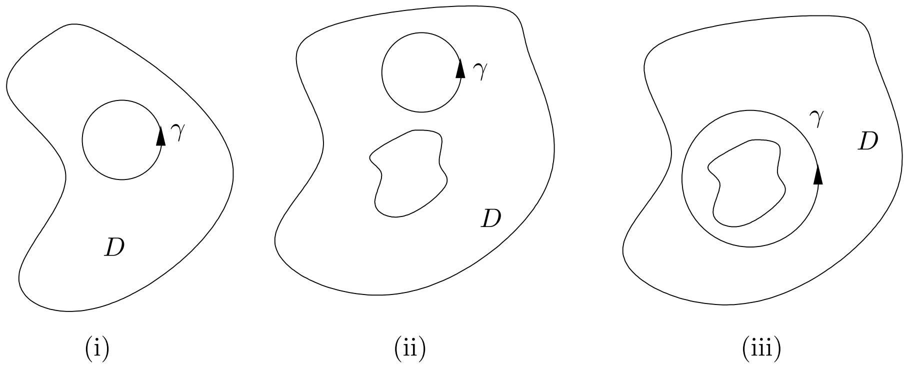

Figure 4.5.2: In (i) and (ii), γ has winding number zero around every point outside D, so the hypotheses of Cauchy's Theorem (Theorem 4.5.5 hold. In (iii) γ has winding number 1 around points inside the 'hole' in D, hence the hypothesis of Cauchy's Theorem do not hold.

$$
\begin{aligned}
&= \int_\gamma u\,dx - v\,dy + i\int_\gamma v\,dx + u\,dy \\
&= \int\int_\Gamma \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy + \int\int_\Gamma \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy \\
&= 0
\end{aligned}
$$

as, by the Cauchy-Riemann equations, $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$ hold everywhere on Γ. □

Remark. In many ways, this proof is cheating: Green's Theorem is a deep theorem and not easy to prove. There are direct proofs of Cauchy's theorem, but they are lengthy and difficult. (The idea is to build D up from smaller pieces, often starting with the case when D is a triangle; see Stewart and Tall, p.143.)

Another reason for why the above proof is cheating is that Green's theorem requires the partial derivatives in (4.5.2) to be continuous. In general, the statement of Cauchy's Theorem only requires the partial derivatives to exist in D (i.e. we do not need to assume that they are continuous). In fact, as we shall see, the existence of the derivative on a domain forces the derivative (and so the partial derivatives) to be continuous (indeed, if the derivative exists on a domain then the function is differentiable infinitely many times). However the proof of this fact uses Cauchy's Theorem.

There are many variants of Cauchy's Theorem. Here we give just two simple modifications.

Our first variant deals with simply connected domains. Heuristically, a domain is simply connected if it does not have any holes in it. (For example, in Figure 4.5.2(i) the domain D is simply connected; however the domains D in Figures 4.5.2(ii) and (iii) are not simply connected as they have holes in them.) More precisely:

Definition. A domain D is *simply connected* if for all closed contours γ in D and for all $z \notin D$, we have $w(\gamma, z) = 0$.

Theorem 4.5.6 (Cauchy's Theorem for simply connected domains)
Suppose that D is a simply connected domain and f is a holomorphic function on D. Then for any closed contour γ in D we have that $\int_\gamma f = 0$.

More generally, we can ask about integrating around several closed contours.

Theorem 4.5.7 (The Generalised Cauchy Theorem)
Let D be a domain and let f be holomorphic on D. Let $\gamma_1, \ldots, \gamma_n$ be closed contours in D. Suppose that

$$w(\gamma_1, z) + \cdots + w(\gamma_n, z) = 0 \text{ for all } z \notin D.$$

Then

$$\int_{\gamma_1} f + \cdots + \int_{\gamma_n} f = 0.$$

Remark. The hypotheses of the Generalised Cauchy Theorem (Theorem 4.5.7) give one way of extending Cauchy's Theorem to non-simply connected domains. Consider the example in Figure 4.5.3. Here, if z is 'outside' D then clearly $w(\gamma_1, z) = w(\gamma_2, z) = 0$. If z is in the 'hole' in D then $w(\gamma_1, z) = 1, w(\gamma_2, z) = -1$ so that $w(\gamma_1, z) + w(\gamma_2, z) = 0$. Hence the hypotheses of the Generalised Cauchy Theorem hold.

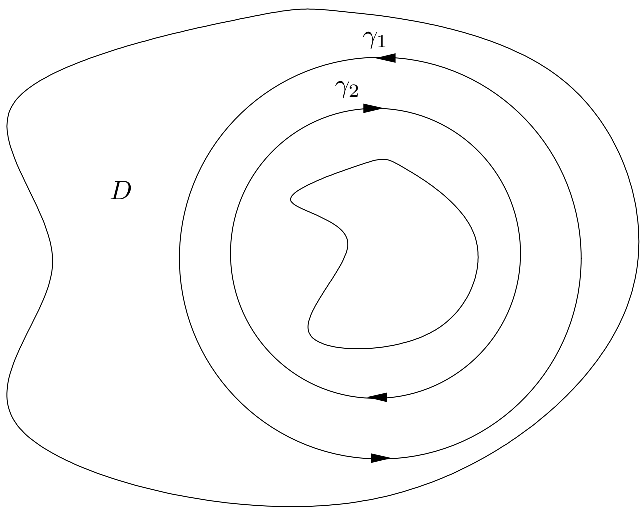

Figure 4.5.3: An example of closed contours that satisfy the hypotheses in the Generalised Cauchy Theorem (Theorem 4.5.7).

Proof of Theorem 4.5.7. Suppose that γ_r starts and ends at $z_j \in D$, $1 \leq j \leq n$. Choose any $z_0 \in D$ and contours $\sigma_1, \ldots, \sigma_n$ in D which join z_0 to $z_1, \ldots, z_n$, respectively. (See Figure 4.5.4.) Note that, for each j, $\sigma_j + \gamma_j - \sigma_j$ is a closed contour that starts and ends at z_0 and, moreover, that for $z \notin D$ we have $w(\sigma_j + \gamma_j - \sigma_j, z) = w(\gamma_j, z)$. We see that

$$\gamma = \sigma_1 + \gamma_1 - \sigma_1 + \cdots + \sigma_n + \gamma_n - \sigma_n$$

is a closed contour that starts and ends at z_0. Let $z \notin D$. Then, using Proposition 4.5.4,

$$w(\gamma, z) \quad = \quad w(\sigma_1 + \gamma_1 - \sigma_1 + \cdots + \sigma_n + \gamma_n - \sigma_n, z)$$

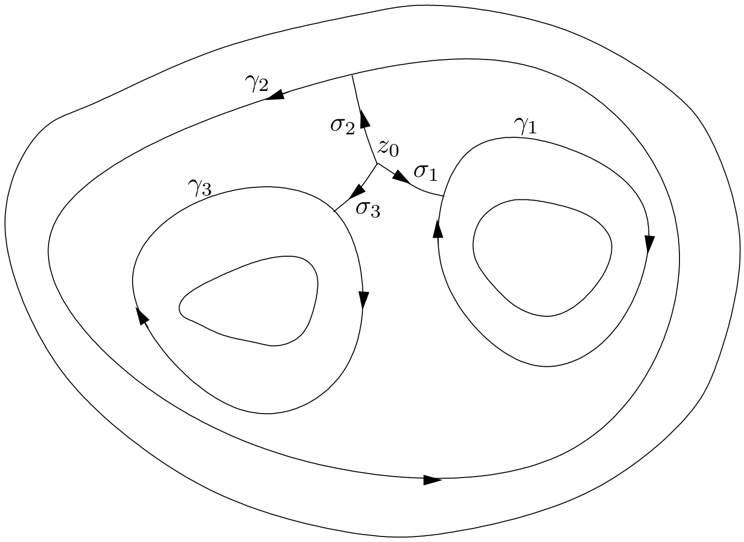

Figure 4.5.4: The path γ is formed by starting at z_0, traversing σ_1, then around γ_1, then back along σ_1, then along σ_2, around γ_2, back along σ_2, along σ_3, around γ_3 and back along σ_3, ending at z_0.

$$\begin{aligned} &= \sum_{j=1}^{n} w(\sigma_j + \gamma_j - \sigma_j, z) \\ &= \sum_{j=1}^{n} w(\gamma_j, z) \\ &= 0. \end{aligned}$$

Hence by Cauchy's Theorem $\int_\gamma f = 0$. Hence

$$\sum_{j=1}^{n} \left(\int_{\sigma_j} f + \int_{\gamma_j} f + \int_{-\sigma_j} f \right) = \sum_{j=1}^{n} \int_{\gamma_j} f$$

as $\int_{-\sigma_j} f = -\int_{\sigma_j} f$. $\square$

Exercises for Part 4

Exercise 4.1
Draw the following paths:

(i) $\gamma(t) = e^{-it}$, $0 \leq t \leq \pi$,

(ii) $\gamma(t) = 1 + i + 2e^{it}$, $0 \leq t \leq 2\pi$,

(iii) $\gamma(t) = t + i\cosh t$, $-1 \leq t \leq 1$,

(iv) $\gamma(t) = \cosh t + i\sinh t$, $-1 \leq t \leq 1$.

Exercise 4.2
Find the values of

$$\int_\gamma x - y + ix^2 \, dz$$

where $z = x + iy$ and γ is:

(i) the straight line joining 0 to $1 + i$;

(ii) the imaginary axis from 0 to i;

(iii) the line parallel to the real axis from i to $1 + i$.

Exercise 4.3
Let

$$\begin{aligned} \gamma_1(t) &= 2 + 2e^{it}, \ 0 \leq t \leq 2\pi, \\ \gamma_2(t) &= i + e^{-it}, \ 0 \leq t \leq \pi/2. \end{aligned}$$

Draw the paths γ_1, γ_2.

From the definition $\int_\gamma f = \int_a^b f(\gamma(t))\gamma'(t)\,dt$, calculate

$$\text{(i)} \int_{\gamma_1} \frac{dz}{z-2}, \ \text{(ii)} \int_{\gamma_2} \frac{dz}{(z-i)^3}.$$

Exercise 4.4
Let γ denote the circular path with centre 1 and radius 1, described once anticlockwise and starting at the point 2. Let $f(z) = |z|^2$. Write down a parametrisation of γ. Hence calculate $\int_\gamma |z|^2 \, dz$

Exercise 4.5
For each of the following functions find an anti-derivative and calculate the integral along any smooth path from 0 to i:

(i) $f : \mathbb{C} \to \mathbb{C}$, $f(z) = z^2 \sin z$;

(ii) $f : \mathbb{C} \to \mathbb{C}$, $f(z) = ze^{iz}$.

Exercise 4.6
Calculate $\int_\gamma |z|^2 \, dz$ where

(i) γ denotes the contour that goes vertically from 0 to i then horizontally from i to $1+i$;

(ii) γ denotes the contour that goes horizontally from 0 to 1 then vertically from 1 to $1+i$.

What does this tell you about possibility of the existence of an anti-derivative for $f(z) = |z|^2$?

Exercise 4.7
Let γ denote the semi-circular path along a circle with centre 0 and radius 3 which starts at 3 and ends at -3. Write down a parametrisation of γ.

Let $f(z) = 1/z^2$ defined on the domain $D = \mathbb{C} \setminus \{0\}$. Calculate $\int_\gamma f$.

Write down a parametrisation of $-\gamma$. Calculate $\int_{-\gamma} f$ from the definition of the contour integral and check that, in this case, $\int_{-\gamma} f = -\int_\gamma f$.

Exercise 4.8
Prove Proposition 4.3.2(iv): Let D be a domain, γ a contour in D, and let $f : D \to \mathbb{C}$ be continuous. Let $-\gamma$ denote the reversed path of γ. Show that

$$\int_{-\gamma} f = -\int_\gamma f.$$

Exercise 4.9
Let $f, g : D \to \mathbb{C}$ be holomorphic. Let γ be a smooth path in D starting at z_0 and ending at z_1. Prove the complex analogue of the integration by parts formula:

$$\int_\gamma fg' = f(z_1)g(z_1) - f(z_0)g(z_0) - \int_\gamma f'g.$$

Exercise 4.10
Calculate (by eye) the winding number around any point which is not on the path.

Exercise 4.11
Let

$$\begin{aligned} \gamma_1(t) &= -1 + \frac{1}{2}e^{it}, \ 0 \leq t \leq 2\pi, \\ \gamma_2(t) &= 1 + \frac{1}{2}e^{it}, \ 0 \leq t \leq 2\pi, \\ \gamma(t) &= 2e^{it}, \ 0 \leq t \leq 2\pi. \end{aligned}$$

Let $f(z) = 1/(z^2 - 1)$. Use the Generalised Cauchy Theorem to deduce that

$$\int_\gamma f \, dz = \int_{\gamma_1} f \, dz + \int_{\gamma_2} f \, dz.$$

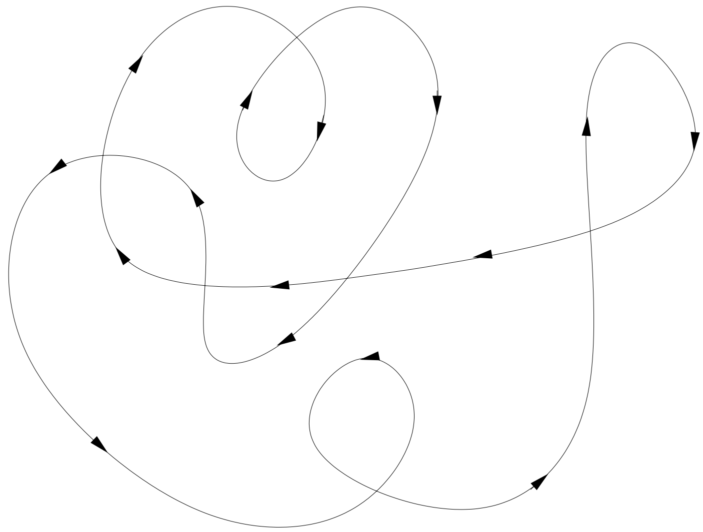

Figure 4.6.1: See Exercise 4.10.

Exercise 4.12
Let γ_1 denote the unit circle centred at 0, radius 1, described anti-clockwise. Let $f(z) = 1/z$. Show that $\int_{\gamma_1} f = 2\pi i$. Let γ_2 be the closed contour as illustrated in Figure 4.6.2. Use the Generalised Cauchy Theorem on an appropriate domain to calculate $\int_{\gamma_2} f$.

Exercise 4.13
Let D be the domain $\mathbb{C} \setminus \{z_1, z_2\}$. Suppose that $\gamma, \gamma_1, \gamma_2$ are closed contours in D as illustrated in Figure 4.6.3. Suppose that

$$\int_{\gamma_1} f = 3 + 4i, \quad \int_{\gamma_2} f = 5 + 6i.$$

Use the Generalised Cauchy Theorem to calculate $\int_\gamma f$.

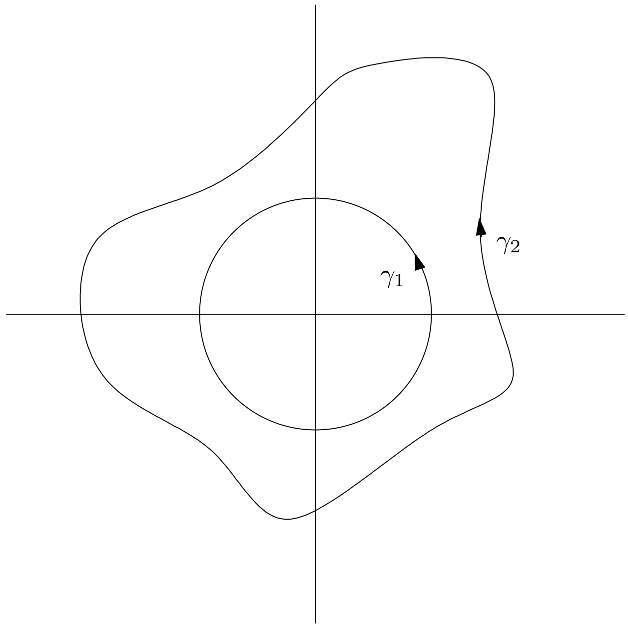

Figure 4.6.2: Here γ_1 denotes the unit circle described anticlockwise and γ_2 is an arbitrary closed contour that winds once around 0.

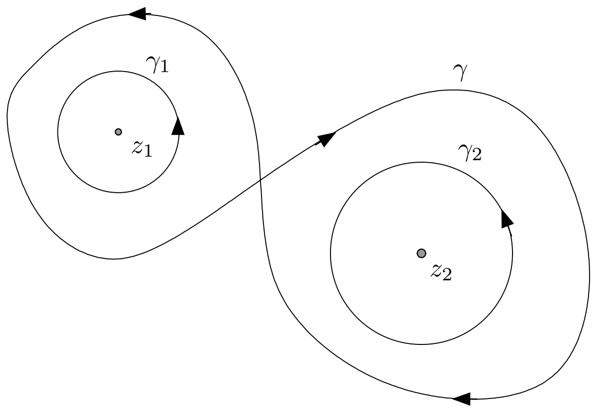

Figure 4.6.3: See Exercise 4.13.

5. Cauchy's Integral Formula and Taylor's Theorem

§5.1 Cauchy's Integral Formula

One of the most remarkable facts in complex analysis is Cauchy's Integral Formula. This says that, in a sense, one can differentiate a function just by knowing how to integrate it. (This partly explains why complex analysis is so much easier than real analysis. In real analysis, we say that a function is C^r if it can be differentiated r times and the rth derivative is continuous. Then $C^1 \supset C^2 \supset \cdots$ and we think of a function that is C^r for a large r as being 'nice'. If we differentiate a C^r function then we obtain a C^{r-1} function, i.e. differentiation takes nice functions and makes them slightly 'less nice'. Integration, however, works the other way: the indefinite integral of a C^r function is C^{r+1}. Hence integration makes nice functions 'even nicer'. In terms of complex analysis, this distinction into C^r functions does not have any meaning: as we shall see, if a function is differentiable once then it is differentiable infinitely many times!)

Theorem 5.1.1 (Cauchy's Integral Formula for a circle)
Suppose that f is holomorphic on the disc $\{z \in \mathbb{C} \mid |z - z_0| < R\}$. For $0 < r < R$ let C_r be the path $C_r(t) = z_0 + re^{it}$, $0 \le t \le 2\pi$ (so that C_r is the circle with centre z_0 and radius r). Then for $|w - z_0| < r$ we have that

$$f(w) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - w}\, dz. \tag{5.1.1}$$

Remark. Equation (5.1.1) has the following remarkable corollary: if we know the value of the function f along the closed path C_r then we know the values of the function at all points inside the disc C_r. This does not have an analogue in real analysis.

Remark. Theorem 5.1.1 is formulated in terms of the function being holomorphic on a disc and integrating around circles. This is not necessary, and a more general version of Cauchy's Integral Formula holds provided that f is holomorphic on a simply connected domain D and we replace C_r by an appropriate simple closed loop. (A closed loop γ is called *simple* if, for every point z not on γ, the winding number is either $w(\gamma, z) = 0$ or $w(\gamma, z) = 1$.)

Proof. Fix w such that $|w - z_0| < r$. Consider the function

$$g(z) = \frac{f(z) - f(w)}{z - w}.$$

Then g is differentiable in the domain $D = \{z \in \mathbb{C} \mid |z - z_0| < R, z \neq w\}$. Define the circle S_ε to be the circle centred at w and of radius $\varepsilon > 0$.

$$S_\varepsilon(t) = w + \varepsilon e^{it},\ 0 \le t \le 2\pi.$$

Then, provided $\varepsilon > 0$ is sufficiently small, both C_r and S_ε lie inside D.

We apply the Generalised Cauchy Theorem (Theorem 4.5.7) to the contours S_ε and $-C_r$. Suppose that z is not in the domain D. Then either $|z - z_0| > R$ or $z = w$. In the first case, if $|z - z_0| > R$ then $w(S_\varepsilon, z) = w(C_r, z) = 0$. In the second case, if $z = w$ then $w(S_\varepsilon, z) = 1$ and $w(-C_r, z) = -1$. Hence we have that $w(S_\varepsilon, z) + w(-C_r, z) = 0$ for all $z \notin D$. Noting that $\int_{-C_r} g = -\int_{C_r} g$ we have that, by the Generalised Cauchy Theorem (Theorem 4.5.7),

$$\int_{C_r} g(z)\,dz = \int_{S_\varepsilon} g(z)\,dz. \tag{5.1.2}$$

Now, from the definition of g, we have that $\lim_{z\to w} g(z) = f'(w)$. As $|f'(w)|$ is finite, it follows that $g(z)$ is bounded for z sufficiently close to w, i.e. there exist $\delta > 0$ and $M > 0$ such that if $0 < |w - z| < \delta$ then $|g(z)| < M$.

Hence, if $\varepsilon < \delta$, the Estimation Lemma (Lemma 4.4.2) implies that

$$\left| \int_{S_\varepsilon} g(z)\,dz \right| \leq M 2\pi\varepsilon.$$

By (5.1.2) it follows that

$$\left| \int_{C_r} g(z)\,dz \right| \leq M 2\pi\varepsilon,$$

and since we can take $\varepsilon > 0$ to be arbitrarily small, it follows that

$$\int_{C_r} g(z)\,dz = 0. \tag{5.1.3}$$

Recalling that $g(z) = (f(z) - f(w))/(z - w)$ and that $f(w)$ is constant, we can substitute this expression for g into (5.1.3) to obtain

$$\begin{aligned}
\int_{C_r} \frac{f(z)}{z-w}\,dz &= \int_{C_r} \frac{f(w)}{z-w}\,dz \\
&= f(w) \int_{C_r} \frac{1}{z-w}\,dz \\
&= f(w) 2\pi i w(C_r, w) \\
&= f(w) 2\pi i
\end{aligned}$$

as C_r winds once anticlockwise around w. Hence

$$f(w) = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z-w}\,dz.$$

□

§5.2 Taylor series

The integral formula allows us to express a differentiable function as a power series (the Taylor series expansion). Hence by Theorem 3.3.2 it follows that if f is differentiable once then it is differentiable arbitrarily many times.

Theorem 5.2.1 (Taylor's Theorem)
Suppose that f is holomorphic in the domain D. Then all of the higher derivatives of f exist in D and, on any disc

$$\{z \in \mathbb{C} \mid |z - z_0| < R\} \subset D,$$

f has a Taylor series expansion given by

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n.$$

Furthermore, if $0 < r < R$ and $C_r(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(z)}{(z - z_0)^{n+1}}\, dz.$$

Remark. This version of Taylor's Theorem is false in the case of real analysis in the following sense: there are functions that are differentiable an arbitrary number of times but that are not equal to their Taylor series. For example, if

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

then f is differentiable arbitrarily many times. However, one can check (by differentiation from first principles) that $f^{(n)}(0) = 0$ for all n, so f has Taylor series 0 at 0. As $f \neq 0$ near 0, it follows that f is not equal to its Taylor series.

Definition. If, for each $z_0 \in D$, a function $f : D \to \mathbb{C}$ is equal to its Taylor series at z_0 on some open disc then we say that f is *analytic*. (It follows from Theorem 5.2.1 that all complex differentiable functions are analytic; however the example in the remark above shows that not all infinitely real-differentiable functions are analytic.)

Proof of Theorem 5.2.1. First recall that for any $w \in \mathbb{C}$ we have

$$1 + w + \cdots + w^m = \frac{1 - w^{m+1}}{1 - w}.$$

Put $w = h/(z - z_0)$. Then

$$\begin{aligned} 1 + \frac{h}{z - z_0} + \cdots + \frac{h^m}{(z - z_0)^m} &= \frac{1 - \left(\dfrac{h}{z - z_0}\right)^{m+1}}{1 - \dfrac{h}{z - z_0}} \\ &= \frac{\left(1 - \left(\dfrac{h}{z - z_0}\right)^{m+1}\right)}{z - z_0 - h} \times (z - z_0). \end{aligned}$$

Hence

$$\frac{1}{z - (z_0 + h)} = \frac{1}{z - z_0 - h} = \frac{1}{z - z_0} + \frac{h}{(z - z_0)^2} + \cdots + \frac{h^m}{(z - z_0)^{m+1}} + \frac{h^{m+1}}{(z - z_0)^{m+1}(z - z_0 - h)}.$$

Fix h such that $0 < |h| < R$ and suppose, for the moment, that $|h| < r < R$. Then Cauchy's Integral formula, together with the above identity, gives

$$\begin{aligned} &f(z_0 + h) \\ &\quad = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z - (z_0 + h)}\, dz \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_{C_r} f(z)\left(\frac{1}{z-z_0}+\frac{h}{(z-z_0)^2}+\cdots+\frac{h^m}{(z-z_0)^{m+1}}\right. \\
&\qquad \left. + \frac{h^{m+1}}{(z-z_0)^{m+1}(z-z_0-h)}\right) dz \\
&= \sum_{n=0}^{m} a_n h^n + A_m.
\end{aligned}
$$

where

$$a_n = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

and

$$A_m = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)h^{m+1}}{(z-z_0)^{m+1}(z-z_0-h)}\,dz.$$

We show that $A_m \to 0$ as $m \to \infty$.

As f is differentiable on C_r, it is bounded. So there exists $M > 0$ such that $|f(z)| \leq M$ for all z on C_r.

By the reverse triangle inequality, using the facts that $|h| < r = |z - z_0|$ for z on C_r, we have that

$$|z - z_0 - h| \geq ||z - z_0| - |h|| = r - |h|.$$

Hence, by the Estimation Lemma (Lemma 4.4.2)

$$|A_m| \leq \frac{1}{2\pi}\frac{M|h|^{m+1}}{r^{m+1}(r-|h|)}2\pi r = \frac{M|h|}{r-|h|}\left(\frac{|h|}{r}\right)^m.$$

Since $|h| < r$, this tends to zero as $m \to \infty$. Hence

$$f(z_0+h) = \sum_{n=0}^{\infty} a_n h^n$$

for $|h| < R$ with

$$a_n = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

provided that r satisfies $|h| < r < R$. However, the integral is unchanged if we vary r in the whole range $0 < r < R$. Hence this formula is valid for the whole of this range of r.

Finally, we put $h = z - z_0$. Then we have that

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

for $|z - z_0| < R$, with a_n given as above. From Theorem 3.3.2 we know that a power series can be differentiated term-by-term as many times as we please and that

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

□

One immediate consequence of Taylor's Theorem is that the coefficients of the Taylor series expansion for f are uniquely defined. More precisely, we have the following result.

Proposition 5.2.2
Suppose that

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = \sum_{n=0}^{\infty} b_n(z-z_0)^n \tag{5.2.1}$$

for all $z \in \mathbb{C}$ such that $|z-z_0| < R$. Then $a_n = b_n$ for all $n \in \mathbb{N}$.

Proof. Subtracting the right-hand side from the left-hand side in (5.2.1), it is sufficient to prove that if

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n = 0 \tag{5.2.2}$$

for all $z \in \mathbb{C}$ with $|z-z_0| < R$ then $a_n = 0$. Differentiate both sides of (5.2.2) k times to see that, for each $k \in \mathbb{N}$, $\sum_{n=k}^{\infty} n(n-1)\cdots(n-(k-1))(z-z_0)^{n-k} = 0$. Putting $z = z_0$ into this expression for each k then gives that $a_k = 0$. □

Often, if we are given a function f and want to calculate the Taylor series of f on some disc, then we may be able to use Proposition5.2.2 to obtain the Taylor series of f without using the formula for the coefficients a_n given in Theorem 5.2.1.

Example. We can find the Taylor series for $f(z) = \sin^2 z$ as follows.

Recall that $\cos z = \sum_{n=0}^{\infty}(-1)^n z^{2n}/(2n!)$ and that this converges for all $z \in \mathbb{C}$. Also recall that $\sin^2 z = (1-\cos 2z)/2$. Hence

$$\begin{aligned}
\sin^2 z &= \frac{1-\cos 2z}{2} \\
&= \frac{1}{2}\left(1 - \sum_{n=0}^{\infty}(-1)^n \frac{(2z)^{2n}}{(2n!)}\right) \\
&= \frac{2z^2}{2!} - \frac{2^3 z^4}{4!} + \cdots + \frac{(-1)^{n+1}2^{n-1}}{(2n)!} z^{2n} + \cdots.
\end{aligned}$$

As this is a power series that is equal to $f(z)$ and is valid for all $z \in \mathbb{C}$, by Proposition 5.2.2 this must be the Taylor series of f on $\mathbb{C}$.

§5.3 Applications of Cauchy's Integral Formula

Cauchy's Integral Formula has many applications; here we give just three.

§5.3.1 Cauchy's Estimate

As a consequence of the formula for the nth derivative of f in terms of an integral given in Taylor's Theorem, we have the following estimate.

Lemma 5.3.1 (Cauchy's Estimate)
Suppose that f is holomorphic on $\{z \in \mathbb{C} \mid |z-z_0| < R\}$. If $0 < r < R$ and $|f(z)| \leq M$ for all z such that $|z-z_0| = r$ then, for all $n \geq 0$,

$$|f^{(n)}(z_0)| \leq \frac{Mn!}{r^n}.$$

Proof. By Theorem 5.2.1 we know that

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz.$$

By the Estimation Lemma (Lemma 4.4.2),

$$\begin{aligned} |f^{(n)}(z_0)| &= \frac{n!}{2\pi} \left| \int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\, dz \right| \\ &\leq \frac{n!}{2\pi} \frac{M}{r^{n+1}} 2\pi r \\ &= \frac{Mn!}{r^n}. \end{aligned}$$

□

§5.3.2 Liouville's Theorem

Theorem 5.3.2 (Liouville's Theorem)
Suppose that f is holomorphic and bounded on the whole of $\mathbb{C}$. Then f is a constant.

Remark. By bounded we mean that there exists $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$.

Remark. This theorem has no analogue in real analysis. It is easy to think of functions $f : \mathbb{R} \to \mathbb{R}$ that are differentiable and bounded but not constant. (For example $f(x) = \sin x$.)

Proof. Choose M such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Let $z_0 \in \mathbb{C}$. Since f is holomorphic on the whole of $\mathbb{C}$, it is holomorphic in the disc $\{z \in \mathbb{C} \mid |z - z_0| < R\}$ of radius R centred at z_0 for R as large as we please. By Cauchy's Estimate (Lemma 5.3.1), we have for $0 < r < R$

$$|f'(z_0)| \leq \frac{M}{r}.$$

Since we can choose R as large as we please, so we can choose r as large as we please. Hence we can let $r \to \infty$. Hence $f'(z_0) = 0$ for every $z_0 \in \mathbb{C}$. Hence f is a constant. □

§5.3.3 The Fundamental Theorem of Algebra

Consider the equation $x - n = 0$ where $n \in \mathbb{N}$. This equation always has solutions $x \in \mathbb{N}$ (indeed, $x = n$). If, however, we consider $x + n = 0$, $n \in \mathbb{N}$, then we need to introduce negative integers to be able to solve this equation. More generally, consider the equation $px - q = 0$ where $p, q \in \mathbb{Z}$; then we need to introduce rational numbers $\mathbb{Q}$ to be able to solve this equation. Continuing this theme, one can see that one needs to introduce surds (to solve $x^2 - 2 = 0$) and complex numbers (to solve $x^2 + 1 = 0$). Let us ask the ultimate question along these lines: if we have a polynomial equation where the coefficients are complex numbers, do we need to invent a larger class of numbers to be able to solve this equation or will complex numbers suffice? The answer is that complex numbers are sufficient.

Theorem 5.3.3 (The Fundamental Theorem of Algebra)
Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$ with coefficients $a_j \in \mathbb{C}$. Then there exists $\alpha \in \mathbb{C}$ such that $p(\alpha) = 0$.

Corollary 5.3.4
Let $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ be a polynomial of degree $n \geq 1$ with coefficients $a_j \in \mathbb{C}$. Then we can factorise $p(z)$: there exist $\alpha_j \in \mathbb{C}$, $1 \leq j \leq n$ such that

$$p(z) = \prod_{j=1}^{n} (z - \alpha_j).$$

Proof of Theorem 5.3.3. Suppose for a contradiction that there are no solutions to $p(z) = 0$, i.e. suppose that $p(z) \neq 0$ for all $z \in \mathbb{C}$.

If $p(z) \neq 0$ for all $z \in \mathbb{C}$ then $1/p(z)$ is holomorphic for all $z \in \mathbb{C}$. We shall show that $1/p(z)$ is bounded and then use Liouville's theorem to show that p is constant.

For $z \neq 0$

$$\frac{p(z)}{z^n} = 1 + \frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n} \to 1$$

as $|z| \to \infty$. Hence there exists $K > 0$ such that if $|z| > K$ then

$$\left| \frac{p(z)}{z^n} \right| \geq \frac{1}{2}.$$

Re-arranging this implies that for $|z| > K$ we have that

$$\left| \frac{1}{p(z)} \right| \leq \frac{2}{|z^n|} \leq \frac{2}{K^n}.$$

Hence $1/p(z)$ is bounded if $|z| > K$.

We shall show that this bound continues to hold for $|z| \leq K$. Let $z_0 \in \mathbb{C}$, $|z_0| \leq K$. Let $C_r(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$, denote the circular path with centre z_0 and radius r. By choosing r sufficiently large, we can assume that C_r is contained in $\{z \in \mathbb{C} \mid |z| > K\}$. Hence, for such an r, if z is any point on C_r then $|z| > K$. Hence if z is any point on C_r then $|1/p(z)| \leq 2/K^n$. By Cauchy's Estimate (Lemma 5.3.1) it follows that

$$\left| \frac{1}{p(z_0)} \right| \leq \frac{2}{K^n}.$$

Hence $|1/p(z)| \leq 2/K^n$ for all $z \in \mathbb{C}$, so that p is a bounded holomorphic function on $\mathbb{C}$. By Liouville's Theorem (Theorem 5.3.2), this implies that p is constant, a contradiction. □

Proof of Corollary 5.3.4. Let $p(z)$ be a degree n polynomial with coefficients in $\mathbb{C}$. By Theorem 5.3.3 we can find $\alpha_1 \in \mathbb{C}$ such that $p(\alpha_1) = 0$. Write $p(z) = (z - \alpha_1)q(z)$ where $q(z)$ is a degree $n-1$ polynomial with coefficients in $\mathbb{C}$. The proof then follows by induction on n. □

Exercises for Part 5

Exercise 5.1
Find the Taylor expansion of the following functions around 0 and find the radius of convergence:

$$\text{(i) } \sin^2 z, \ \text{(ii) } (2z+1)^{-1}, \ \text{(iii) } f(z) = e^{z^2}.$$

Exercise 5.2
Calculate the Taylor series expansion of $\mathrm{Log}(1+z)$ around 0. What is the radius of convergence?

Exercise 5.3
Show that every polynomial p of degree at least 1 is surjective (that is, for all $a \in \mathbb{C}$ there exists $z \in \mathbb{C}$ such that $p(z) = a$).

Exercise 5.4
Suppose that f is holomorphic on the whole of $\mathbb{C}$ and suppose that $|f(z)| \leq K|z|^k$ for some real constant $K > 0$ and some positive integer $k \geq 0$. Prove that f is a polynomial function of degree at most k.

(Hint: Calculate the coefficients of z^n, $n \geq k$, in the Taylor expansion of f around 0.)

Exercise 5.5
(Sometimes one can use Cauchy's Integral formula even in the case when f is not holomorphic.)

Let $f(z) = |z+1|^2$. Let $\gamma(t) = e^{it}$, $0 \leq t \leq 2\pi$ be the path that describes the unit circle with centre 0 anticlockwise.

(i) Show that f is not holomorphic on any domain that contains γ. (Hint: use the Cauchy-Riemann Theorem.)

(ii) Find a function g that is holomorphic on some domain that contains γ and such that $f(z) = g(z)$ at all points on the unit circle γ. (It follows that $\int_\gamma f = \int_\gamma g$.) (Hint: recall that if $w \in \mathbb{C}$ then $|w|^2 = w\bar{w}$.)

(iii) Use Cauchy's Integral formula to show that

$$\int_\gamma |z+1|^2 \, dz = 2\pi i.$$

6. Laurent series and singularities

§6.1 Introduction

We have already seen that a holomorphic function f can be expressed as a Taylor series: i.e. if f is differentiable on a domain D and $z_0 \in D$ then we can write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{6.1.1}$$

for suitable coefficients a_n, and this expression is valid for z such that $|z - z_0| < R$, for some $R > 0$. The idea of Laurent series is to generalise (6.1.1) to allow negative powers of $(z - z_0)$. This turns out to be a remarkably useful tool.

§6.2 Laurent series

Definition. A Laurent series is a series of the form

$$\sum_{n=-\infty}^{\infty} a_n (z - z_0)^n. \tag{6.2.1}$$

As (6.2.1) is a doubly infinite sum, we need to take care as to what it means. We define (6.2.1) to mean

$$\sum_{n=1}^{\infty} a_{-n}(z - z_0)^{-n} + \sum_{n=0}^{\infty} a_n (z - z_0)^n = \Sigma^- + \Sigma^+.$$

The first question to address is when does (6.2.1) converge? For this, we need both Σ^- and Σ^+ to converge.

Now Σ^+ converges for $|z - z_0| < R_2$ for some $R_2 \geq 0$, where R_2 is the radius of convergence of Σ^+.

We can recognise Σ^- as a power series in $(z - z_0)^{-1}$. This has a radius of convergence equal to, say, $R_1^{-1} \geq 0$. That is, Σ^- converges when $|(z - z_0)^{-1}| < R_1^{-1}$. In other words, Σ^- converges when $|z - z_0| > R_1$.

Combining these, we see that if $0 \leq R_1 < R_2 \leq \infty$ then (6.2.1) converges on the annulus

$$\{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}.$$

See Figure 6.2.1.

The following theorem says that if we have a function f that is holomorphic on an annulus then it can be expressed as a Laurent series. (Compare this with Taylor's Theorem: if f is holomorphic on a disc then it can be expressed as a Taylor series.) Moreover, we can obtain an expression for the coefficients a_n in terms of the function f.

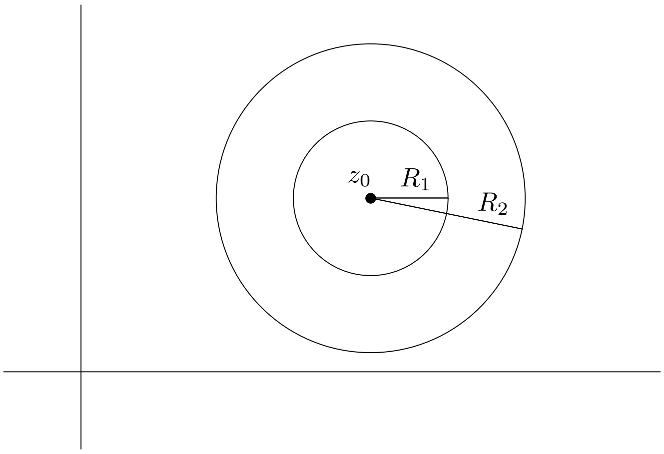

Figure 6.2.1: An annulus in $\mathbb{C}$ with centre z_0 and radii $R_1 < R_2$.

Theorem 6.2.1 (Laurent's theorem)
Suppose that f is holomorphic on the annulus $\{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$, where $0 \leq R_1 < R_2 \leq \infty$. Then we can write f as a Laurent series: on $\{z \in \mathbb{C} \mid R_1 < |z-z_0| < R_2\}$ we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} a_{-n} (z - z_0)^{-n} \tag{6.2.2}$$

Moreover, let $R_1 < r < R_2$ and let $C_r(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$ be the circular path around z_0 of radius r. Then

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z - z_0)^{n+1}} \, dz \tag{6.2.3}$$

for $n \in \mathbb{Z}$.

Remark. Note that in this case we cannot conclude that $a_n = f^{(n)}(z_0)/n!$ as we do not know that f is differentiable at z_0 (indeed, it may not even be defined at z_0).

Remark. The proof is similar to the proof of Taylor's Theorem and can be found in Stewart and Tall's book (Theorem 11.1).

We call the series (6.2.2) the *Laurent series of $f(z)$ about z_0* or the *Laurent expansion of $f(z)$*.

We call

$$\sum_{n=-\infty}^{-1} a_n (z - z_0)^n$$

the *principal part* of the Laurent series. Thus the principal part of a Laurent series is the part that contains all the negative powers of $(z - z_0)$.

§6.2.1 Calculating Laurent series

The following result tells us that the coefficients in the Laurent series expansion are uniquely determined (compare with the analogous result for Taylor series in Proposition 5.2.2).

Proposition 6.2.2
Suppose that

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = \sum_{n=\infty}^{\infty} b_n(z-z_0)^n \tag{6.2.4}$$

for all $z \in \mathbb{C}$ *such that* $R_1 < |z-z_0| < R_2$. *Then* $a_n = b_n$ *for all* $n \in \mathbb{Z}$.

Proof. Subtracting one side of (6.2.4) from the other tells us that it is sufficient to prove that if

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n = 0 \tag{6.2.5}$$

for all $z \in \mathbb{C}$ such that $R_1 < |z-z_0| < R_2$ then $a_n = 0$ for all $n \in \mathbb{Z}$. Let $f(z) = \sum_{n=-\infty}^{\infty} a_n(z-z_0)^n$. Then f is identically equal to 0. By Laurent's Theorem we have that

$$a_n = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

where C_r is a circular path with centre z_0 and radius r (with $R_1 < r < R_2$), described once anticlockwise. As f is identically equal to zero, the integrand in this expression is identically equal to 0. Hence $a_n = 0$. □

Remark. Suppose that $f : D \to \mathbb{C}$ is a holomorphic function. Suppose that the two (differenet) annuli $\{z \in \mathbb{C} \mid R_1 < |z-z_0| < R_2\}, \{z \in \mathbb{C} \mid R_1' < |z-z_0'| < R_2'\} \subset D$. Then by Laurent's Theorem, we can expand f on these two annuli as two Laurent series:

$$\sum_{n=-\infty}^{\infty} a_n(z-z_0)^n, \quad \sum_{n=-\infty}^{\infty} a_n'(z-z_0')^n,$$

respectively. As these two annuli are different, there is no reason why the coefficients a_n, a_n' should be the same. Indeed, we shall see some specific examples of this below.

Given a specific function f that is holomorphic on an annulus, we want to be calculate the Laurent series of f; that is, we want to be able to calculate the coefficients a_n. If we were to appeal directly to Theorem 6.2.1 we would have to evaluate the integral in (6.2.3). In general, this is difficult or time-consuming (Exercise 7.12 in the next section leads you through one example of this). Instead, we can appeal to Proposition 6.2.4: given a function f that is holomorphic on an annulus $R_1 < |z-z_0| < R_2$, if we can find an expression of the form (6.2.1) that si equal to f on this annulus then it must be the Laurent series.

Example. Let $f(z) = e^z + e^{1/z}$. Recall that $e^z = \sum_{n=0}^{\infty} z^n/n!$ for all $z \in \mathbb{C}$. Hence $e^{1/z} = \sum_{n=0}^{\infty} z^{-n}/n!$ for all $z \neq 0$. Hence

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n = \cdots + \frac{1}{n!z^n} + \cdots + \frac{1}{2!z^2} + \frac{1}{z} + 2 + z + \frac{z^2}{2} + \cdots + \frac{z^n}{n!} + \cdots.$$

where

$$a_n = \frac{1}{n!} \text{ for } n \geq 1,\ a_0 = 2,\ a_{-n} = \frac{1}{n!} \text{ for } n \geq 1.$$

This expansion if valid for all $z \neq 0$, i.e. $R_1 = 0, R_2 = \infty$.

Example. Let
$$f(z) = \frac{1}{z} + \frac{1}{1-z}$$
and let us calculate the Laurent series at $z_0 = 0$.

Now $1/z$ is already a Laurent series at 0 (the only non-zero coefficient is $a_{-1} = 1$). Note that this converges if $z \neq 0$ (in this case, as there is only one term, checking convergence just means checking when this formula makes sense!).

Now, by summing a geometric progression, we have that $1/(1-z) = \sum_{n=0}^{\infty} z^n$ and this power series converges for $|z| < 1$.

Hence $f(z)$ has Laurent series
$$f(z) = \frac{1}{z} + 1 + z + z^2 + z^3 + \cdots = \sum_{n=-1}^{\infty} z^n$$
and this expression is valid on the annulus $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$.

Example. Let
$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$
We will expand f as three different Laurent series about $z_0 = 0$, valid in three different annuli.

First note that we can write
$$\frac{1}{z-1} = \frac{-1}{1-z} = -\sum_{n=0}^{\infty} z^n \tag{6.2.6}$$
(summing a geometric progression) and that this is valid for $|z| < 1$. We can also write
$$\frac{1}{z-1} = \frac{1}{z}\frac{1}{\left(1-\frac{1}{z}\right)} = \frac{1}{z}\sum_{n=0}^{\infty} z^{-n} = \sum_{n=1}^{\infty} \frac{1}{z^n}$$
by again noting that $\sum_{n=0}^{\infty} z^{-n} = 1/(1-z^{-1})$ is the sum to infinity of a geometric progression with common ratio z^{-1}. This converges for $|z^{-1}| < 1$, i.e. $|z| > 1$. Hence
$$\frac{1}{z-1} = \sum_{n=1}^{\infty} \frac{1}{z^n} \tag{6.2.7}$$
and this is valid for $|z| > 1$.

Similarly, we can write
$$\frac{1}{z-2} = \frac{-1}{2-z} = -\frac{1}{2}\left(\frac{1}{1-\frac{z}{2}}\right) = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \tag{6.2.8}$$
by noting that $\sum_{n=0}^{\infty}(z/2)^n = 1/(1-z/2)$ is the sum of a geometric progression with common ratio $z/2$. This expansion is valid when $|z/2| < 1$, i.e. when $|z| < 2$.

We can also write
$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{\left(1-\frac{2}{z}\right)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^{-n} = \sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n} \tag{6.2.9}$$

by recognising the middle term as the sum of a geometric progression with common ratio $(z/2)^{-1}$. This converges when $|(2/z)^{-1}| < 1$, i.e. when $|z| > 2$.

Using (6.2.6) and (6.2.8) we see that we can expand

$$f(z) = -\sum_{n=0}^{\infty} z^n + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n = \sum_{n=0}^{\infty}\left(-1+\frac{1}{2^{n+1}}\right)z^n$$

and this is valid on the annulus $\{z \in \mathbb{C} \mid 0 \leq |z| < 1\}$.

Using (6.2.7) and (6.2.8) we can expand

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty}\frac{1}{z^n} + \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \\ &= \cdots + \frac{1}{z^n} + \cdots + \frac{1}{z} + \frac{1}{2} + \frac{z}{2^2} + \cdots + \frac{z^n}{2^{n+1}} + \cdots \end{aligned}$$

and this is valid on the annulus $\{z \in \mathbb{C} \mid 1 < |z| < 2\}$.

Using (6.2.7) and (6.2.9) we can expand

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty}\frac{1}{z^n} - \sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n} \\ &= \sum_{n=1}^{\infty}\frac{1-2^{n-1}}{z^n} \end{aligned}$$

and this is valid on the annulus $\{z \in \mathbb{C} \mid 2 < |z| < \infty\}$.

In the above examples we have expanded functions as Laurent series on annuli centred at the origin. If we want to expand a function $f(z)$ as a Laurent series on an annulus centred at z_0 then it is often convenient to first change co-ordinates to $w = z - z_0$, calculate the Laurent series in terms of w, and then change co-ordinates back to z

Example. Let

$$f(z) = \frac{e^z}{(z-1)^2}.$$

We will expand f as a Laurent series on the annulus $\{z \in \mathbb{C} \mid 0 < |z-1| < \infty\}$.

We first change co-ordinates and let $w = z - 1$. Then $z = 1 + w$ and we are interested in expanding

$$\frac{e^{1+w}}{w^2}.$$

Now $e^{1+w} = ee^w = e\sum_{n=0}^{\infty} w^n/n!$. Hence

$$\frac{e^{1+w}}{w^2} = \frac{e}{w^2}\sum_{n=0}^{\infty}\frac{w^n}{n!} = \frac{e}{w^2} + \frac{e}{w} + \frac{e}{2!} + \frac{e}{3!}w + \frac{e}{4!}w^2 + \cdots + \frac{e}{n!}w^{n-2} + \cdots,$$

and this is valid provided that $w \neq 0$. Changing co-ordinates back to z we obtain

$$f(z) = \frac{e}{(z-1)^2} + \frac{e}{z-1} + \frac{e}{2!} + \frac{e}{3!}(z-1) + \frac{e}{4!}(z-1)^2 + \cdots + \frac{e}{n!}(z-1)^{n-2} + \cdots,$$

valid for $z \neq 1$, i.e. on $\{z \in \mathbb{C} \mid 0 < |z-1|\}$.

§6.3 Singularities

Definition. A *singularity* of a function $f(z)$ is a point z_0 at which $f(z)$ is not differentiable.

Remark. Here is a common way for a singularity to occur: if f is not defined at z_0 then it cannot be differentiable at z_0.

Example. If $f(z) = 1/z$ then f is not defined at the origin (we are not allowed to divide by 0). Hence f has a singularity at $z = 0$.

Suppose that f has a singularity at z_0.

Definition. If there exists a punctured disc $0 < |z - z_0| < R$ such that f is differentiable on this punctured disc then we say that z_0 is an *isolated singularity of* f.

Example. In the above example, 0 is an isolated singularity of $f(z) = 1/z$.

In this course we will only be interested in isolated singularities. Suppose that f has an isolated singularity at z_0. Then f is holomorphic on an annulus of the form $\{z \in \mathbb{C} \mid 0 < |z - z_0| < R\}$. We expand f as a Laurent series around z_0 on this annulus to obtain

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n},$$

and this is valid for $0 < |z - z_0| < R$. Consider the principal part of the Laurent series

$$\sum_{n=1}^{\infty} b_n (z - z_0)^{-n}. \tag{6.3.1}$$

There are three possibilities: the principal part of f may have

(i) no terms,

(ii) a finite number of terms,

(iii) an infinite number of terms.

§6.3.1 Removable singularities

Suppose that f has an isolated singularity at z_0 and that the principal part of the Laurent series (6.3.1) has no terms in it. In this case, for $0 < |z - z_0| < R$ we have that

$$f(z) = a_0 + a_1(z - z_0) + \cdots + a_n(z - z_0)^n + \cdots.$$

The radius of convergence of this power series is at least R, and so $f(z)$ extends to a function that is differentiable at z_0.

Example. Let

$$f(z) = \frac{\sin z}{z}, z \neq 0.$$

Then f has an isolated singularity at 0 as $f(z)$ is not defined at $z = 0$. However, we know that

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$$

for $z \neq 0$. Define $f(0) = 1$. Then $f(z)$ is differentiable for all $z \in \mathbb{C}$. Hence f has a removable singularity at $z = 0$.

§6.3.2 Poles

Suppose that f has an isolated singularity at z_0 and that the principal part of the Laurent series (6.3.1) has finitely many terms in it. In this case, for $0 < |z - z_0| < R$ we can write

$$f(z) = \frac{b_m}{(z-z_0)^m} + \cdots + \frac{b_1}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

where $b_m \neq 0$. In this case, we say that f has a *pole of order m at z_0*. A pole of order 1 is called a *simple pole*.

Example. Let

$$f(z) = \frac{\sin z}{z^4}, \ z \neq 0.$$

Then f has an isolated singularity at $z = 0$. We can write

$$\frac{\sin z}{z^4} = \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z} + \frac{1}{5!}z - \frac{1}{7!}z^3 + \cdots.$$

Hence f has a pole of order 3 at $z = 0$.

We will often consider functions $f : D \to \mathbb{C}$ defined on a domain D that are differentiable except at finitely many points in D and f has either removable singularities or poles at these points.

Definition. Let D be a domain. A function $f : D \to \mathbb{C}$ is said to be *meromorphic* if f is differentiable on D except at finitely many points, and these points are either removable singularities or poles.

§6.3.3 Isolated essential singularities

Suppose that f has an isolated singularity at z_0 and that the principal part of the Laurent series (6.3.1) has infinitely many terms in it. In this case we say that f has an *isolated essential singularity*.

Isolated essential singularities are difficult to deal with and we will not consider them in this course.

Example. Let $f(z) = \sin 1/z$, $z \neq 0$. Then f has a singularity at $z = 0$ and

$$\sin\left(\frac{1}{z}\right) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \cdots.$$

Hence f has an isolated essential singularity at $z = 0$.

Exercises for Part 6

Exercise 6.1
Find the Laurent expansions of the following around $z = 0$:

(i) $(z-3)^{-1}$, valid for $3 < |z| < \infty$;

(ii) $1/(z(1-z))$, valid for $0 < |z| < 1$;

(iii) $z^3 e^{1/z}$, valid for $0 < |z| < \infty$;

(iv) $\cos(1/z)$, valid for $0 < |z| < \infty$.

Exercise 6.2
Find Laurent expansions for the function

$$f(z) = \frac{1}{z+1} + \frac{1}{z-3}.$$

valid on the annuli

$$\text{(i) } 0 \leq |z| < 1, \text{ (ii) } 1 < |z| < 3, \text{ (iii) } 3 < |z| < \infty.$$

Exercise 6.3
(i) Find a Laurent series expansion for

$$f(z) = \frac{1}{z^2(z-1)}$$

valid for $0 < |z| < 1$.

(ii) Find a Laurent series expansion for

$$f(z) = \frac{1}{z^2(z-1)}$$

valid for $0 < |z-1| < 1$.

(Hint: introduce $w = z - 1$ and recall that $1/(1-w)^2 = \sum_{n=1}^{\infty} nw^{n-1}$, provided that $|w| < 1$.)

Exercise 6.4
Let $f(z) = (z-1)^{-2}$. Find Laurent series for f valid on the following annuli:

(i) $\{z \in \mathbb{C} \mid 0 < |z-1| < \infty\}$,

(ii) $\{z \in \mathbb{C} \mid 0 \leq |z| < 1\}$,

(iii) $\{z \in \mathbb{C} \mid 1 < |z| < \infty\}$.

Exercise 6.5
Find the poles and their orders of the functions

$$\text{(i)}\ \frac{1}{z^2+1},\ \text{(ii)}\ \frac{1}{z^4+16},\ \text{(iii)}\ \frac{1}{z^4+2z^2+1},\ \text{(iv)}\ \frac{1}{z^2+z-1}.$$

Exercise 6.6
Describe the type of singularity at 0 of each of the following functions:

$$\text{(i)}\ \sin(1/z),\ \text{(ii)}\ z^{-3}\sin^2 z,\ \text{(iii)}\ \frac{\cos z - 1}{z^2}.$$

Exercise 6.7
Let D be a domain and let $z_0 \in D$. Suppose that f is holomorphic on $D \setminus \{z_0\}$ and is bounded on $D \setminus \{z_0\}$ (that is, there exists $M > 0$ such that $|f(z)| \leq M$ for all $z \in D \setminus \{z_0\}$). Show that f has a removable singularity at z_0.

7. Cauchy's Residue Theorem

§7.1 Introduction

One of the more remarkable applications of integration in the complex plane in general, and Cauchy's Theorem in particular, is that it gives a method for calculating real integrals that, up until now, would have been difficult or even impossible (assuming that you only had the tools of 1st year calculus or A-level mathematics to hand). As another application: you may remember from Real Analysis or Sequences and Series that you studied whether an infinite series $\sum_{n=0}^{\infty} a_n$ converged or not. However, in only very few examples were you able to say what the limit actually is! Using complex analysis, it becomes very easy to evaluate infinite series such as $\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90$.

§7.2 Zeros and poles of holomorphic functions

Recall that a function f has a singularity at z_0 if f is not differentiable at z_0. We will only consider the case when f has poles as singularities. In the examples we have seen so far $f(z)$ has a pole at z_0 because we have been able to write $f(z) = p(z)/q(z)$ and $q(z_0) = 0$ (so that f is not even defined at z_0). Thus it makes sense to first study zeros of functions.

Definition. A function f defined on a domain D has a *zero* at $z_0 \in D$ if $f(z_0) = 0$.

We will only be interested in isolated zeros. Intuitively, a function f has an isolated zero at z_0 if there are no other zeros nearby. More formally, we have the following definition.

Definition. A function f defined on a domain D has an *isolated zero* at z_0 if $f(z_0) = 0$ and there exists $\varepsilon > 0$ such that $f(z) \neq 0$ for all z such that $0 < |z - z_0| < \varepsilon$.

Let $f : D \to \mathbb{C}$ be holomorphic and suppose that f has an isolated zero at z_0. By Taylor's Theorem (Theorem 5.2.1), we can expand f as a Taylor series in some neighbourhood around z_0. That is we can wrote

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \tag{7.2.1}$$

for all z in some disc that contains z_0.

Definition. We say that f has a *zero of order m at z_0* if $a_0 = a_1 = \cdots = a_{m-1} = 0$ but $a_m \neq 0$. We say that z_0 is a *simple zero* if it is a zero of order 1.

Example. (i) Let $f(z) = z^2$. Then f has a zero of order 2 at 0.

(ii) Let $f(z) = z(z + 2i)^3$. Then f has a zero of order 1 at 0 and a zero of order 3 at $-2i$.

(iii) Let $f(z) = z^2 + 4$. Then, noting that $z^2 + 4 = (z - 2i)(z + 2i)$, we see that f has simple zeros at $\pm 2i$.

Remark. The coefficients a_n in the Taylor expansion are given by $a_n = f^{(n)}(z_0)/n!$. Thus f has a zero of order m at z_0 if and only if $f^{(k)}(z_0) = 0$ for $0 \leq k \leq m-1$ but $f^{(m)}(z_0) \neq 0$. In partiocular, if $f(z_0) = 0$ but $f'(z_0) \neq 0$ then z_0 is a simple zero.

Example. (i) Let $f(z) = \sin z$. Then $f(z)$ has zeros at $k\pi$, $k \in \mathbb{Z}$. Note that $f'(k\pi) = \cos k\pi = (-1)^k \neq 0$. Hence all the zeros are simple zeros.

(ii) Let $f(z) = 1 - \cos z$. Then $f(z)$ has zeros at $2k\pi$, $k \in \mathbb{Z}$. Now $f'(z) = \sin z$ and $f'(2k\pi) = 0$, but $f''(2k\pi) = \cos 2k\pi = 1 \neq 0$. Hence all the zeros have order 2.

Lemma 7.2.1
Suppose that the holomorphic function f has a zero of order m at z_0. Then, on some disc centred at z_0, we can write

$$f(z) = (z - z_0)^m g(z)$$

where g is a holomorphic function on an open disc centred on z_0 and $g(z_0) \neq 0$.

Proof. By (7.2.1) we can write

$$f(z) = a_m(z - z_0)^m + a_{m+1}(z - z_0)^{m+1} + \cdots = (z - z_0)^m \sum_{n=0}^{\infty} a_{n+m}(z - z_0)^n$$

where $a_m \neq 0$. Take $g(z) = \sum_{n=0}^{\infty} a_{n+m}(z - z_0)^n$. Then g is holomorphic on an open disc centred on z_0 and $g(z_0) = a_m \neq 0$. □

We can now link poles of a function $f(z) = p(z)/q(z)$ with zeros of the function q.

Lemma 7.2.2
Suppose that $f(z) = p(z)/q(z)$ where

(i) p is holomorphic and $p(z_0) \neq 0$,

(ii) q is holomorphic and q has a zero of order m at z_0.

Then f has a pole of order m at z_0.

Proof. By Lemma 7.2.1, we can write $q(z) = (z - z_0)^m r(z)$ where r is holomorphic and $r(z_0) \neq 0$. Define $g(z) = p(z)/r(z)$. Then $g(z)$ is holomorphic at z_0, and so we can expand it as a Taylor series at z_0 as

$$g(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$$

and this expression is valid in some disc $|z - z_0| < R$, for some $R > 0$. Then

$$\begin{aligned}
f(z) &= \frac{p(z)}{q(z)} \\
&= \frac{p(z)}{(z - z_0)^m r(z)} \\
&= \frac{g(z)}{(z - z_0)^m} \\
&= \frac{1}{(z - z_0)^m} \sum_{n=0}^{\infty} a_n(z - z_0)^n \\
&= \frac{a_0}{(z - z_0)^m} + \frac{a_1}{(z - z_0)^{m-1}} + \frac{a_2}{(z - z_0)^{m-2}} + \cdots.
\end{aligned}$$

However, $a_0 = g(z_0) = p(z_0)/r(z_0) \neq 0$, as $p(z_0) \neq 0$. Hence f has a pole of order m at z_0. $\square$

Example. (i) Let
$$f(z) = \frac{\sin z}{(z-3)^2}.$$
Then f has a pole of order 2 at $z = 3$. This is because $\sin z \neq 0$ when $z = 3$ and $(z-3)^2$ has a zero of order 2 at $z = 3$.

(ii) Let
$$f(z) = \frac{z+3}{\sin z}.$$
Then f has a simple pole at $k\pi$ for each $k \in \mathbb{Z}$. This is because $\sin z$ has a simple zero at $z = k\pi$ for each $k \in \mathbb{Z}$ but $z + 3 \neq 0$ when $z = k\pi$.

§7.3 Residues and Cauchy's Residue Theorem

We begin with the following important definition.

Definition. Suppose that f is holomorphic on a domain D except for an isolated singularity at $z_0 \in D$. Suppose that on $\{z \in \mathbb{C} \mid 0 < |z - z_0| < R\} \subset D$, f has Laurent expansion
$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}.$$
The *residue* of f at z_0 is defined to be
$$\mathrm{Res}(f, z_0) = b_1.$$

That is, the residue of f at the isolated singularity z_0 is the coefficient of $(z - z_0)^{-1}$ in the Laurent expansion.

Let $0 < r < R$. By Laurent's Theorem (Theorem 6.2.1) we have the alternative expression
$$\mathrm{Res}(f, z_0) = \frac{1}{2\pi i} \int_{C_r} f(z)\, dz$$
where $C_r(t) = z_0 + re^{it}$, $0 \leq t \leq 2\pi$ is a circular anticlockwise path around z_0 in the annulus of convergence. This shows that residues are related to integration.

Cauchy's Residue Theorem relies on using Cauchy's Theorem in just the right way. In particular, we have to be careful about the paths that we integrate over. We make the following definition.

Definition. A closed contour γ is said to be a *simple closed loop* if, for every point z not on γ, the winding number is either $w(\gamma, z) = 0$ or $w(\gamma, z) = 1$. If $w(\gamma, z) = 1$ then we say that z is *inside* γ.

Thus a simple closed loop is a loop that goes round anticlockwise in a loop once, and without intersecting itself; see Figure 7.3.1. In practice, we will look at simple closed loops that are made up of line segments and arcs of circles.

We can now state the main result of this section.

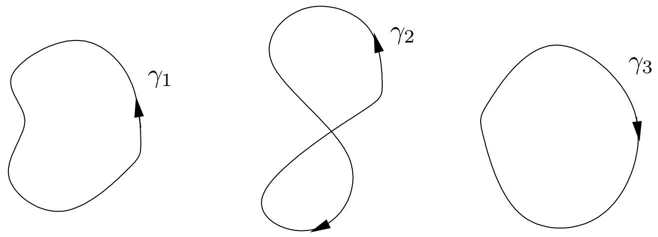

Figure 7.3.1: Here γ_1 is a simple closed loop. The closed loops γ_2 and γ_3 are not simple because there are points where the winding number is -1.

Theorem 7.3.1 (Cauchy's Residue Theorem)
Let D be a domain containing a simple closed loop γ and the points inside γ. Suppose that f is meromorphic on D with finitely many poles at $z_1, z_2, \ldots, z_n$ inside γ. Then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f, z_j).$$

Remark. A word of warning: you will have noticed that many expressions in complex analysis have a factor of $2\pi i$ in them. A very common mistake is to either miss a $2\pi i$ out, or put one in by mistake.

We shall defer the proof of Cauchy's Residue Theorem until later.

§7.4 Calculating residues

In order to use Cauchy's Residue Theorem we need to be able to easily calculate residues. In some cases, ad hoc manipulations have to be used to calculate the Laurent series, but there are many cases where one can calculate them more systematically.

First recall that if $f(z)$ has Laurent series

$$f(z) = \frac{b_m}{(z - z_0)^m} + \cdots + \frac{b_1}{(z - z_0)} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with $b_m \neq 0$ then we say that f has a pole of order m at z_0. We say that a pole of order 1 is a *simple pole*.

Remark. If we can write $f(z) = p(z)/q(z)$ where p and q are differentiable and $p(z) \neq 0$ when $q(z) = 0$ then the poles of f occur at the zeros of q. Moreover f has a pole of order m at z_0 if q has a zero of order m at z_0.

It is easy to calculate the residue at a simple pole.

Lemma 7.4.1
(i) If f has a simple pole at z_0 then

$$\operatorname{Res}(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z).$$

(ii) *If* $f(z) = p(z)/q(z)$ *where* p, q *are differentiable,* $p(z_0) \neq 0, q(z_0) = 0$ *but* $q'(z_0) \neq 0$, *then*

$$\mathrm{Res}(f, z_0) = \frac{p(z_0)}{q'(z_0)}.$$

Proof. (i) If f has a simple pole at z_0 then it has a Laurent series

$$f(z) = \frac{b_1}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

valid on some punctured disc $0 < |z - z_0| < R$. Hence

$$(z - z_0) f(z) = b_1 + \sum_{n=0}^{\infty} a_n (z - z_0)^{n+1}$$

so that $\lim_{z \to z_0} (z - z_0) f(z) = b_1$.

(ii) The hypotheses imply that f has a simple pole at z_0. By part (i) and the fact that $q(z_0) = 0$, the residue is

$$\lim_{z \to z_0} \frac{(z - z_0) p(z)}{q(z)} = \lim_{z \to z_0} \frac{p(z)}{\left(\frac{q(z) - q(z_0)}{z - z_0} \right)} = \frac{p(z_0)}{q'(z_0)}.$$

□

Example. For example, let

$$f(z) = \frac{\cos \pi z}{(1 - z^3)}.$$

This has a simple pole at $z = 1$ and satisfies the hypothesis of Lemma 7.4.1. Hence

$$\mathrm{Res}(f, 1) = \frac{\cos \pi}{(-3) \times 1^2} = \frac{1}{3}.$$

We can generalise Lemma 7.4.1 to poles of order m.

Lemma 7.4.2
Suppose that f *has a pole of order* m *at* z_0. *Then*

$$\mathrm{Res}(f, z_0) = \lim_{z \to z_0} \left(\frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m f(z) \right) \right).$$

Proof. If f has a pole of order m at z_0 then it has a Laurent series

$$f(z) = \frac{b_m}{(z - z_0)^m} + \cdots + \frac{b_1}{z - z_0} + \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

valid for $0 < |z - z_0| < R$, for some $R > 0$. Hence

$$(z - z_0)^m f(z) = b_m + (z - z_0) b_{m-1} + \cdots + (z - z_0)^{m-1} b_1 + \sum_{n=0}^{\infty} a_n (z - z_0)^{m+n}.$$

Differentiating this $m - 1$ times gives

$$\frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) = (m-1)! b_1 + \sum_{n=0}^{\infty} \frac{(m+n)!}{(n+1)!} a_n (z - z_0)^{n+1}.$$

Dividing by $(m - 1)!$ and letting $z \to z_0$ gives the result. □

Example. Let
$$f(z) = \left(\frac{z+1}{z-1}\right)^3.$$
This has a pole of order 3 at $z = 1$. To calculate the residue we note that $(z-1)^3 f(z) = (z+1)^3$. Hence
$$\frac{1}{2!}\frac{d^2}{dz^2}\left((z-1)^3 f(z)\right) = \frac{6}{2!}(z+1) \to \frac{6}{2!} \times 2 = 6$$
as $z \to 1$. Hence $\mathrm{Res}(f, 1) = 6$.

Let us check this by calculating the Laurent series. First let us change variables by writing $w = z - 1$. Then $z = w + 1$ and we can write

$$\begin{aligned}
\left(\frac{z+1}{z-1}\right)^3 &= \frac{(w+2)^3}{w^3} \\
&= \frac{w^3 + 6w^2 + 12w + 8}{w^3} \\
&= \frac{8}{w^3} + \frac{12}{w^2} + \frac{6}{w} + 1 \\
&= \frac{8}{(z-1)^3} + \frac{12}{(z-1)^2} + \frac{6}{(z-1)} + 1.
\end{aligned}$$

Hence f has a pole of order 3 at $z = 1$ and we can read off $\mathrm{Res}(f, 1) = 6$ as the coefficient of $1/(z-1)$.

In other cases, one has to manipulate the formula for f to calculate the residue.

Example. Let
$$f(z) = \frac{1}{z^2 \sin z}.$$
This has singularities whenever the denominator is zero. Hence the singularities are at $z = 0, k\pi$. We will use Laurent series to calculate the residue at $z = 0$.

Recalling the power series for $\sin z$ we can write

$$\begin{aligned}
f(z) &= \frac{1}{z^2 \sin z} \\
&= \frac{1}{z^2\left(z - \frac{z^3}{6} + \cdots\right)} \\
&= \frac{1}{z^3}\left(1 - \frac{z^2}{6} + \cdots\right)^{-1} \\
&= \frac{1}{z^3}\left(1 + \frac{z^2}{6} + \cdots\right) \\
&= \frac{1}{z^3} + \frac{1}{6z} + \cdots
\end{aligned}$$

where we have omitted higher order terms. (Note that when doing computations such as these, one can usually ignore terms that will not contribute to the coefficient of $1/z$.) Hence $\mathrm{Res}(f, 0) = 1/6$.

For the poles at $k\pi$, $k \neq 0$, we could change variables to $w = z - k\pi$ and calculate the Laurent series. Alternatively, we can use Lemma 7.4.1(ii). First note that we can write

$$f(z) = \frac{p(z)}{q(z)}$$

where $p(z) = 1$ and $q(z) = z^2 \sin z$. Now, for $k \neq 0$, $k\pi$ is a simple zero of $\sin z$ (as $\sin' k\pi = \cos k\pi \neq 0$) and so is a simple zero of $q(z)$. Hence

$$\operatorname{Res}(f, k\pi) = \frac{p(k\pi)}{q'(k\pi)} = \frac{(-1)^k}{(k\pi)^2}$$

as $q'(z) = 2z \sin z + z^2 \cos z$ so that $q'(k\pi) = (k\pi)^2 \cos k\pi = (-1)^k (k\pi)^2$.

§7.5 Applications

§7.5.1 Easy examples

We shall evaluate some simple integrals around the circular contours $C_2(t) = 2e^{it}$, $0 \leq t \leq 2\pi$ and $C_4(t) = 4e^{it}$, $0 \leq t \leq 2\pi$. Thus C_2 is the circle of radius 2 centred at 0 described anticlockwise, and C_4 is the circle of radius 4 centred at 0 described anticlockwise. Hence both C_2 and C_4 are simple closed loops.

Consider the function

$$f(z) = \frac{3}{z-1}$$

Then f has a pole at $z = 1$ and no other poles. We can read off from the definition of f that $\operatorname{Res}(f, 1) = 3$. As the pole at $z = 1$ lies inside C_2, by Cauchy's Residue Theorem we have that

$$\int_{C_2} f \, dz = 2\pi i \operatorname{Res}(f, 1) = 6\pi i.$$

Similarly, the pole at $z = 1$ lies inside C_4, hence

$$\int_{C_4} f \, dz = 2\pi i \operatorname{Res}(f, 1) = 6\pi i.$$

See Figure 7.5.1.

Now consider the function

$$f(z) = \frac{1}{z^2 + (i-3)z - 3i}.$$

Then f has a pole when the denominator has a zero. To find the poles we first factorise the denominator

$$z^2 + (i-3)z - 3i = (z-3)(z+i)$$

(to do this we could either use the quadratic formula or inspired guesswork). Thus f has simple poles $z = 3$ and $z = -i$. Using Lemma 7.4.1 we can calculate that

$$\operatorname{Res}(f, -i) = \frac{-1}{3+i}, \ \operatorname{Res}(f, 3) = \frac{1}{3+i}.$$

See Figure 7.5.2.

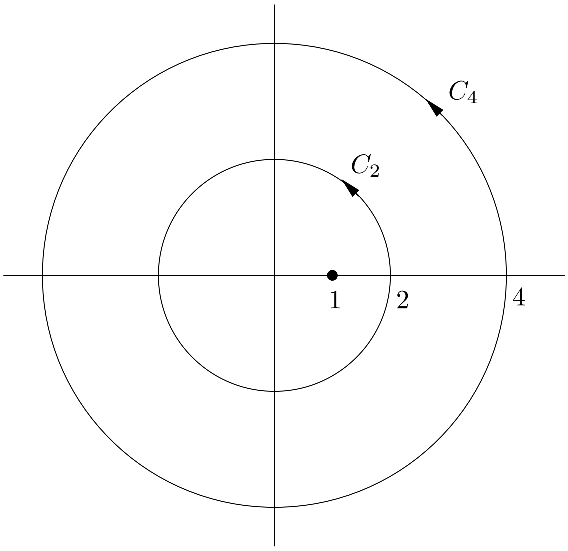

Figure 7.5.1: The function $f(z) = 3/(z-1)$ has a pole at $z = 1$ which lies inside both C_2 and C_4.

Now consider $\int_{C_2} f\,dz$. The pole $z = -i$ is inside C_2 but the pole $z = 3$ is outside. Hence

$$\begin{aligned}\int_{C_2} f\,dz &= 2\pi i\,\mathrm{Res}(f,-i) = 2\pi i\left(\frac{-1}{3+i}\right) = \frac{-2\pi i(3-i)}{10} \\ &= \frac{-2\pi - 6\pi i}{10} = \frac{-\pi}{5}(1+3i).\end{aligned}$$

Now consider $\int_{C_4} f\,dz$. In this case, both the poles at $z = -i$ and $z = 3$ lie inside C_4. Hence

$$\int_{C_4} f\,dz = 2\pi i\,(\mathrm{Res}(f,-i) + \mathrm{Res}(f,3)) = 2\pi i\left(\frac{-1}{3+i} + \frac{1}{3+i}\right) = 0.$$

§7.5.2 Infinite real integrals

In this section we shall show how to use Cauchy's Residue Theorem to calculate some infinite real integrals, i.e. integrals of the form

$$\int_{-\infty}^{\infty} f(x)\,dx \tag{7.5.1}$$

where f is a real-valued function defined on the real line.

First we need to make precise what (7.5.1) means. Formally, we say that $\int_{-\infty}^{\infty} f(x)\,dx$ exists if

$$\lim_{A,B\to\infty} \int_{-A}^{B} f(x)\,dx \tag{7.5.2}$$

converges, where the limits can be taken in either order. We then define $\int_{-\infty}^{\infty} f(x)\,dx$ to be equal to this limit.

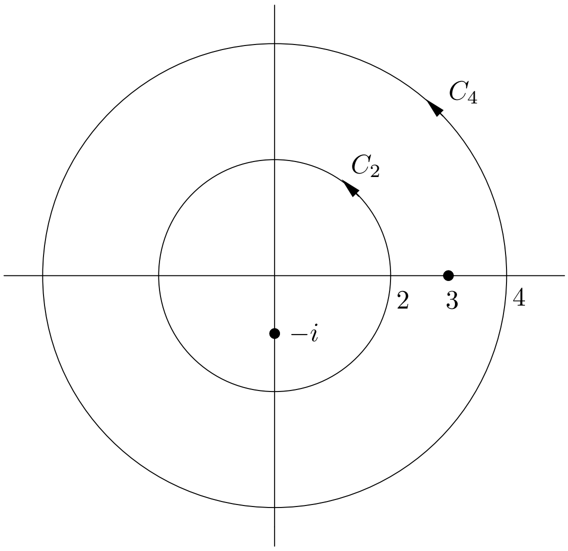

Figure 7.5.2: The function $f(z) = 1/(z^2 + (i-3)z - 3i)$ has simple poles at $z = -i$ and $z = 3$.

If $\int_{-\infty}^{\infty} f(x)\, dx$ exists then it is equal to its *principal value*, defined by

$$\mathcal{P} \int_{-\infty}^{\infty} f(x)\, dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx. \tag{7.5.3}$$

However, there are many functions f for which the principal value of the integral (7.5.3) exists but (7.5.2) does not. For example, take $f(x) = x$. Then

$$\int_{-R}^{R} f(x)\, dx = \int_{-R}^{R} x\, dx = \frac{1}{2}x^2\Big|_{x=-R}^{R} = \frac{R^2}{2} - \frac{R^2}{2} = 0$$

and so converges to 0 as $R \to \infty$. Hence $\mathcal{P}\int_{-\infty}^{\infty} x\, dx = 0$. However

$$\int_{-A}^{B} f(x)\, dx = \int_{-A}^{B} x\, dx = \frac{1}{2}x^2\Big|_{x=-A}^{B} = \frac{B^2}{2} - \frac{A^2}{2}$$

does not converge if we first let B tend to ∞ and then let A tend to ∞. Hence $\int_{-\infty}^{\infty} x\, dx$ does not exist.

The following gives a criterion for (7.5.2) to converge.

Lemma 7.5.1
Suppose that $f : \mathbb{R} \to \mathbb{C}$ is a continuous function and there exist constants $K > 0$, $C > 0$ and $r > 1$ such that for $|x| \geq K$ we have

$$|f(x)| \leq \frac{C}{|x|^r}. \tag{7.5.4}$$

Then $\int_{-\infty}^{\infty} f(x)\, dx$ exists and is equal to its principal value $\mathcal{P}\int_{-\infty}^{\infty} f(x)\, dx$.

Instead of giving a general theorem, let us consider an example that will illustrate the method. We will show how to use Cauchy's Residue Theorem to evaluate

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\,dx \tag{7.5.5}$$

(the fact that 1 and 4 are squares will make the calculations notationally easier, but this is not essential to the method).

First note that the complex contour integral $\int_{[-R,R]} f$ is equal to the real integral $\int_{-R}^{R} f(x)\,dx$. To see this, first recall from (4.2.1) that $[-R,R]$ denotes the straight line path from $-R$ to R and that this has parametrisation $\gamma(t)=t$, $-R \leq t \leq R$. Hence

$$\int_{[-R,R]} f = \int_{-R}^{R} f(\gamma(t))\gamma'(t)\,dt = \int_{-R}^{R} f(t)\,dt.$$

Note that there exists a constant $C>0$ such that

$$\left|\frac{1}{(x^2+1)(x^2+4)}\right| \leq \frac{C}{x^4}.$$

Hence, by Lemma 7.5.1, the infinite integral $\int_{-\infty}^{\infty} 1/(x^2+1)(x^2+4)\,dx$ exists and is equal to its principal value

$$\mathcal{P}\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\,dx = \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{(x^2+1)(x^2+4)}\,dx.$$

We will calculate the principal value of integral using Cauchy's Residue Theorem.

Let

$$f(z) = \frac{1}{(z^2+1)(z^2+4)}$$

(note that we have introduced a complex variable). Let $[-R,R]$ denote the path along the real axis that starts at $-R$ and ends at R. This has parametrisation t, $-R \leq t \leq R$. Note that we can equate the real integral (7.5.5) with the complex integral as follows:

$$\int_{-R}^{R} \frac{1}{(x^2+1)(x^2+4)}\,dx = \int_{[-R,R]} f(z)\,dz.$$

To use Cauchy's Residue Theorem, we need a closed contour. Introduce a semi-circular path $S_R(t) = Re^{it}$, $0 \leq t \leq \pi$ and the 'D-shaped' contour $\Gamma_R = [-R,R] + S_R$ (see Figure 7.5.3).

Now Γ_R is a simple closed loop. To use Cauchy's Residue Theorem, we need to know the poles and residues of $f(z)$. Now

$$f(z) = \frac{1}{(z^2+1)(z^2+4)} = \frac{1}{(z-i)(z+i)(z-2i)(z+2i)}.$$

Hence $f(z)$ has simple poles at $z=+i,-i,+2i,-2i$. If we take $R>2$ then the poles at $z=i,2i$ lie inside Γ_R (note that the poles at $z=-i,-2i$ lie outside Γ_R). Now by Lemma 7.4.1,

$$\begin{aligned}
\mathrm{Res}(f,i) &= \lim_{z\to i}(z-i)f(z) \\
&= \lim_{z\to i} \frac{1}{(z+i)(z-2i)(z+2i)} \\
&= \frac{1}{6i}
\end{aligned}$$

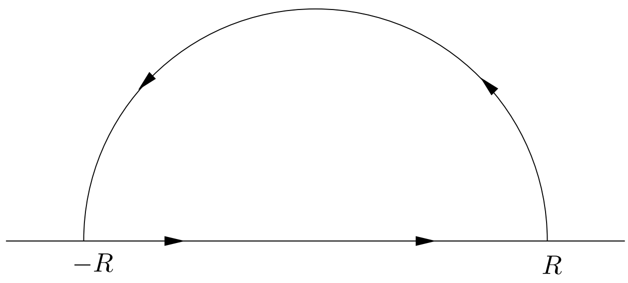

Figure 7.5.3: The 'D-shaped' contour Γ_R. It starts at $-R$, travels along the real axis to R, and then anticlockwise along the semicircle S_R with centre 0 and radius R.

and

$$\begin{aligned}\operatorname{Res}(f, 2i) &= \lim_{z\to 2i}(z-2i)f(z) \\ &= \lim_{z\to 2i}\frac{1}{(z-i)(z+i)(z+2i)} \\ &= \frac{-1}{12i}.\end{aligned}$$

Hence by Cauchy's Residue Theorem

$$\begin{aligned}\int_{[-R,R]} f(z),\, dz + \int_{S_R} f(z)\, dz &= \int_{\Gamma_R} f(z)\, dz \\ &= 2\pi i\left(\operatorname{Res}(f,i) + \operatorname{Res}(f,2i)\right) \\ &= 2\pi i\left(\frac{1}{6i} - \frac{1}{12i}\right) = \frac{\pi}{6}.\end{aligned}$$

If we can show that

$$\lim_{R\to\infty}\int_{S_R} f(z)\, dz = 0 \tag{7.5.6}$$

then we will have that

$$\int_{-\infty}^{\infty}\frac{1}{(x^2+1)(x^2+4)}\, dx = \lim_{R\to\infty}\int_{[-R,R]} f(z)\, dz = \frac{\pi}{6}.$$

To complete the calculation, we show that (7.5.6) holds. We shall use the Estimation Lemma. Let z be a point on S_R. Note that $|z| = R$. Hence

$$|(z^2+1)(z^2+4)| \geq (R^2-1)(R^2-4)$$

so that

$$\left|\frac{1}{(z^2+1)(z^2+4)}\right| \leq \frac{1}{(R^2-1)(R^2-4)}.$$

Hence, by the Estimation Lemma,

$$\begin{aligned}\left|\int_{S_R} f(z)\, dz\right| &\leq \frac{1}{(R^2-1)(R^2-4)}\operatorname{length}(S_R) \\ &= \frac{\pi R}{(R^2-1)(R^2-4)} \\ &\to 0\end{aligned}$$

as $R \to \infty$, which is what we wanted to check.

Remark. As a general method, to evaluate

$$\int_{-R}^{R} f(x)\,dx$$

one uses the following steps:

(i) Check that $f(x)$ satisfies the hypotheses of Lemma 7.5.1.

(ii) Construct a 'D-shaped' contour Γ_R as in Figure 7.5.3.

(iii) Find the poles and residues of $f(z)$ that lie inside Γ_R when R is large.

(iv) Use Cauchy's Residue Theorem to write down $\int_{\Gamma_R} f(z)\,dz$.

(v) Split this integral into an integral over $[-R, R]$ and an integral over S_R. Use the Estimation Lemma to conclude that the integral over S_R converges to 0 as $R \to \infty$.

For a particular example, one may need to make small modifications to the above process, but the general method is normally as above.

Remark. It is very easy to lose minus signs or factors of $2\pi i$ when doing these computations. You should always check that your answer makes sense. For example, if I had missed out a factor of i in the above then I would have obtained an expression of the form

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\,dx = \frac{i}{6}.$$

This is obviously wrong: the left-hand side is a real number, whereas the (incorrect) right-hand side is imaginary. Similarly, in this example the integrand on the left-hand side is a positive function, and so the integral must be positive; hence if the right-hand side is negative then there must be a mistake somewhere in the calculation.

§7.5.3 Trigonometric integrals

We can use Cauchy's Residue Theorem to calculate integrals of the form

$$\int_0^{2\pi} Q(\cos t, \sin t)\,dt \tag{7.5.7}$$

where Q is some function. (Integrands such as $\cos^4 t \sin^3 t - 7\sin t$, or $\cos t + \sin^2 t$, etc, fall into this category.)

The first step is to turn (7.5.7) into a complex integral. Set $z = e^{it}$. Then

$$\cos t = \frac{z+z^{-1}}{2}, \ \sin t = \frac{z-z^{-1}}{2i}.$$

Also $[0, 2\pi]$ transforms into the unit circle $C_1(t) = e^{it}$, $0 \leq t \leq 2\pi$. Finally, note that $dz = ie^{it}\,dt$ so that

$$dt = \frac{dz}{iz}.$$

Hence

$$\int_0^{2\pi} Q(\cos t, \sin t)\,dt = \int_{C_1} Q\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz}.$$

Then in principle we can evaluate this integral by finding the poles of

$$Q\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{1}{iz}$$

inside C_1, together with their associated residues, and then use Cauchy's Residue Theorem.

Instead of stating a general theorem, we shall compute some examples to illustrate the method.

Example. We shall how to compute

$$\int_0^{2\pi} (\cos^3 t + \sin^2 t)\, dt.$$

Let $z = e^{it}$ so that $dt = dz/iz$. Then

$$\begin{aligned}
&\int_0^{2\pi} (\cos^3 t + \sin^2 t)\, dt \\
&= \int_{C_1} \left(\left(\frac{z+z^{-1}}{2}\right)^3 + \left(\frac{z-z^{-1}}{2i}\right)^2\right)\frac{dz}{iz} \\
&= \int_{C_1} \left(\frac{z^3}{8} + \frac{3z}{8} + \frac{3z^{-1}}{8} + \frac{z^{-3}}{8} - \frac{z^2}{4} + \frac{1}{2} - \frac{z^{-2}}{4}\right)\frac{dz}{iz} \\
&= \int_{C_1} \frac{1}{i}\left(\frac{z^2}{8} + \frac{3}{8} + \frac{3z^{-2}}{8} + \frac{z^{-4}}{8} - \frac{z}{4} + \frac{z^{-1}}{2} - \frac{z^{-3}}{4}\right) dz
\end{aligned}$$

The integrand has a pole of order 4 at $z = 0$ with residue $1/2i$, and no other poles. Hence

$$\int_0^{2\pi} (\cos^3 t + \sin^2 t)\, dt = 2\pi i \frac{1}{2i} = \pi.$$

Example. We shall compute

$$\int_0^{2\pi} \cos t \sin t\, dt.$$

Again, substituting $z = e^{it}$ we have that

$$\begin{aligned}
&\int_0^{2\pi} \cos t \sin t\, dt \\
&= \int_{C_1} \frac{1}{4i}(z+z^{-1})(z-z^{-1})\frac{dz}{iz} \\
&= \int_{C_1} \frac{1}{4i}(z^2 - z^{-2})\frac{dz}{iz} \\
&= \int_{C_1} \left(\frac{-1}{4}\right)\left(z - \frac{1}{z^3}\right) dz.
\end{aligned}$$

The integrand has a pole of order 3 at $z = 0$ with residue 0. There are no other poles. Hence

$$\int_0^{2\pi} \cos t \sin t\, dt = 0.$$

Remark. The above illustrates a more general method. For example, one can also evaluate integrals of the form
$$\int_0^\pi Q(\cos t, \sin t)\, dt$$
by using the substitution $z = e^{2it}$. In this case, as t varies from 0 to π then z describes the unit circle in $\mathbb{C}$ with centre 0 and radius 1 described anti-clockwise.

§7.5.4 Summation of series

Recall that $\cot \pi z = \cos \pi z / \sin \pi z$. Then $\cot \pi z$ has a pole whenever $\sin \pi z = 0$, i.e. whenever $z = n$, $n \in \mathbb{Z}$. First note that $\sin \pi z$ has a simple zero at $z = n$ (as $\sin' \pi z = \pi \cos \pi z \neq 0$ when $z = n$). Hence $\cot \pi z$ has a simple pole at $z = n$. By Lemma 7.4.1(ii) we have
$$\operatorname{Res}(\cot \pi z, n) = \frac{\cos \pi n}{\pi \cos \pi n} = \frac{1}{\pi}.$$
This suggests a method for summing infinite series of the form $\sum_{n=1}^\infty a_n$. Let $f(z)$ be a meromorphic function defined on $\mathbb{C}$ such that $f(n) = a_n$. Consider the function $f(z) \cot \pi z$. Then, if $f(n) \neq 0$, we have
$$\operatorname{Res}(f(z) \cot \pi z, n) = \frac{a_n}{\pi}$$
and we can use Cauchy's Residue Theorem to calculate $\sum_{n=1}^\infty a_n$. For example, we will show how to use this method to calculate $\sum_{n=1}^\infty 1/n^2$.

There are two technicalities to overcome. First of all, we need to choose a good contour to integrate around. We will want to use the Estimation Lemma along this contour, so we will need some bounds on $|f(z) \cot(\pi z)|$. Secondly, $f(z)$ may have poles of its own and these will need to be taken into account. (In the above example, to calculate $\sum_{n=1}^\infty 1/n^2$ we will take $f(z) = 1/z^2$, which has a pole at $z = 0$.)

Instead of choosing a D-shaped contour, here we use a square contour. Let C_N denote the square in $\mathbb{C}$ with vertices at
$$\left(N+\frac{1}{2}\right) - i\left(N+\frac{1}{2}\right),\ \left(N+\frac{1}{2}\right) + i\left(N+\frac{1}{2}\right),$$
$$-\left(N+\frac{1}{2}\right) + i\left(N+\frac{1}{2}\right),\ -\left(N+\frac{1}{2}\right) - i\left(N+\frac{1}{2}\right)$$
(see Figure 7.5.4). This is a square with each side having length $2N + 1$. (The factors of $1/2$ are there so that the sides of this square do not pass through the integer points on the real axis.)

Lemma 7.5.2
There is a bound, independent of N, on $\cot \pi z$ where $z \in C_N$, i.e. there exists $M > 0$ such that for all N and all $z \in C_N$, we have $|\cot \pi z| \leq M$.

Proof. Consider the square C_N. This has two horizontal sides and two vertical sides, parallel to the real and imaginary axes, respectively.

Consider first the horizontal sides. Let $z = x + iy$ be a point on one of the horizontal sides of C_N. Then $|y| \geq 1/2$. Hence
$$|\cot \pi z| = \left| \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \right|$$

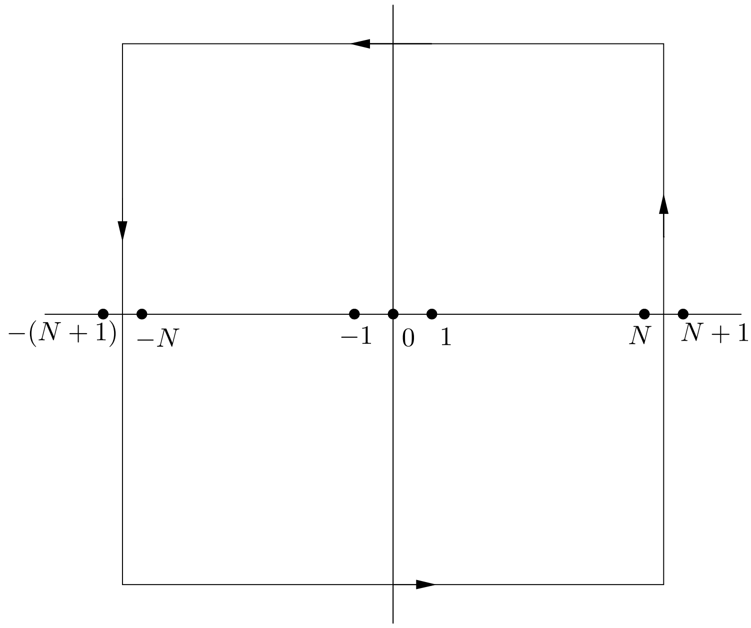

Figure 7.5.4: The square contour C_N.

$$\begin{aligned}
&\leq \left|\frac{\left|e^{i\pi z}\right| + \left|e^{-i\pi z}\right|}{\left|e^{i\pi z}\right| - \left|e^{-i\pi z}\right|}\right| \\
&\leq \left|\frac{e^{-\pi y} + e^{\pi y}}{e^{-\pi y} - e^{\pi y}}\right| \\
&= \coth|\pi y| \\
&\leq \coth\left(\frac{\pi}{2}\right)
\end{aligned}$$

as $|y| \geq 1/2$.

Consider now the vertical sides. If z is on a vertical side of C_N then

$$z = \pm\left(N + \frac{1}{2}\right) + iy.$$

Hence

$$\begin{aligned}
|\cot \pi z| &= \left|\frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}}\right| \\
&= \left|\frac{e^{2\pi iz} + 1}{e^{2\pi iz} - 1}\right| \\
&= \left|\frac{e^{i\pi - 2\pi y} + 1}{e^{i\pi - 2\pi y} - 1}\right| \\
&= \left|\frac{-e^{-2\pi iy} + 1}{-e^{-2\pi iy} - 1}\right| \\
&= \frac{1 - e^{-2\pi y}}{1 + e^{2\pi y}} \\
&\leq 1.
\end{aligned}$$

Hence $|\cot \pi z| \leq \max\{1, \coth \pi/2\}$ for all $z \in C_N$. □

Instead of stating a general theorem on how to use Cauchy's Residue Theorem to evaluate infinite sums, we will work through an example to illustrate the method. Very similar techniques and slight modifications to the argument work for many other examples.

We will evaluate $\sum_{n=0}^{\infty} 1/n^2$. Let $f(z) = 1/z^2$ and consider the function

$$f(z)\cot \pi z = \frac{\cot \pi z}{z^2} = \frac{\cos \pi z}{z^2 \sin \pi z}.$$

This has a pole whenever the denominator has a zero. These occur when $z^2 \sin \pi z = 0$, i.e. when $z = n$, $n \in \mathbb{Z}$. Note that when $n \neq 0$ we have a simple pole and when $n = 0$ we have a pole of order 3.

Let us calculate the residue when $n \neq 0$. We use Lemma 7.4.1(ii). Then

$$\begin{aligned} \operatorname{Res}\left(\frac{\cot \pi z}{z^2}, n\right) &= \frac{\cos \pi n}{\pi n^2 \cos \pi n + 2n \sin \pi n} \\ &= \frac{1}{\pi n^2}. \end{aligned}$$

Now consider the pole at $z = 0$. There are (at least) three ways to work out the residue here, and for completeness we'll discuss them all. Firstly, we can write

$$\begin{aligned} &\frac{1}{z^2}\frac{\cos \pi z}{\sin \pi z} \\ &= \frac{1}{z^2}\left(1 - \frac{(\pi z)^2}{2!} + \frac{(\pi z)^4}{4!} - \cdots\right)\left((\pi z) - \frac{(\pi z)^3}{3!} + \frac{(\pi z)^5}{5!} - \cdots\right)^{-1} \\ &= \frac{1}{z^2}\frac{1}{\pi z}\left(1 - \frac{(\pi z)^2}{2!} + \frac{(\pi z)^4}{4!} - \cdots\right)\left(1 - \frac{(\pi z)^2}{3!} + \frac{(\pi z)^4}{5!} - \cdots\right)^{-1} \\ &= \frac{1}{z^2}\frac{1}{\pi z}\left(1 - \frac{(\pi z)^2}{2!} + \frac{(\pi z)^4}{4!} - \cdots\right)\left(1 + \frac{(\pi z)^2}{3!} - \frac{(\pi z)^4}{5!} + \cdots\right) \\ &= \frac{1}{\pi z^3}\left(1 - \frac{(\pi z)^2}{3} + \cdots\right) \end{aligned}$$

so that $\operatorname{Res}(\cot \pi z/z^2, 0) = -\pi/3$. (We used the expansion $(1-x)^{-1} = 1+x+x^2+\cdots$.) Note that to calculate the residue we need only calculate the coefficient of the term involving $1/z$; hence we need to be very careful when manipulating these infinite sums to ensure that we account for all the possible terms which may contribute towards $1/z$.

Alternatively, as another method for calculating the residue at 0, we can use the following power series expansion for $\cot z$:

$$\cot z = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \cdots.$$

Hence

$$\frac{\cot \pi z}{z^2} = \frac{1}{\pi z^3} - \frac{\pi}{3z} - \frac{\pi^3 z}{45} - \frac{2\pi^5 z^3}{945} - \cdots$$

from which it is clear that $z = 0$ is a pole of order 3 with residue $-\pi/3$.

Finally, as a third method of calculating the residue at 0, one could use Lemma 7.4.2.

Now let C_N be the square contour illustrated above. Note that each side of the square has length $2N + 1$. Hence the length of C_N is $4(2N + 1)$.

Note that the poles that lie inside C_N occur at $z = 0, \pm 1, \cdots, \pm N$. By Cauchy's Residue Theorem we have that

$$2\pi i \sum_{n=-N}^{N} \operatorname{Res}\left(\frac{\cot \pi z}{z^2}, n\right) = \int_{C_N} \frac{\cot \pi z}{z^2}\,dz.$$

Recall from Lemma 7.5.2 that $|\cot \pi z| \leq M$ on C_N, where M is independent of N. Also note that $|1/z^2| \leq 1/N^2$ for z on C_N. By the Estimation Lemma we have

$$\left|\int_{C_N} \frac{\cot \pi z}{z^2}\,dz\right| \leq \frac{M}{N^2}\,\text{length}\, C_N = \frac{M}{N^2} 4(2N+1)$$

which tends to 0 as $N \to \infty$. Hence

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \operatorname{Res}\left(\frac{\cot \pi z}{z^2}, n\right) = 0. \tag{7.5.8}$$

Now

$$\begin{aligned} &\sum_{n=-N}^{N} \operatorname{Res}\left(\frac{\cot \pi z}{z^2}, n\right) \\ = &\sum_{n=-N}^{-1} \operatorname{Res}\left(\frac{\cot \pi z}{z^2}, n\right) + \operatorname{Res}\left(\frac{\cot \pi z}{z^2}, 0\right) + \sum_{n=1}^{N} \operatorname{Res}\left(\frac{\cot \pi z}{z^2}, n\right) \\ = &\ 2\sum_{n=1}^{N} \frac{1}{\pi n^2} - \frac{\pi}{3} \end{aligned}$$

and combining this with (7.5.8) we see that

$$2\sum_{n=1}^{\infty} \frac{1}{\pi n^2} - \frac{\pi}{3} = 0.$$

This rearranges to give

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

§7.6 Proof of Cauchy's Residue Theorem

Let us first recall the statement of Cauchy's Residue Theorem:

Theorem 7.6.1 (Cauchy's Residue Theorem)
Let D be a domain containing a simple closed loop γ and the points inside γ. Suppose that f is holomorphic on D except for finitely many poles at $z_1, z_2, \ldots, z_n$ inside γ. Then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f, z_j).$$

Proof. The proof is a simple application of the Generalised Cauchy Theorem (Theorem 4.5.7).

Since D is open, for each $j = 1, \ldots, n$, we can find circles

$$S_j(t) = z_j + \varepsilon_j e^{it},\ 0 \leq t \leq 2\pi$$

centred at z_j and of radii ε_j, each described once anticlockwise, such that S_j and the points inside S_j lie in D and such that S_j contains no singularity other than z_j (see Figure 7.6.1).

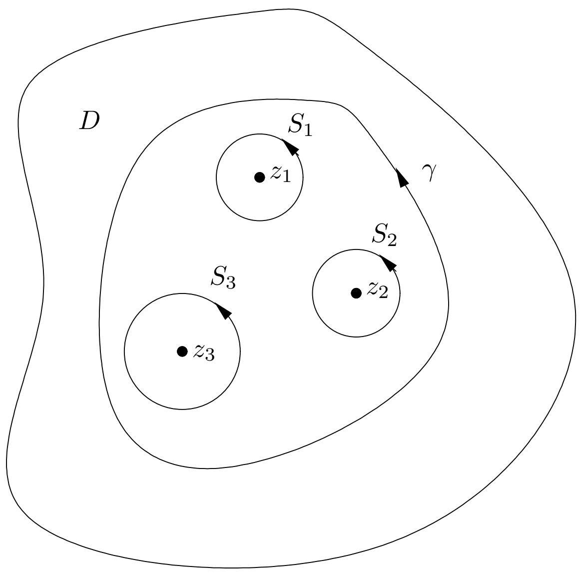

Figure 7.6.1: Here we have 3 poles at z_1, z_2, z_3 inside γ. The circles S_1, S_2, S_3 (centred on z_1, z_2, z_3, respectively) have been chosen so that they lie inside γ and do not intersect each other.

Let $D' = D \setminus \{z_1, \ldots, z_n\}$. We claim that the collection of paths

$$-\gamma, S_1, \ldots, S_n$$

satisfy the hypotheses of the Generalised Cauchy Theorem (Theorem 4.5.7) with respect to D': i.e. their winding numbers sum to zero for every point not in D'.

To see this, first note that

$$w(-\gamma, z) = w(S_j, z) = 0 \text{ for } z \notin D.$$

Hence the hypotheses of the Generalised Cauchy Theorem hold for points z not in D. It remains to consider points in D that are not in D', i.e. the poles z_j.

Since each pole z_j lies inside γ, we have that

$$w(-\gamma, z_j) = -w(\gamma, z_j) = -1.$$

Moreover,

$$w(S_k, z_j) = \begin{cases} 0 \text{ if } k \neq j \\ 1 \text{ if } k = j. \end{cases}$$

Hence

$$w(-\gamma, z_j) + w(S_1, z_j) + \cdots + w(S_n, z_j) = 0.$$

Hence, by the Generalised Cauchy Theorem,

$$\int_{-\gamma} f + \int_{S_1} f + \cdots + \int_{S_n} f = 0.$$

By Laurent's Theorem (Theorem 6.2.1) we have that

$$\mathrm{Res}(f, z_j) = \frac{1}{2\pi i} \int_{S_j} f(z)\, dz.$$

Hence

$$\begin{aligned} \int_{\gamma} f &= \int_{S_1} f + \cdots + \int_{S_n} f \\ &= 2\pi i \left(\mathrm{Res}(f, z_1) + \cdots + \mathrm{Res}(f, z_n) \right), \end{aligned}$$

concluding the proof. □

Exercises for Part 7

Exercise 7.1
Using Lemma 7.2.2 to determine the poles of the following functions. For each pole, use Lemmas 7.4.1, and 7.4.2 (as appropriate) to calculate the residue at each pole.

$$\text{(i)}\ \frac{1}{z(1-z^2)},\ \text{(ii)}\ \tan z,\ \text{(iii)}\ \frac{z}{1+z^4},\ \text{(iv)}\ \left(\frac{z+1}{z^2+1}\right)^2.$$

Exercise 7.2
Determine the singularities of the following functions. By considering Taylor series, calculate the residue at each singularity.

$$\text{(i)}\ \frac{\sin z}{z^2},\ \text{(ii)}\ \frac{\sin^2 z}{z^4}.$$

Exercise 7.3
(i) Let $f(z) = 1/z(1-z)^2$. Then f has singularities at 0 and 1. Expand f as a Laurent series at 0 and as a Laurent series at 1. In each case, read off from the Laurent series the order of the pole and the residue of the pole.

(Hint: recall that $1/(1-z)^2 = 1 + 2z + 3z^2 + \cdots + nz^{n-1} + \cdots$ if $|z| < 1$.)

(ii) Check your answer by using Lemmas 7.2.2, 7.4.1 and 7.4.2.

Exercise 7.4
Suppose that $f, g : D \to \mathbb{C}$ are holomorphic and that $z_0 \in D$. Suppose that f has a zero of order n at z_0 and g has a zero of order m at z_0. Show that $f(z)g(z)$ has a zero of order $n+m$ at z_0.

Exercise 7.5
Let C_r be the circle $C_r(t) = re^{it}$, $0 \leq t \leq 2\pi$, with centre 0 and radius r. Use Cauchy's Residue Theorem to evaluate the integrals

$$\text{(i)}\ \int_{C_4} \frac{1}{z^2-5z+6}\,dz,\ \text{(ii)}\ \int_{C_{5/2}} \frac{1}{z^2-5z+6}\,dz,\ \text{(iii)}\ \int_{C_2} \frac{e^{az}}{1+z^2}\,dz\ (a \in \mathbb{R}).$$

Exercise 7.6
(a) Consider the following real integral:

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx.$$

(i) Explain why this integral is equal to its principal value.

(ii) Use Cauchy's Residue Theorem to evaluate this integral. (How would you have done this without using complex analysis?)

(b) (i) Now evaluate, using Cauchy's Residue Theorem, the integral

$$\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2+1}\,dx.$$

(ii) By taking real and imaginary parts, calculate

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2+1}\,dx, \quad \int_{-\infty}^{\infty} \frac{\sin 2x}{x^2+1}\,dx.$$

(Why is it obvious, without having to use complex integration, that one of these integrals is zero?)

(iii) Why does the 'D-shaped' contour used in the lectures for calculating such integrals fail when we try to integrate

$$\int_{-\infty}^{\infty} \frac{e^{-2ix}}{x^2+1}\,dx?$$

By choosing a different contour, explain how one could evaluate this integral using Cauchy's Residue Theorem.

Exercise 7.7

Use Cauchy's Residue Theorem to evaluate the following real integrals:

$$\text{(i)} \int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+3)}\,dx, \quad \text{(ii)} \int_{-\infty}^{\infty} \frac{1}{28+11x^2+x^4}\,dx.$$

Exercise 7.8

By considering the function

$$f(z) = \frac{e^{iz}}{z^2+4z+5}$$

integrated around a suitable contour, show that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}\,dx = \frac{-\pi \sin 2}{e}.$$

Exercise 7.9

Consider the integral

$$\int_0^{2\pi} \frac{1}{13+5\cos t}\,dt$$

(i) Use the substitution $z = e^{it}$ to show that

$$\int_0^{2\pi} \frac{1}{13+5\cos t}\,dt = \frac{2}{i}\int_{C_1} \frac{1}{5z^2+26z+5}\,dz$$

where C_1 is the circle with centre 0, radius 1, described once anticlockwise.

(ii) Show that $f(z) = 1/(5z^2+26z+5)$ has simple poles at $z=-5$ and $z=-1/5$. Show that $\mathrm{Res}(f,-1/5) = 1/24$.

(iii) Use Cauchy's Residue Theorem to show that

$$\int_0^{2\pi} \frac{1}{13 + 5\cos t}\, dt = \frac{\pi}{6}.$$

Exercise 7.10
Convert the following real integrals into complex integrals around the unit circle in the complex plane. Hence use Cauchy's Residue Theorem to evaluate them.

$$\text{(i)} \int_0^{2\pi} 2\cos^3 t + 3\cos^2 t\, dt, \ \text{(ii)} \int_0^{2\pi} \frac{1}{1+\cos^2 t}\, dt.$$

Exercise 7.11
Use the method of summation of series to show that $\sum_{n=1}^{\infty} 1/n^4 = \pi^4/90$.
Why doesn't the method work for evaluating $\sum_{n=1}^{\infty} 1/n^3$?

Remark. Cauchy's Residue Theorem has a lot of applications. The exercises below provide more practice at using Cauchy's Residue Theorem to calculate integrals, summations, etc. You may find them useful if you want more practice at using Cauchy's Residue Theorem. If you are focussing solely on doing well in the exam then you should concentrate on Exercises 7.1–7.11 above.

Exercise 7.12
(This exercise shows how to use Cauchy's Residue Theorem to explicitly calculate the coefficients in Laurent's Theorem using the formula therein.)

Recall that Laurent's Theorem (Theorem 6.2.1) says the following: Suppose that f is holomorphic on the annulus $\{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$. Then f can be written as a Laurent series on this annulus in the form

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

The coefficients are given by

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z - z_0)^{n+1}}\, dz$$

and $C_r(t) = z + re^{it}$, $0 \leq t \leq 2\pi$, denotes the circular path around z_0 of radius r where r is chosen such that $R_1 < r < R_2$.

By using Cauchy's Residue Theorem to evaluate a_n, determine the Laurent series for the function

$$f(z) = \frac{1}{z(z-1)}$$

valid on the annuli

(i) $0 < |z| < 1$,

(ii) $1 < |z| < \infty$,

(iii) $0 < |z - 1| < 1$,

(iv) $1 < |z - 1| < \infty$.

(Thus we are using Cauchy's Residue Theorem to evaluate $\int_{C_r} f(z)/(z-z_0)^{n+1}\,dz$ around a suitable closed contour C_r of radius r and centred at an appropriate z_0 ($z_0 = 0$ in (i),(ii) and $z_0 = 1$ in (iii),(iv)) by locating the poles of $f(z)/(z-z_0)$ that lie inside C_r and calculating their residues.)

In each case, check your answer by directly calculate the Laurent series using the methods described in §6.2.

Exercise 7.13

(This exercise uses Cauchy's Residue Theorem to calculate an integral that is (I believe) impossible to calculate using common techniques of real analysis/calculus.)

Let $0 < a < b$. Evaluate the integral discussed in §1.1:

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+a^2)(x^2+b^2)}\,dx$$

by integrating a suitable function around a suitable contour.

Exercise 7.14

Suppose $a \neq 0$. Consider the function

$$\frac{\cot \pi z}{z^2+a^2}$$

Show that this function has poles at $z = n$, $n \in \mathbb{Z}$ and $z = \pm ia$. Calculate the residues at these poles.

Hence show that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{2a}\coth \pi a - \frac{1}{2a^2}.$$

Exercise 7.15

(The method used in §7.5.3 can be used evaluate other, more complicated, integrals.)

Let $C_1(t) = e^{it}$, $0 \leq t \leq 2\pi$, denote the unit circle in $\mathbb{C}$ centred at 0 and with radius 1.

(i) Prove, using Cauchy Residue Theorem, that

$$\int_{C_1} \frac{e^z}{z}\,dz = 2\pi i.$$

(ii) By using the substitution $z = e^{it}$, prove that

$$\int_0^{2\pi} e^{\cos t}\cos(\sin t)\,dt = 2\pi, \quad \int_0^{2\pi} e^{\cos t}\sin(\sin t)\,dt = 0.$$

Exercise 7.16

(Sometimes, to calculate an indefinite integral, one has to be rather creative in picking the right contour.)

Let $0 < a < 1$. Show that

$$\int_{-\infty}^{\infty} \frac{e^{az}}{1+e^z} = \frac{\pi}{\sin a\pi}$$

using the following steps.

(i) Show that this integral exists and is equal to its principal value.

(ii) Let $f(z) = e^{az}/(1 + e^z)$. Show that f is holomorphic except for simple poles at $z = (2k + 1)\pi i$, $k \in \mathbb{Z}$. Draw a diagram to illustrate where the poles are. Calculate the residue $\mathrm{Res}(f, \pi i)$.

(iii) On the diagram from (ii), draw the contour $\Gamma_R = \gamma_{1,R} + \gamma_{2,R} + \gamma_{3,R} + \gamma_{4,R}$ where:

$\gamma_{1,R}$ is the horizontal straight line from $-R$ to R,

$\gamma_{2,R}$ is the vertical straight line from R to $R + 2\pi i$,

$\gamma_{3,R}$ is the horizontal straight line from $R + 2\pi i$ to $-R + 2\pi i$,

$\gamma_{4,R}$ is the vertical straight line from $-R + 2\pi i$ to $-R$.

Which poles does Γ_R wind around? Use Cauchy's Residue Theorem to calculate $\int_{\Gamma_R} f$.

(iv) Show, by choosing suitable parametrisations of the paths $\gamma_{1,R}$ and $\gamma_{3,R}$ and direct computation, that $\int_{\gamma_3} f = -e^{2\pi i a} \int_{\gamma_1} f$.

(v) Show, using the Estimation Lemma, that

$$\lim_{R\to\infty} \int_{\gamma_{2,R}} f = \lim_{R\to\infty} \int_{\gamma_{4,R}} f = 0.$$

(vi) Deduce the claimed result.

8. Solutions to Part 1

Solution 1.1

(i) $(3+4i)^2 = 9+24i-16 = -7+24i$.

(ii)

$$\frac{2+3i}{3-4i} = \frac{2+3i}{3-4i}\frac{3+4i}{3+4i} = \frac{(2+3i)(3+4i)}{25} = \frac{-6}{25} + i\frac{17}{25}.$$

(iii)

$$\frac{1-5i}{-1+3i} = \frac{-8}{5} + i\frac{1}{5}.$$

(iv)

$$\frac{1-i}{1+i} - i + 2 = -i - i + 2 = 2-2i.$$

(v)

$$\frac{1}{i} = -i.$$

Solution 1.2

(i) We have $|2i| = 2$, $\arg 2i = \pi/2 + 2k\pi$ for any $k \in \mathbb{Z}$, $\operatorname{Arg} 2i = \pi/2$.

(ii) We have $|-1-i\sqrt{3}| = \sqrt{1+3} = 2$, $\arg(-1-i\sqrt{3}) = \tan^{-1}\frac{-\sqrt{3}}{-1} = -\pi/3 + 2k\pi$ for any $k \in \mathbb{Z}$, $\operatorname{Arg}(-1-i\sqrt{3}) = -\pi/3$.

(iii) We have $|-4| = 4$, $\arg -4 = \pi + 2k\pi$ for any $k \in \mathbb{Z}$, $\operatorname{Arg} -4 = \pi$.

Solution 1.3

(i) Write $z = x+iy$. Then $x^2 + 2ixy - y^2 = -5 + 12i$. Comparing real and imaginary parts gives the simultaneous equations $x^2 - y^2 = -5$, $2xy = 12$. The second equation gives $y = 6/x$ and substituting this into the first gives $x^4 + 5x^2 - 36 = 0$, a quadratic in x^2. Solving this quadratic equation gives $x^2 = 4$, hence $x = \pm 2$. When $x = 2$ we have $y = 3$; when $x = -2$ we have $y = -3$. Hence $z = 2+3i, -2-3i$ are the solutions.

(ii) A bare-hands computation as in (i) will work, but is very lengthy. The trick is to instead first complete the square. Write

$$z^2 + 4z + 12 - 6i = (z+2)^2 + 8 - 6i.$$

Write $z+2 = x+iy$. Then $(x+iy)^2 = -8+6i$. Solving this as in (i) gives $x = 1, y = 3$ or $x = -1, y = -3$. Hence $z = -1+3i$ or $z = -3-3i$.

Solution 1.4

(i) Let $z = a+ib$, $w = c+id$. Then $\mathrm{Re}(z+w) = \mathrm{Re}(a+ib+c+id) = \mathrm{Re}((a+c)+i(b+d)) = a+c = \mathrm{Re}(z)+\mathrm{Re}(w)$. Similarly $\mathrm{Re}(z-w) = \mathrm{Re}(z) - \mathrm{Re}(w)$.

(ii) Note that $\mathrm{Im}(z+w) = \mathrm{Im}(a+ib+c+id) = \mathrm{Im}((a+c)+i(b+d)) = b+d = \mathrm{Im}(z)+\mathrm{Im}(w)$. Similarly $\mathrm{Im}(z-w) = \mathrm{Im}(z) - \mathrm{Im}(w)$.

Almost any two complex numbers picked at random will give an example for which $\mathrm{Re}(zw) \neq \mathrm{Re}(z)\,\mathrm{Re}(w)$. For example, choose $z = i$, $w = -i$. Then $zw = 1$. However, $\mathrm{Re}(zw) = 1 \neq \mathrm{Re}(z)\,\mathrm{Re}(w) = 0 \times 0 = 0$. Similarly, $\mathrm{Im}(zw) = 0 \neq \mathrm{Im}(z)\,\mathrm{Im}(w) = 1 \times (-1) = -1$.

Solution 1.5

Throughout write $z = a+ib$, $w = c+id$.

(i) $\overline{z+w} = \overline{(a+ib)+(c+id)} = \overline{(a+c)+i(b+d)} = (a+c)-i(b+d) = (a-ib)+(c-id) = \bar{z}+\bar{w}$. Similarly for $\overline{z-w}$.

(ii) $\overline{zw} = \overline{(a+ib)(c+id)} = \overline{(ac-bd)+i(ad+bc)} = (ac-bd)-i(ad+bc) = (a-ib)(c-id) = \overline{z}\overline{w}$.

(iii) First note that $\overline{\left(\frac{1}{z}\right)} = \overline{\left(\frac{1}{a+ib}\right)} = \overline{\left(\frac{a-ib}{a^2+b^2}\right)} = \frac{a+ib}{a^2+b^2}$. We also have $\frac{1}{\bar{z}} = \frac{1}{a-ib} = \frac{a+ib}{a^2+b^2}$, so the result follows.

(iv) $z+\bar{z} = (x+iy)+(x-iy) = 2x = 2\,\mathrm{Re}(z)$.

(v) $z-\bar{z} = (x+iy)-(x-iy) = 2iy = 2i\,\mathrm{Im}(z)$.

Solution 1.6

Let $z, w \in \mathbb{C}$. Then

$$|z| = |z-w+w| \leq |z-w| + |w|$$

by the reverse triangle inequality. Hence

$$|z| - |w| \leq |z-w|.$$

Similarly, $|w| - |z| \leq |z-w|$. Hence

$$||z| - |w|| \leq |z-w|.$$

Solution 1.7

(i) Writing $z = x+iy$ we obtain $\mathrm{Re}(z) = \{(x,y) \mid x > 2\}$, i.e. a half-plane.

(ii) Here we have the open strip $\{(x,y) \mid 1 < y < 2\}$.

(iii) The condition $|z| < 3$ is equivalent to $x^2+y^2 < 9$; hence the set is the open disc of radius 3 centred at the origin.

(iv) Write $z = x+iy$. We have $|x+iy-1| < |x+iy+1|$, i.e. $(x-1)^2+y^2 < (x+1)^2+y^2$. Multiplying this out (and noting that the ys cancel) gives $x > 0$, i.e. an open half-plane.

Solution 1.8

(i) We have

$$\begin{aligned} zw &= rs\left((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)\right) \\ &= rs\left(\cos(\theta+\phi) + i\sin(\theta+\phi)\right). \end{aligned}$$

Hence $\arg zw = \theta + \phi = \arg z + \arg w$.

(ii) From (i) we have that $\arg z^2 = 2\arg z$. By induction $\arg z^n = n(\arg z)$. Put $z = \cos\theta + i\sin\theta$ so that $\arg z = \theta$. Note that $|z|^2 = \cos^2\theta + \sin^2\theta = 1$. Hence $|z^n| = 1$. Hence $z^n = \cos n\theta + i\sin n\theta$.

(iii) Applying De Moivre's theorem in the case $n = 3$ gives

$$\begin{aligned} (\cos\theta + i\sin\theta)^3 &= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta \\ &= \cos 3\theta + i\sin 3\theta. \end{aligned}$$

Hence, comparing real and imaginary parts and using the fact that $\cos^2\theta + \sin^2\theta = 1$, we obtain

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta,$$

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta.$$

Similarly,

$$\cos 4\theta = \sin^4\theta - 6\cos^2\theta\sin^2\theta + \cos^4\theta,$$

$$\sin 4\theta = -4\cos\theta\sin^3\theta + 4\cos^3\theta\sin\theta.$$

Solution 1.9

Let $w_0 = re^{i\theta}$ and suppose that $z^n = w_0$. Write $z = \rho e^{i\phi}$. Then $z^n = \rho^n e^{in\phi}$. Hence $\rho^n = r$ and $n\phi = \theta + 2k\pi$, $k \in \mathbb{Z}$. Thus we have that $\rho = r^{1/n}$ and we get distinct values of the argument ϕ of z when $k = 0, 1, \ldots, n-1$. Hence

$$z = r^{1/n} e^{i\left(\frac{\theta + 2k\pi}{n}\right)}, \ k = 0, 1, \ldots, n-1.$$

Solution 1.10

Take $z_1 = z_2 = -1 + i$. Then $\mathrm{Arg}(z_1) = \mathrm{Arg}(z_2) = 3\pi/4$. However, $z_1 z_2 = -2i$ and $\mathrm{Arg}(z_1 z_2) = -\pi/2$. (Draw a picture!) In this case, $\mathrm{Arg}(z_1 z_2) \neq \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$.

(More generally, any two points z_1, z_2 for which $\mathrm{Arg}(z_1) + \mathrm{Arg}(z_2) \notin (-\pi, \pi]$ will work.)

Solution 1.11

As far as I know it isn't possible to evaluate this integral using some combination of integration by substitution, integration by parts, etc. However, there is one technique that may work (I haven't tried it), and it's one that was a favourite of Richard Feynman (Nobel laureate in physics, safe-cracker, and bongo-player, amongst many other talents). Feynman claimed to have never learned complex analysis but could perform many real integrals using a trick called 'differentiation under the integral sign'. See `http://www.math.uconn.edu/~kconrad/blurbs/analysis/diffunderint.pdf` for an account of this, if you're interested.

9. Solutions to Part 2

Solution 2.1
Drawing a picture and describing informally whether a set is a domain or not will be sufficient in this course.

(i) This set is domain. Let $D = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. Then D is open: if $z_0 \in D$ and $z \in \mathbb{C}$ is sufficiently close to z_0 (i.e. $|z-z_0|$ is small) then $z \in D$. Also, the straight-line between any two points in D lies in D; hence there is a polygonal arc between any two points in D.

(If one wanted to show in more detail that D is open then one could argue as follows. Let $D = \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$. Let $z_0 \in D$. We have to find $\varepsilon > 0$ such that $B_\varepsilon(z_0) \subset D$. To do this, write $z_0 = x_0 + iy_0$ and let $\varepsilon = y_0/2 > 0$. Suppose that $z = x + iy \in B_\varepsilon(z_0)$. Then

$$|y - y_0| \le \sqrt{|x-x_0|^2 + |y-y_0|^2} = |z - z_0| \le \frac{y_0}{2}.$$

Hence

$$-\frac{y_0}{2} < y - y_0 < \frac{y_0}{2}$$

so that $y > y_0/2$ i.e. $\mathrm{Im}(z) > 0$. Hence $z \in D$. See Figure 9.1(i).)

(ii) This set is domain. Let $D = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0,\ |z| < 2\}$. Intuitively, we can see why D is open. Let $z_0 \in D$ so that $\mathrm{Re}(z_0) > 0$ and $|z_0| < 2$. If z is very close to z_0 (in the sense that $|z - z_0|$ is small) then $\mathrm{Re}(z) > 0$ and $|z| < 2$; hence $z \in D$. (We could calculate exactly how small one needs $|z - z_0|$ to be; this is done below for completeness.) Also, the straight-line between any two points in D lies in D; hence there is a polygonal arc between any two points in D. Hence D is a domain.

(The detailed argument to show that D is open is as follows. We have to find $\varepsilon > 0$ such that $B_\varepsilon(z_0) \subset D$. That one can do this is clear from Figure 9.1(ii). In order to produce ε we argue as follows. Let

$$\varepsilon = \min\left\{\frac{x_0}{2}, \frac{2-|z_0|}{2}\right\} > 0$$

(note that $|z_0| < 2$ as $z_0 \in D$). Let $z = x + iy \in B_\varepsilon(z_0)$ so that $|z - z_0| < x_0/2$ and $|z - z_0| \le (2 - |z_0|)/2$. Then arguing as in (i) we see that

$$|x - x_0| \le \sqrt{|x-x_0|^2 + |y-y_0|^2} = |z - z_0| \le \frac{x_0}{2}$$

so that

$$-\frac{x_0}{2} < x - x_0 < \frac{x_0}{2}$$

from which it follows that $x > x_0/2 > 0$, i.e. $\mathrm{Re}(z) > 0$. We also have that

$$|z| = |z - z_0 + z_0| \le |z - z_0| + |z_0| \le \frac{2-|z_0|}{2} + |z_0| = 1 + \frac{|z_0|}{2} < 2$$

as $|z_0| < 2$. Hence $|z| < 2$. It follows that $z \in D$.)

(iii) Let $D = \{z \in \mathbb{C} \mid |z| \leq 6\}$. This set is not open and so is not a domain. If we take the point $z_0 = 6$ on the real axis, then no matter how small $\varepsilon > 0$ is, there are always points in $B_\varepsilon(z_0)$ that are not in D. See Figure 9.1(iii).

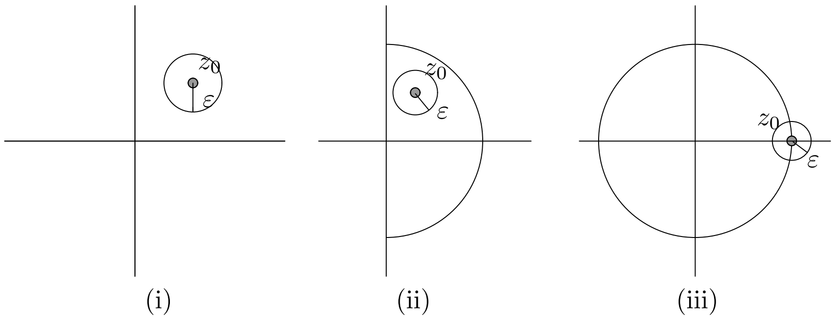

Figure 9.1: See Solution 2.1.

(iv) Let $D = \{z \in \mathbb{C} \mid |z| < 2\} \cup \{z \in \mathbb{C} \mid |z| > 4\}$. This set is open, but it is not a domain. Let z_1 be such that $|z_1| < 2$ and let z_2 be such that $|z_2| > 4$. Then there is no polygonal arc between z_1 and z_2 that is within D.

Solution 2.2

(i) For any $z_0 \in \mathbb{C}$ we have

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{z^2 + z - (z_0^2 + z_0)}{z - z_0} \\ &= \lim_{z \to z_0} \frac{(z - z_0)(z + z_0 + 1)}{z - z_0} \\ &= \lim_{z \to z_0} z + z_0 + 1 \\ &= 2z_0 + 1 \end{aligned}$$

so that $f'(z) = 2z + 1$.

(ii) For $z_0 \neq 0$ we have

$$\begin{aligned} f'(z_0) &= \lim_{z \to z_0} \frac{1/z - 1/z_0}{z - z_0} \\ &= \lim_{z \to z_0} \frac{z_0 - z}{z_0 z(z - z_0)} \\ &= \lim_{z \to z_0} \frac{-1}{z_0 z} \\ &= \frac{-1}{z_0^2} \end{aligned}$$

so that $f'(z) = -1/z^2$.

(iii) For each $z_0 \in \mathbb{C}$ we have

$$f'(z_0) = \lim_{z \to z_0} \frac{(z^3 - z^2) - (z_0^3 - z_0^2)}{z - z_0}$$

$$\begin{aligned} &= \lim_{z\to z_0} \frac{(z-z_0)(z^2+z_0z+z_0^2-z-z_0)}{z-z_0} \\ &= \lim_{z\to z_0} z^2+z_0z+z_0^2-z-z_0 \\ &= 3z_0^2-2z_0 \end{aligned}$$

so that $f'(z) = 3z^2 - 2z$.

Notice that the complex derivatives are identical to and can be computed in the same way as their real analogues ('bring down the power and knock one off the power', etc).

Solution 2.3

(i) Throughout write $z = x + iy$.

(a) Note that $f(z) = (x+iy)^2 = x^2 + 2ixy - y^2$. Hence $u(x,y) = x^2 - y^2$, $v(x,y) = 2xy$.

(b) Note that for $z \neq 0$

$$f(x+iy) = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} + i\frac{-y}{x^2+y^2}$$

so that $u(x,y) = x/(x^2+y^2)$, $v(x,y) = -y/(x^2+y^2)$.

(ii) (a) Here

$$\frac{\partial u}{\partial x} = 2x = \frac{\partial v}{\partial y}, \ \frac{\partial u}{\partial y} = -2y = -\frac{\partial v}{\partial x}$$

so that the Cauchy-Riemann equations are satisfied.

(b) Here

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{-x^2+y^2}{(x^2+y^2)^2}, \ \frac{\partial u}{\partial y} = \frac{-2xy}{(x^2+y^2)^2}, \\ \frac{\partial v}{\partial x} &= \frac{2xy}{(x^2+y^2)^2}, \ \frac{\partial v}{\partial y} = \frac{-x^2+y^2}{(x^2+y^2)^2}. \end{aligned}$$

Hence $\partial u/\partial x = \partial v/\partial y$ and $\partial u/\partial y = -\partial v/\partial x$ so that the Cauchy-Riemann equations hold.

(iii) When $f(z) = |z|$ we have $f(x+iy) = \sqrt{x^2+y^2}$ so that $u(x,y) = \sqrt{x^2+y^2}$, $v(x,y) = 0$. Then for $(x,y) \neq (0,0)$ we have

$$\frac{\partial u}{\partial x} = \frac{x}{(x^2+y^2)^{1/2}}, \ \frac{\partial u}{\partial y} = \frac{y}{(x^2+y^2)^{1/2}}, \ \frac{\partial v}{\partial x} = 0, \ \frac{\partial v}{\partial y} = 0.$$

If the Cauchy-Riemann equations hold then $x/(x^2+y^2)^{1/2} = 0$, $y/(x^2+y^2)^{1/2} = 0$, which imply that $x = y = 0$, which is impossible as we are assuming that $(x,y) \neq (0,0)$.

At $(x,y) = (0,0)$ we have

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{|h|}{h}$$

which does not exist. (To see this, note that if $h \to 0$, $h > 0$, then $|h|/h \to 1$; however, if $h \to 0$, $h < 0$, then $|h|/h = -h/h \to -1$.)

Hence f is not differentiable anywhere.

Solution 2.4

(i) Here

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \ \frac{\partial v}{\partial y} = 3x^2 - 3y^2,$$

and

$$\frac{\partial u}{\partial y} = -6xy, \ \frac{\partial v}{\partial x} = 6xy$$

so that the Cauchy-Riemann equations hold.

(ii) Here

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{4}{(x^2+y^2)^5}(-x^5 + 10x^3y^2 - 5xy^4), \\ \frac{\partial v}{\partial y} &= \frac{4}{(x^2+y^2)^5}(-x^5 + 10x^3y^2 - 5xy^4), \end{aligned}$$

and

$$\begin{aligned} \frac{\partial u}{\partial y} &= \frac{4}{(x^2+y^2)^5}(-5x^4 + 10x^2y^3 - y^5), \\ \frac{\partial v}{\partial x} &= \frac{4}{(x^2+y^2)^5}(5x^4 - 10x^2y^3 + y^5) \end{aligned}$$

so that the Cauchy-Riemann equations hold.

Let $z_0 \in \mathbb{C}$. In both cases, the partial derivatives of u and v exist at z_0. The partial derivatives of u and v are continuous at z_0. The Cauchy-Riemann equations holds at z_0. Thus u and v satisfy the hypotheses of Proposition 2.5.2. Hence $f = u + iv$ is differentiable at z_0. As $z_0 \in \mathbb{C}$ is arbitrary, we see that f is holomorphic on $\mathbb{C}$.

Solution 2.5

(i) Recall that

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z}.$$

As $f(z) = 0$ for the function in the question we need to investigate the limit

$$\lim_{z \to 0} \frac{f(z)}{z}.$$

Put $z = x + ix$ with $x > 0$. Then $f(x) = x$ and

$$\lim_{z \to 0} \frac{f(z)}{z} = \lim_{x \to 0} \frac{x}{x + ix} = \frac{1}{1+i}.$$

However, if $z = x - ix$, $x > 0$ then

$$\lim_{z \to 0} \frac{f(z)}{z} = \lim_{x \to 0} \frac{x}{x - ix} = \frac{1}{1-i}.$$

Hence there is no limit of $(f(z) - f(0))/z$ as $z \to 0$.

(ii) For $f(x+iy) = \sqrt{|xy|}$ we have $u(x,y) = \sqrt{|xy|}$ and $v(x,y) = 0$. Then clearly

$$\frac{\partial v}{\partial x}(0,0) = \frac{\partial v}{\partial y}(0,0).$$

Now

$$\frac{\partial u}{\partial x}(0,0) = \lim_{h\to 0} \frac{u(h,0)-u(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

and

$$\frac{\partial u}{\partial y}(0,0) = \lim_{k\to 0} \frac{u(0,k)-u(0,0)}{k} = \lim_{k\to 0} \frac{0}{k} = 0.$$

Hence the Cauchy-Riemann equations are satisfied.

This does not contradict Proposition 2.5.2 because the partial derivative $\partial u/\partial x$ is not continuous at $(0,0)$. To see this, note that for $x > 0, y > 0$,

$$\frac{\partial u}{\partial x}(x,y) = \frac{\sqrt{y}}{2\sqrt{x}}$$

so that

$$\lim_{(x,y)\to(0,0)} \frac{\partial u}{\partial x}(x,y)$$

does not exist.

Solution 2.6
By the Cauchy-Riemann equations we have that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$$

so that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Similarly

$$\begin{aligned}\frac{\partial^2 v}{\partial x^2} &= \frac{\partial}{\partial x}\frac{\partial v}{\partial x} = -\frac{\partial}{\partial x}\frac{\partial u}{\partial y} = -\frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial^2 u}{\partial y \partial x} \\ &= -\frac{\partial}{\partial y}\frac{\partial u}{\partial x} = -\frac{\partial}{\partial y}\frac{\partial v}{\partial y} = -\frac{\partial^2 v}{\partial y^2}\end{aligned}$$

so that

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

Solution 2.7
Let $f(z) = z^3$ and write $z = x+iy$ so that

$$f(x+iy) = (x+iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3).$$

Hence $u(x,y) = x^3 - 3xy^2$ and $v(x,y) = 3x^2y - y^3$.

Now

$$\frac{\partial^2 u}{\partial x^2} = 6x,\ \frac{\partial^2 u}{\partial y^2} = -6x$$

and
$$\frac{\partial^2 v}{\partial x^2} = 6y,\ \frac{\partial^2 v}{\partial y^2} = -6y$$
so that
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$
Hence both u and v are harmonic.

Solution 2.8
Suppose we know that $u(x,y) = x^5 - 10x^3y^2 + 5xy^4$. Then
$$\frac{\partial u}{\partial x} = 5x^4 - 30x^2y^2 + 5y^4 = \frac{\partial v}{\partial y}.$$
Integrating with respect to y gives
$$v(x,y) = 5x^4y - 10x^2y^3 + y^5 + \alpha(x) \tag{9.0.1}$$
for some function $\alpha(x)$ that depends only on x and not on y. (Recall that we are looking for an anti-partial derivative and $\partial\alpha(x)/\partial y = 0$.)

Similarly,
$$\frac{\partial u}{\partial y} = -20x^3y + 20xy^3 = -\frac{\partial v}{\partial x}$$
and integrating with respect to x gives
$$v(x,y) = 5x^4y - 10x^2y^3 + \beta(y) \tag{9.0.2}$$
for some arbitrary function $\beta(y)$.

Comparing (9.0.1) and (9.0.2) we see that $y^5 + \alpha(x) = \beta(y)$, i.e.
$$\alpha(x) = \beta(y) - y^5.$$
The right-hand side depends only on y and the left-hand side depends only on x. This is only possible if both $\alpha(x)$ and $\beta(y) - y^5$ is a constant. Hence
$$v(x,y) = 5x^4y - 10x^2y^3 + y^5 + c$$
for some constant $c \in \mathbb{R}$.

Solution 2.9
From Exercise 2.6, we know that if u is the real part of a holomorphic function then u is harmonic, i.e. u satisfies Laplace's equation. Note that
$$\frac{\partial^2 u}{\partial x^2} = 6x, \quad \frac{\partial^2 u}{\partial y^2} = -2kx$$
so that
$$0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = (6-2k)x.$$
Hence $k = 3$.

It remains to show that in the case $k = 3$, u is the real part of a holomorphic function. We argue as in Exercise 2.8. First note that if $u(x, y) = x^3 - 3xy^2 + 12xy - 12x$ then

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 12y - 12 = \frac{\partial v}{\partial y}.$$

Hence

$$v(x, y) = 3x^2y - y^3 + 6y^2 - 12y + \alpha(x) \tag{9.0.3}$$

for some arbitrary function $\alpha(x)$ depending only on x. Similarly

$$\frac{\partial u}{\partial y} = -6xy + 12x = -\frac{\partial v}{\partial y}$$

so that

$$v(x, y) = 3x^2y - 6x^2 + \beta(y) \tag{9.0.4}$$

for some arbitrary function $\beta(y)$ depending only on y. Comparing (9.0.3) and (9.0.4) we see that

$$\alpha(x) + 6x^2 = \beta(y) + y^3 - 6y^2 + 12y;$$

as the left-hand side depends only on x and the right-hand side depends only on y, the above two expressions must be equal to a constant $c \in \mathbb{R}$. Hence

$$v(x, y) = 3x^2y - 6x^2 - y^3 + 6y^2 - 12y + c.$$

Note that the partial derivatives for both u and v exist and are continuous at every point in $\mathbb{C}$ and the Cauchy-Riemann equations hold at every point in $\mathbb{C}$, it follows from the converse of the Cauchy-Riemann Theorem that $f(x + iy) = u(x, y) + iv(x, y)$ is a holomorphic function on $\mathbb{C}$.

Solution 2.10

Suppose that $f(x + iy) = u(x, y) + iv(x, y)$ and $u(x, y) = c$, a constant. Then $\partial u/\partial x = 0$. Hence by the Cauchy-Riemann equations $\partial v/\partial y = 0$. Integrating with respect to y gives that $v(x, y) = \alpha(x)$ for some function $\alpha(x)$ that depends only on x.

Similarly, $\partial u/\partial y = 0$. Hence by the Cauchy-Riemann equations $-\partial v/\partial x = 0$. Integrating with respect to x gives that $v(x, y) = \beta(y)$ for some function $\beta(y)$ that depends only on y.

Hence

$$v(x, y) = \alpha(x) = \beta(y).$$

As $\alpha(x)$ depends only on x and $\beta(y)$ depends only on y, this is only possible if both $\alpha(x)$ and $\beta(y)$ are constant. Hence $v(x, y)$ is constant and it follows that f is constant.

Solution 2.11

Suppose that $f(x + iy) = u(x) + iv(y)$ where the real part depends only on x and the imaginary part depends only on y. Then

$$\frac{\partial u}{\partial x} = u'(x),\ \frac{\partial v}{\partial y} = v'(y).$$

By the Cauchy-Riemann equations, $u'(x) = v'(y)$. As the left-hand side of this equation depends only on x and the right-hand side depends only on y, we must have that

$$u'(x) = v'(y) = \lambda$$

for some real constant λ. From $u'(x) = \lambda$ we have that $u(x) = \lambda x + c_1$, for some constant $c_1 \in \mathbb{R}$. From $v'(y) = \lambda$ we have that $v(y) = \lambda y + c_2$ for some constant $c_2 \in \mathbb{R}$. Let $a = c_1 + ic_2$. Then $f(z) = \lambda z + a$.

Solution 2.12

Suppose that $f(z) = u(x, y) + iv(x, y)$ and $2u(x, y) + v(x, y) = 5$. Partially differentiating the latter expression with respect to x gives

$$2\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0$$

and using the Cauchy-Riemann equations gives

$$2\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} = 0.$$

Similarly, partially differentiating $2u(x, y) + v(x, y) = 5$ with respect to y and using the Cauchy-Riemann equations gives

$$\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0.$$

This gives us two simultaneous equations in $\partial u/\partial x$ and $\partial u/\partial y$. Solving these equations gives

$$\frac{\partial u}{\partial x} = 0, \quad \frac{\partial u}{\partial y} = 0.$$

From $\partial u/\partial x = 0$ it follows that $u(x, y) = \alpha(y)$, an arbitrary function of y. From $\partial u/\partial y = 0$ it follows that $u(x, y) = \beta(x)$, an arbitrary function of x. This is only possible if u is constant.

If u is constant then

$$0 = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

(so that v depends only on x) and

$$0 = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

(so that v depends only on y). Hence v must also be constant.

10. Solutions to Part 3

Solution 3.1
Let $z_n \in \mathbb{C}$. Let

$$s_n = \sum_{k=0}^{n} z_k, \ x_n = \sum_{k=0}^{n} \mathrm{Re}(z_k), \ y_n = \sum_{k=0}^{n} \mathrm{Im}(z_k)$$

denote the partial sums of z_n, $\mathrm{Re}(z_n)$, $\mathrm{Im}(z_n)$, respectively. Let $s = \sum_{k=0}^{\infty} z_n$, $x = \sum_{k=0}^{\infty} \mathrm{Re}(z_k)$, $y = \sum_{k=0}^{\infty} \mathrm{Im}(z_k)$, if these exist.

Suppose that $\sum_{n=0}^{\infty} z_n$ is convergent. Let $\varepsilon > 0$. Then there exists N such that if $n \geq N$ we have $|s - s_n| < \varepsilon$. As

$$|x - x_n| \leq |s - s_n| < \varepsilon,$$

and

$$|y - y_n| \leq |s - s_n| < \varepsilon$$

(using the facts that $|\mathrm{Re}(w)| \leq |w|$ and $|\mathrm{Im}(w)| \leq |w|$ for any complex number w), provided $n \geq N$, it follows that $\sum_{k=0}^{\infty} \mathrm{Re}(z_k)$ and $\sum_{k=0}^{\infty} \mathrm{Im}(z_k)$ exist.

Conversely, suppose that $\sum_{k=0}^{\infty} \mathrm{Re}(z_k)$ and $\sum_{k=0}^{\infty} \mathrm{Im}(z_k)$ exist. Let $\varepsilon > 0$. Choose N_1 such that if $n \geq N_1$ then $|x - x_n| < \varepsilon/2$. Choose N_2 such that if $n \geq N_2$ then $|y - y_n| < \varepsilon/2$. Then if $n \geq \max\{N_1, N_2\}$ we have that

$$|z - z_n| \leq |x - x_n| + |y - y_n| < \varepsilon.$$

Hence $\sum_{k=0}^{\infty} z_k$ converges.

Solution 3.2
Recall that a formula for the radius of convergence R of $\sum a_n z^n$ is given by $1/R = \lim_{n\to\infty} |a_{n+1}|/|a_n|$ (if this limit exists).

(i) Here $a_n = 2^n/n$ so that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{2^{n+1}}{n+1}\frac{n}{2^n} = \frac{2n}{n+1} \to 2 = \frac{1}{R}$$

as $n \to \infty$. Hence the radius of convergence is $R = 1/2$.

(ii) Here $a_n = 1/n!$ so that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0 = \frac{1}{R}$$

as $n \to \infty$. Hence the radius of convergence is $R = \infty$ and the series converges for all $z \in \mathbb{C}$.

(iii) Here $a_n = n!$ so that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)!}{n!} = n \to \infty = \frac{1}{R}$$

as $n \to \infty$. Hence the radius of convergence is $R = 0$ and the series converges for $z = 0$ only.

(iv) Here $a_n = n^p$ so that

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(n+1)^p}{n^p} = \left(\frac{n+1}{n}\right)^p \to 1^p = 1 = \frac{1}{R}$$

as $n \to \infty$. Hence the radius of convergence is $R = 1$.

Solution 3.3
To see that the expression in Proposition 3.2.2(i) does not converge, note that

$$\left|\frac{a_{n+1}}{a_n}\right| = \begin{cases} \dfrac{2^n}{3^{n+1}} & \text{if } n \text{ is even,} \\ \dfrac{3^n}{2^{n+1}} & \text{if } n \text{ is odd.} \end{cases}$$

Hence $\lim_{n\to\infty} |a_{n+1}/a_n| = 0$ if we let $n \to \infty$ through the subsequence of even values of n but $\lim_{n\to\infty} |a_{n+1}/a_n| = \infty$ if we let $n \to \infty$ through the subsequence of odd values of n. Hence $\lim_{n\to\infty} |a_{n+1}/a_n|$ does not exist.

To see that the expression in Proposition 3.2.2(ii) does not converge, note that

$$|a_n|^{1/n} = \begin{cases} 1/2 \text{ if } n \text{ is even,} \\ 1/3 \text{ if } n \text{ is odd.} \end{cases}$$

Hence $\lim_{n\to\infty} |a_{n+1}/a_n|$ does not exist.

Note, however, that $a_n \leq 1/2^n$ for all n. Hence

$$\left|\sum_{n=0}^{\infty} a_n z^n\right| \leq \left|\sum_{n=0}^{\infty} \frac{z^n}{2^n}\right| \leq \sum_{n=0}^{\infty} \left|\frac{z}{2}\right|^n,$$

which converges provided that $|z/2| < 1$, i.e. if $|z| < 2$. Hence, by the comparison test, $\sum_{n=0}^{\infty} a_n z^n$ converges for $|z| < 2$.

Solution 3.4
(i) We know that for $|z| < 1$

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$

(this is the sum of a geometric progression). Hence

$$\left(\frac{1}{1-z}\right)^2 = \left(\frac{1}{1-z}\right)\left(\frac{1}{1-z}\right) = \left(\sum_{n=0}^{\infty} z^n\right)\left(\sum_{n=0}^{\infty} z^n\right).$$

Using Proposition 3.1.2 we can easily see that the coefficient of z^{n-1} in the above product is equal to n. Hence

$$\left(\frac{1}{1-z}\right)^2 = \sum_{n=1}^{\infty} n z^{n-1}.$$

(ii) Using Proposition 3.1.2 we see that

$$\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{n=0}^{\infty}\frac{w^n}{n!}=\sum_{n=0}^{\infty}c_n$$

where

$$c_n=\sum_{r=0}^{n}\frac{1}{r!(n-r)!}z^r w^{n-r}.$$

The claimed result follows by noting that

$$c_n=\frac{1}{n!}\sum_{r=0}^{n}\left(\begin{array}{c} n \\ r \end{array}\right) z^r w^{n-r}=\frac{1}{n!}(z+w)^n$$

by the Binomial Theorem.

Solution 3.5

Let $f(z)=\sum_{n=0}^{\infty}z^n$. Then $f(z)$ defines a power series with radius of convergence 1. Moreover, $f(z)=1/(1-z)$ by summing the geometric progression.

By Theorem 3.3.2 we can differentiate $\sum_{n=0}^{\infty}z^n=1/(1-z)$ k-times and obtain a power series that converges for $|z|<1$. We obtain, for each $k\geq 1$,

$$\sum_{n=k-1}^{\infty}n(n-1)\cdots(n-(k-2))z^{n-(k-1)}=(k-1)!(1-z)^{-k}$$

for $|z|<1$. Dividing both sides by $(k-1)!$ gives the result.

Solution 3.6

(i) We have that $e^{iz}=\cos z+i\sin z$ and $e^{-iz}=\cos z-i\sin z$. Adding these expressions gives $2\cos z=e^{iz}+e^{iz}$ so that $\cos z=(e^{iz}+e^{-iz})/2$.

(ii) Subtracting the above expressions for e^{iz} and e^{-iz} gives $2i\sin z=e^{iz}-e^{-iz}$ so that $\sin z=(e^{iz}-e^{-iz})/2i$.

(iii) It is easier to start with the right-hand side:

$$\begin{aligned}
&\sin z\cos w+\cos z\sin w\\
&= \frac{1}{4i}(e^{iz}-e^{-iz})(e^{iw}+e^{-iw})+\frac{1}{4i}(e^{iz}+e^{-iz})(e^{iw}-e^{-iw})\\
&= \frac{1}{2i}\left(e^{i(z+w)}-e^{-i(z+w)}\right)\\
&= \sin(z+w).
\end{aligned}$$

(iv) Similarly,

$$\begin{aligned}
&\cos z\cos w-\sin z\sin w\\
&= \frac{1}{4}(e^{iz}+e^{-iz})(e^{iw}+e^{-iw})-\frac{1}{4i^2}(e^{iz}-e^{-iz})(e^{iw}-e^{-iw})\\
&= \frac{1}{2}\left(e^{i(z+w)}+e^{-i(z+w)}\right)\\
&= \cos(z+w).
\end{aligned}$$

Solution 3.7

(i) We have

$$\begin{aligned}
\sin z &= \sin(x+iy) \\
&= \frac{1}{2i}\left(e^{i(x+iy)} - e^{-i(x+iy)}\right) \\
&= \frac{1}{2i}(e^{ix}e^{-y} - e^{-ix}e^{y}) \\
&= \frac{1}{2i}((e^{-y}\cos x - e^{y}\cos x) + i(e^{-y}\sin x + e^{y}\sin x)) \\
&= \left(\frac{e^{-y}\sin x + e^{y}\sin x}{2}\right) + i\left(\frac{e^{y}\cos x - e^{-y}\cos x}{2}\right) \\
&= \sin x\cosh y + i\cos x\sinh y.
\end{aligned}$$

Hence the real and imaginary parts of $\sin z$ are $u(x,y) = \sin x\cosh y$ and $v(x,y) = \cos x\sinh y$, respectively.

Now

$$\frac{\partial u}{\partial x} = \cos x\cosh y,\ \frac{\partial v}{\partial y} = \cos x\cosh y,$$

and

$$\frac{\partial u}{\partial y} = \sin x\sinh y,\ \frac{\partial v}{\partial x} = -\sin x\sinh y,$$

so that the Cauchy-Riemann equations are satisfied.

(ii) Here we have that

$$\begin{aligned}
\cos z &= \cos(x+iy) \\
&= \frac{1}{2}\left(e^{i(x+iy)} + e^{-i(x+iy)}\right) \\
&= \frac{1}{2}(e^{ix}e^{-y} + e^{-ix}e^{y}) \\
&= \frac{1}{2}((e^{-y}\cos x + e^{y}\cos x) + i(e^{-y}\sin x - e^{y}\sin x)) \\
&= \cos x\cosh y - i\sin x\sinh y.
\end{aligned}$$

Hence the real and imaginary parts of $\cos z$ are $u(x,y) = \cos x\cosh y$ and $v(x,y) = -\sin x\sinh y$, respectively.

Now

$$\frac{\partial u}{\partial x} = -\sin x\cosh y,\ \frac{\partial v}{\partial y} = -\sin x\cosh y,$$

and

$$\frac{\partial u}{\partial y} = \cos x\sinh y,\ \frac{\partial v}{\partial x} = -\cos x\sinh y,$$

so that the Cauchy-Riemann equations are satisfied.

Alternatively, one could note that

$$\begin{aligned}
\cos z &= \sin\left(z + \frac{\pi}{2}\right) \\
&= \sin\left(x + \frac{\pi}{2}\right)\cosh y + i\cos\left(x + \frac{\pi}{2}\right)\sinh y \\
&= \cos x\cosh y - i\sin x\sinh y.
\end{aligned}$$

(iii) Here we have that

$$\begin{aligned}
\sinh z &= \sinh(x+iy) \\
&= \frac{1}{2}\left(e^{x+iy} - e^{-(x+iy)}\right) \\
&= \frac{1}{2}(e^x e^{iy} - e^{-x}e^{-iy}) \\
&= \frac{1}{2}((e^x \cos y - e^{-x}\cos y) + i(e^x \sin y + e^{-x}\sin y)) \\
&= \sinh x \cos y + i \cosh x \sin y.
\end{aligned}$$

Hence the real and imaginary parts of $\sinh z$ are $u(x,y) = \sinh x \cos y$ and $v(x,y) = \cosh x \sin y$, respectively.

Now

$$\frac{\partial u}{\partial x} = \cosh x \cos y, \ \frac{\partial v}{\partial y} = \cosh x \cos y,$$

and

$$\frac{\partial u}{\partial y} = -\sinh x \sin y, \ \frac{\partial v}{\partial x} = \sinh x \sin y,$$

so that the Cauchy-Riemann equations are satisfied.

(One can also argue, assuming the results of (i), by using the fact that $\sinh z = -i \sin iz$.)

(iv) Here we have that

$$\begin{aligned}
\cosh z &= \cosh(x+iy) \\
&= \frac{1}{2}\left(e^{x+iy} + e^{-(x+iy)}\right) \\
&= \frac{1}{2}(e^x e^{iy} + e^{-x}e^{-iy}) \\
&= \frac{1}{2}((e^x \cos y + e^{-x}\cos y) + i(e^x \sin y - e^{-x}\sin y)) \\
&= \cosh x \cos y + i \sinh x \sin y.
\end{aligned}$$

Hence the real and imaginary parts of $\cosh z$ are $u(x,y) = \cosh x \cos y$ and $v(x,y) = \sinh x \sin y$, respectively.

Now

$$\frac{\partial u}{\partial x} = \sinh x \cos y, \ \frac{\partial v}{\partial y} = \sinh x \cos y,$$

and

$$\frac{\partial u}{\partial y} = -\cosh x \sin y, \ \frac{\partial v}{\partial x} = \cosh x \sin y,$$

so that the Cauchy-Riemann equations are satisfied.

(Alternatively, using the results of (ii), one can use the fact that $\cosh z = \cos iz$ to derive this.)

Solution 3.8

(i) A complex-valued function takes real values if and only its imaginary part equals 0.

For $\exp z$: note that if $z = x + iy$ then

$$e^z = e^x \cos y + ie^x \sin y$$

and this is real if and only $e^x \sin y = 0$. As $e^x > 0$ for all $x \in \mathbb{R}$, this is zero if and only if $\sin y = 0$, i.e. $y = k\pi$, $k \in \mathbb{Z}$.

Using the results of the previous exercise, $\sin z$ is real if and only if $\cos x \sinh y = 0$, i.e. either $x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$ or $y = 0$.

Similarly, the imaginary part of $\cos z$ is $-\sin x \sinh y$ and this equals zero if and only if $x = k\pi$, $k \in \mathbb{Z}$, or $y = 0$.

The imaginary part of $\cosh z$ is $\sinh x \sin y$ and this equals zero if and only if $x = 0$ or $y = k\pi$, $k \in \mathbb{Z}$.

The imaginary part of $\sinh z$ is $\cosh x \sin y$ and this equals zero if and only if $y = k\pi$, $k \in \mathbb{Z}$ (as $\cosh x > 0$ for all $x \in \mathbb{R}$).

(ii) A complex-valued function takes purely imaginary values if and only if its real part is zero.

Now $e^z = e^x \cos y + ie^x \sin y$ has zero real part if and only if $e^x \cos y = 0$, i.e. if $\cos y = 0$. Hence e^z takes purely imaginary values when $y = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.

The real part of $\sin z$ is $\sin x \cosh y$ and this equals zero if and only if $\sin x = 0$, i.e. $x = k\pi$, $k \in \mathbb{Z}$.

The real part of $\cos z$ is $\cos x \cosh y$ and this equals zero if and only if $\cos x = 0$, i.e. $x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.

The real part of $\sinh z$ is $\sinh x \cos y$ and this equals zero if and only if either $x = 0$ or $y = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.

The real part of $\cosh z$ is $\cosh x \cos y$ and this equals zero if and only if $y = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$.

Solution 3.9

Let $z = x+iy$. Then $\sin z = 0$ if and only if both the real parts and imaginary parts of $\sin z$ are equal to 0. This happens if and only if $\sin x \cosh y = 0$ and $\cos x \sinh y = 0$. We know $\cosh y > 0$ for all $y \in \mathbb{R}$ so the first equation gives $x = k\pi$, $k \in \mathbb{Z}$. Now $\cos k\pi = (-1)^k$ so the second equation gives $\sinh y = 0$, i.e. $y = 0$. Thus the solutions to $\sin z = 0$ are $z = k\pi$, $k \in \mathbb{Z}$.

For $\cos z$, we note that the real part of $\cos z$ is $\cos x \cosh y$ and the imaginary part is $-\sin x \sinh y$. Now $\cos x \cosh y = 0$ implies $\cos x = 0$ so that $x = (k + 1/2)\pi$, $k \in \mathbb{Z}$. As $\sin(k + 1/2)\pi = (-1)^k$, it follows that the second equation gives $\sinh y = 0$, i.e. $y = 0$. Hence the solutions to $\cos z = 0$ are $z = (k + 1/2)\pi$, $k \in \mathbb{Z}$.

Solution 3.10

(i) Let $z = x + iy$ and suppose that $1 + e^z = 0$. Then

$$e^z = e^x \cos y + ie^x \sin y = -1.$$

Comparing real and imaginary parts we have that $e^x \cos y = -1$ and $e^x \sin y = 0$. As $e^x > 0$ for all $x \in \mathbb{R}$ the second equation gives that $\sin y = 0$, i.e. $y = k\pi$, $k \in \mathbb{Z}$. Substituting this into the first equation gives $(-1)^k e^x = -1$. When k is even this gives $e^x = -1$ which has no real solutions. When k is odd this gives $e^x = 1$, i.e. $x = 0$. Hence the solutions are $z = (2k+1)\pi i$, $k \in \mathbb{Z}$.

(ii) Let $z = x + iy$ and suppose that $1 + i - e^z = 0$. Then

$$e^z = e^x \cos y + ie^x \sin y = 1 + i$$

and comparing real and imaginary parts gives

$$e^x \cos y = 1, \ e^x \sin y = 1.$$

As $e^x > 0$ for $x \in \mathbb{R}$, it follows that $\cos y = \sin y$, i.e. either $y = \pi/4 + 2k\pi$ or $y = 5\pi/4 + 2k\pi$, $k \in \mathbb{Z}$. In the first case, $\cos(\pi/4 + 2k\pi) = \sin(\pi/4 + 2k\pi) = 1/\sqrt{2}$ and so we have $e^x = \sqrt{2}$; hence $x = \log\sqrt{2}$. In the second case, $\cos(5\pi/4 + 2k\pi) = \sin(5\pi/4 + 2k\pi) = -1/\sqrt{2}$ so that $e^x = -\sqrt{2}$, which has no real solutions. Hence $z = \log\sqrt{2} + i(\pi/4 + 2k\pi)$, $k \in \mathbb{Z}$.

Solution 3.11

(i) Write $z = x + iy$. Suppose that $\sin(z + p) = \sin z$ for all $z \in \mathbb{C}$, for some $p \in \mathbb{C}$. Putting $z = 0$ we get $\sin p = \sin 0 = 0$, so that $p = k\pi$, $k \in \mathbb{Z}$. Now

$$\begin{aligned}
\sin(z + n\pi) &= \sin(z + (n-1)\pi + \pi) \\
&= \sin(z + (n-1)\pi)\cos\pi + \cos(z + (n-1)\pi)\sin\pi \\
&= -\sin(z + (n-1)\pi).
\end{aligned}$$

Continuing inductively, we see that $\sin(z + n\pi) = (-1)^n \sin z$. Hence $\sin(z + n\pi) = \sin z$ if and only if n is even. Hence the periods of sin are $p = 2\pi n$, $n \in \mathbb{Z}$.

(ii) Suppose that $\exp(z + p) = \exp z$ for all $z \in \mathbb{C}$. Putting $z = 0$ gives $\exp p = \exp 0 = 1$. Put $p = x + iy$. Then

$$\exp p = e^x \cos y + ie^x \sin y = 1$$

and comparing real and imaginary parts gives $e^x \cos y = 1$, $e^x \sin y = 0$. As $e^x > 0$ for all $x \in \mathbb{R}$, the second equation gives $\sin y = 0$, i.e. $y = n\pi$, $n \in \mathbb{Z}$. The first equation then gives $(-1)^n e^x = 1$. When n is odd this has no real solutions. When n is even this gives $e^x = 1$, i.e. $x = 0$. Hence the periods of exp are $2n\pi i$, $n \in \mathbb{Z}$.

Solution 3.12

Let $z_1 = r_1 e^{i\theta_1}$, $z_2 = r_2 e^{i\theta_2} \in \mathbb{C} \setminus \{0\}$. We choose $\theta_1, \theta_2 \in (-\pi, \pi]$ to be the principal value of the arguments of z_1, z_2. Hence

$$\operatorname{Log} z_1 = \ln r_1 + i\theta_1, \ \operatorname{Log} z_2 = \ln r_2 + i\theta_2.$$

Then

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

so that $z_1 z_2$ has argument $\theta_1 + \theta_2$. However, $\theta_1 + \theta_2 \in (-2\pi, 2\pi]$ and is not necessarily a principle value of the argument for $z_1 z_2$. However, by adding or subtracting 2π to $\theta_1 + \theta_2$ we can obtain the principal value of the argument of $z_1 z_2$. Thus

$$\begin{aligned}
\operatorname{Log} z_1 z_2 &= \ln r_1 r_2 + i(\theta_1 + \theta_2 + 2\pi n) \\
&= \operatorname{Log} z_1 + \operatorname{Log} z_2 + 2\pi n
\end{aligned}$$

for some integer $n \in \{-1, 0, 1\}$.

For example, take $z_1 = z_2 = -1+i$. Then $|z_1| = |z_2| = \sqrt{2}$ and $\operatorname{Arg} z_1 = \operatorname{Arg} z_2 = 3\pi/4$. Moreover $z_1 z_2 = -2i$ and the principal value of the argument of $z_1 z_2$ is $-\pi/2$. Hence

$$\operatorname{Log} z_1 = \operatorname{Log} z_2 = \ln\sqrt{2} + \frac{3\pi i}{4}, \ \operatorname{Log} z_1 z_2 = \ln 2 - \frac{\pi i}{2}$$

so that

$$\operatorname{Log} z_1 + \operatorname{Log} z_2 = \ln 2 + \frac{3\pi i}{2}.$$

Hence

$$\operatorname{Log} z_1 z_2 = \operatorname{Log} z_1 + \operatorname{Log} z_2 - 2\pi i.$$

(Any two complex numbers z_1, z_2 where the principal values of the arguments of z_1, z_2 add to either more than π or less than $-\pi$ will also work.)

Solution 3.13
Take $b = z = i$. Then $|i| = 1$, $\arg i = \pi/2 + 2n\pi$ and the principal value of the argument is $\pi/2$. Hence

$$\begin{aligned} \operatorname{Log}(i) &= \ln(1) + i\frac{\pi}{2} = i\frac{\pi}{2} \\ \log(i) &= \ln(1) + i\left(\frac{\pi}{2} + 2n\pi\right) = i\left(\frac{\pi}{2} + 2n\pi\right). \end{aligned}$$

Hence

$$i^i = \exp(i \operatorname{Log} i) = \exp\left(\frac{-\pi}{2}\right)$$

and the subsidiary values are

$$\exp(i \log i) = \exp\left(\frac{-\pi}{2} + 2n\pi\right).$$

Solution 3.14

(i) Using Proposition 3.2.2(i), the radius of convergence of this power series is given by

$$\frac{1}{R} = \lim_{n\to\infty}\left|\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)(\alpha-n)}{(n+1)!}\frac{n!}{\alpha(\alpha-1)\cdots(\alpha-n+1)}\right|,$$

if this limit exists. Note that

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)(\alpha-n)}{(n+1)!}\frac{n!}{\alpha(\alpha-1)\cdots(\alpha-n+1)}\right| &= \lim_{n\to\infty}\left|\frac{\alpha-n}{n+1}\right| \\ &= 1 \end{aligned}$$

Hence the power series has radius of convergence $R = 1$.

(ii) Recall from Theorem 3.3.2 that a power series is holomorphic on its disc of convergence and can be differentiated term by term. Hence for $|z| < 1$ we have

$$f'(z) = \alpha + \alpha(\alpha-1)z + \frac{\alpha(\alpha-1)(\alpha-2)}{2!}z^2 + \cdots + \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n)}{n!}z^n + \cdots.$$

Multiply this by $1+z$. Using Proposition 3.1.2 we see that the coefficient of z^n for $n \geq 1$ in $(1+z)f'(z)$ is given by

$$\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(n-1))(\alpha-n)}{n!} + \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(n-1))}{(n-1)!}$$

which can be rearranged to

$$\frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(n-1))}{(n-1)!}\left(\frac{\alpha-n}{n}+1\right) = \frac{\alpha^2(\alpha-1)(\alpha-2)\cdots(\alpha-(n-1))}{n!},$$

which is α times the coefficient of z^n in the power series $f(z)$. Clearly the constant term of $(1+z)f'(z)$ is α. Hence $(1+z)f'(z) = \alpha f(z)$ for $|z| < 1$.

(iii) Let $g(z) = f(z)/(1+z)^\alpha$. Then for $|z| < 1$ we have that

$$g'(z) = \frac{(1+z)^\alpha f'(z) - \alpha(1+z)^{\alpha-1} f(z)}{(1+z)^{2\alpha}} = 0,$$

using (ii). Hence $g'(z) = 0$ on $\{z \in \mathbb{C} \mid |z| < 1\}$. By Lemma 3.4.1 we have that $g(z)$ is equal to a constant on $\{z \in \mathbb{C} \mid |z| < 1\}$. Noting that $g(0) = 1$ we have that $g(z) = 1$ for all z with $|z| < 1$. Hence $f'(z) = (1+z)^\alpha$ for $|z| < 1$.

11. Solutions to Part 4

Solution 4.1

(i) We have $\gamma(t) = e^{-it} = \cos t - i \sin t$, $0 \le t \le \pi$, so the path is the semicircle of radius 1 centre 0 that starts at 1 and travels clockwise to -1. See Figure 11.1(i).

(ii) Here $\gamma(t) = x(t) + iy(t)$, $x(t) = 1 + 2\cos t$, $y(t) = 1 + 2\sin t$, $0 \le t \le 2\pi$. Hence $(x-1)^2 + (y-1)^2 = 2^2$, i.e. γ is the circle centred at $1+i$ with radius 2, where we start at $2+i$ and travel anticlockwise. See Figure 11.1(ii).

(iii) Here $\gamma(t) = x(t) + iy(t)$ where $x(t) = t$, $y(t) = \cosh t$, $-1 \le t \le 1$, i.e. $y = \cosh x$. Hence γ describes the piece of the graph of cosh from -1 to 1. See Figure 11.1(iii).

(iv) Here $\gamma(t) = x(t) + iy(t)$ where $x(t) = \cosh t$, $y(t) = \sinh t$, $-1 \le t \le 1$. Hence $x(t)^2 - y(t)^2 = \cosh^2 t - \sinh^2 t = 1$, i.e. γ describes a hyperbola from $(\cosh(-1), \sinh(-1))$ to $(\cosh(1), \sinh(1))$. See Figure 11.1(iv).

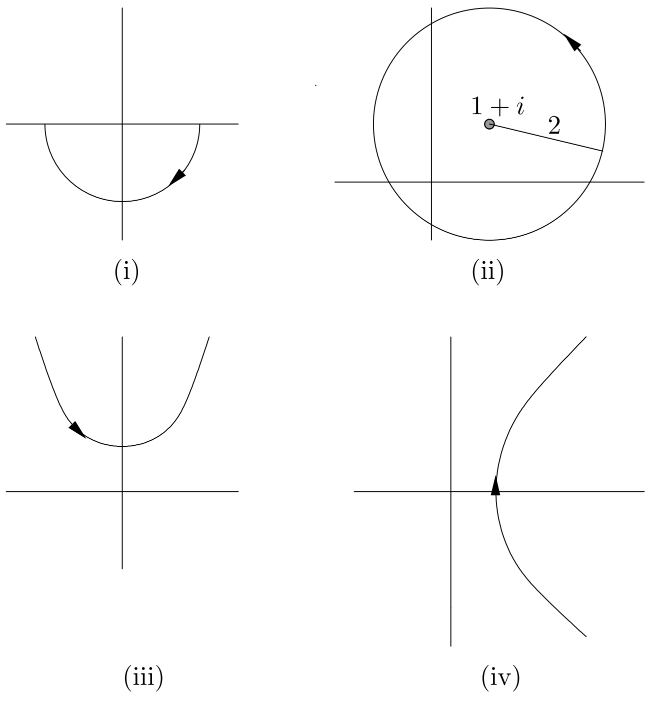

Figure 11.1: See Solution 4.1.

Solution 4.2

Let $f(z) = x - y + ix^2$ where $z = x + iy$.

(i) The straight line from 0 to $1+i$ has parametrisation $\gamma(t) = t + it$, $0 \le t \le 1$. Here $\gamma'(t) = 1+i$ and $f(\gamma(t)) = t - t + it^2 = it^2$. Hence

$$\begin{aligned}\int_\gamma x - y + ix^2 &= \int_0^1 it^2(1+i)\,dt \\ &= \int_0^1 -t^2 + it^2\,dt \\ &= \left[\frac{-1}{3}t^3 + \frac{i}{3}t^3\right]_0^1 \\ &= \frac{-1}{3} + \frac{i}{3}.\end{aligned}$$

(ii) The imaginary axis from 0 to i has parametrisation $\gamma(t) = it$, $0 \le t \le 1$. Here $\gamma'(t) = i$ and $f(\gamma(t)) = -t$. Hence

$$\begin{aligned}\int_\gamma x - y + ix^2 &= \int_0^1 -it\,dt \\ &= \left[\frac{-it^2}{2}\right]_0^1 \\ &= \frac{-i}{2}.\end{aligned}$$

(iii) The line parallel to the real axis from i to $1+i$ has parametrisation $\gamma(t) = t + i$, $0 \le t \le 1$. Here $\gamma'(t) = 1$ and $f(\gamma(t)) = t - 1 + it^2$. Hence

$$\begin{aligned}\int_\gamma x - y + ix^2 &= \int_0^1 (t - 1 + it^2)\,dt \\ &= \left[\frac{t^2}{2} - t + \frac{it^3}{3}\right]_0^1 \\ &= \frac{-1}{2} + \frac{i}{3}.\end{aligned}$$

Solution 4.3

The path γ_1 is the circle of radius 2, centre 2, described anticlockwise. The path γ_2 is the arc of the circle of radius 1, centre i, from $i+1$ to 0, described clockwise.

(i) Let $f(z) = 1/(z-2)$. Note that

$$f(\gamma_1(t)) = \frac{1}{2e^{it}}$$

and

$$\gamma_1'(t) = 2ie^{it}.$$

Hence

$$\int_{\gamma_1} \frac{dz}{z-2} = \int_0^{2\pi} \frac{1}{2e^{it}} 2ie^{it}\,dt = 2\pi i.$$

(ii) Let $f(z) = 1/(z-i)^3$. Note that

$$f(\gamma_2(t)) = \frac{1}{e^{-3it}}$$

and

$$\gamma_2'(t) = -ie^{-it}.$$

Hence

$$\int_{\gamma_2} \frac{dz}{(z-i)^3} = \int_0^{\pi/2} \frac{1}{e^{-3it}}(-ie^{-it})\,dt = -i\int_0^{\pi/2} e^{2it}\,dt = \frac{-i}{2i}\left[e^{2it}\right]_0^{\pi/2} = 1.$$

Solution 4.4
Let γ denote the path determined by the circle with centre 1 and radius 1 described once anti-clockwise and starting from the point 2. Then γ has parametrisation $\gamma(t) = 1 + e^{it} = 1 + \cos t + i\sin t$, $0 \leq t \leq 2\pi$. Here $\gamma'(t) = ie^{it}$. Note that

$$f(\gamma(t)) = |\gamma(t)|^2 = (1+e^{it})(1+e^{-it}) = 2 + e^{it} + e^{-it}$$

where we have used the fact that $|z|^2 = z\bar{z}$ for $z \in \mathbb{C}$.Hence

$$\begin{aligned}
\int_\gamma f(z)\,dz &= \int_0^{2\pi} f(\gamma(t))\gamma'(t)\,dt \\
&= \int_0^{2\pi} (2 + e^{it} + e^{-it})ie^{it}\,dt \\
&= \int_0^{2\pi} 2ie^{it} + ie^{2it} + i\,dt \\
&= 2e^{it} + \frac{1}{2}e^{2it} + it\Big|_0^{2\pi} \\
&= 2\pi i.
\end{aligned}$$

(Alternatively, we could have written $\gamma(t) = 1+\cos t + i\sin t$) and the n write $f(\gamma(t)) = (1+\cos t)^2 + \sin^2 = 2(1+\cos t)$ and $\gamma'(t) = -\sin t + i\cos t$. We can then use standard trig integrals to calculate $\int_0^{2\pi} 2(1+\cos t)(-\sin t + i\cos t)\,dt$.)

Solution 4.5
(i) We want to find a function F such that $F'(z) = f(z)$. We know that differentiation for complex functions obeys the same rules (chain rule, product rule, etc) as for real functions, so we first find an anti-derivative for the real function $f(x) = x^2\sin x$. Note that by integration by parts we have

$$\begin{aligned}
\int x^2 \sin x\,dx &= -x^2\cos x + \int 2x\cos x\,dx \\
&= -x^2\cos x + 2x\sin x - \int 2\sin x\,dx \\
&= -x^2\cos x + 2x\sin x + 2\cos x.
\end{aligned}$$

Let $F(z) = -z^2\cos z + 2z\sin z + 2\cos z$. Then clearly F is defined on $\mathbb{C}$ and one can check that $F'(z) = z^2\sin z$.

Hence, by the Fundamental Theorem of Contour Integration (Theorem 4.3.3), if γ is any smooth path from 0 to i then

$$\begin{aligned}\int_\gamma f = F(i) - F(0) &= -i^2 \cos i + 2i \sin i + 2\cos i - 2\cos 0 \\ &= 3\cosh 1 - 2\sinh 1 - 2.\end{aligned}$$

(ii) Again, let us first find an anti-derivative for $f(x) = xe^{ix}$. Integrating by parts gives

$$\begin{aligned}\int xe^{ix}\,dx &= -ixe^{ix} + \int ie^{ix}\,dx \\ &= -ixe^{ix} + e^{ix}\end{aligned}$$

(noting that $1/i = -i$). Hence $F(z) = -ize^{iz} + e^{iz}$ is an anti-derivative for f. Hence if γ is any smooth path from 0 to i then

$$\int_\gamma f = F(i) - F(0) = (-i^2e^{-1} + e^{-1}) - (0+1) = 2e^{-1} - 1.$$

Solution 4.6

(i) The contour γ that goes vertically from 0 to i and then horizontally from i to $1+i$ is the sum of the two paths

$$\begin{aligned}\gamma_1(t) &= it,\ 0 \leq t \leq 1, \\ \gamma_2(t) &= t + i,\ 0 \leq t \leq 1.\end{aligned}$$

Note that $|\gamma_1(t)|^2 = t^2$, $\gamma'(t) = i$, $|\gamma_2(t)|^2 = t^2 + 1$, $\gamma'(t) = 1$. Hence

$$\begin{aligned}\int_\gamma |z|^2\,dz &= \int_{\gamma_1} |z|^2\,dz + \int_{\gamma_2} |z|^2\,dz \\ &= \int_0^1 it^2\,dt + \int_0^1 t^2 + 1\,dt \\ &= \left[\frac{it^3}{3}\right]_0^1 + \left[\frac{t^3}{3} + t\right]_0^1 \\ &= \frac{4}{3} + \frac{i}{3}.\end{aligned}$$

(ii) Similarly, the contour γ that goes horizontally from 0 to 1 and then vertically from 1 to $1+i$ is the sum of the paths

$$\begin{aligned}\gamma_3(t) &= t,\ 0 \leq t \leq 1 \\ \gamma_4(t) &= 1 + it,\ 0 \leq t \leq 1.\end{aligned}$$

Here $|\gamma_3(t)|^2 = t^2$, $\gamma_3'(t) = 1$, $|\gamma_4(t)|^2 = 1 + t^2$, $\gamma_4'(t) = i$. Hence

$$\begin{aligned}\int_\gamma |z|^2\,dz &= \int_{\gamma_3} |z|^2\,dz + \int_{\gamma_4} |z|^2\,dz \\ &= \int_0^1 t^2\,dt + \int_0^1 (1+t^2)i\,dt \\ &= \left[\frac{1}{3}t^3\right]_0^1 + \left[it + \frac{it^3}{3}\right]_0^1 \\ &= \frac{1}{3} + \frac{4i}{3}.\end{aligned}$$

As the integral from 0 to $1+i$ depends on the choice of path, this tells us (by the Fundamental Theorem of Contour Integration) that $|z|^2$ does not have an anti-derivative.

Solution 4.7

One needs to be very careful with minus signs in this question!

Let γ denote the semi-circular path with centre 0 and radius 3, starting from 3 and travelling anticlockwise to -3. Then γ has a parametrisation given by $\gamma(t) = 3e^{it}$, $0 \leq t \leq \pi$.

Let $f(z) = 1/z^2$. Then

$$f(\gamma(t)) = \frac{1}{(3e^{it})^2} = \frac{1}{9e^{2it}}$$

and $\gamma'(t) = 3ie^{it}$. Hence

$$\begin{aligned}
\int_\gamma f &= \int_0^\pi \frac{1}{9e^{2it}} \times 3ie^{it}\,dt \\
&= \int_0^\pi \frac{i}{3}e^{-it}\,dt \\
&= \left.\frac{-1}{3}e^{-it}\right|_0^\pi \\
&= \frac{-1}{3}\left[e^{-i\pi} - 1\right] = \frac{2}{3}.
\end{aligned}$$

Recall that if $\gamma : [a,b] \to D$ is a parametrisation then $-\gamma$ has the parametrisation

$$-\gamma(t) = \gamma(a+b-t) : [a,b] \to D.$$

Here $-\gamma(t) = \gamma(0+\pi-t)$, $0 \leq t \leq \pi$, i.e. $-\gamma(t) = -3e^{-it}$, $0 \leq t \leq \pi$. One can check that this is correct by noting that $-\gamma(0) = -3$ (so we start at the point where γ ends), $-\gamma(\pi) = -3e^{-i\pi} = 3$ (so we end where γ begins), and $-\gamma(\pi/2) = -3e^{-i\pi/2} = 3i$ (so we are going around the 'top' half of the semi-circle, as we require).

Note that $(-\gamma)'(t) = 3ie^{-it}$ and $f(-\gamma(t)) = 1/(-3e^{-it})^2 = 1/9e^{-2it}$. Hence

$$\begin{aligned}
\int_{-\gamma} f &= \int_0^\pi \frac{1}{9e^{-2it}} \times 3ie^{-it}\,dt \\
&= \int_0^\pi \frac{i}{3}e^{it}\,dt \\
&= \left.\frac{1}{3}e^{it}\right|_0^\pi \\
&= \frac{1}{3}\left[e^{i\pi} - 1\right] = \frac{-2}{3} \\
&= -\int_\gamma f.
\end{aligned}$$

Solution 4.8

Let $\gamma : [a,b] \to D$ be a parametrisation of the contour γ. Then $-\gamma$ has

$$-\gamma(t) = \gamma(a+b-t) : [a,b] \to D$$

as a parametrisation. Note that $(-\gamma)'(t) = \gamma'(a+b-t) = -(\gamma'(a+b-t))$, using the chain rule for differentiation. Hence

$$\begin{aligned}
\int_{-\gamma} f &= \int_a^b f(-\gamma(t))(-\gamma'(t))\,dt \\
&= -\int_a^b f(\gamma(a+b-t))\gamma'(a+b-t)\,dt \\
&= \int_b^a f(\gamma(u))\gamma'(u)\,du \text{ using the substitution } u = a+b-t \\
&= -\int_a^b f(\gamma(t))\gamma'(t)\,dt \\
&= -\int_\gamma f.
\end{aligned}$$

Solution 4.9
Let $\gamma : [a,b] \to \mathbb{C}$ be a parametrisation of γ. Using the formula for integration by parts from real analysis, we can write

$$\begin{aligned}
\int_\gamma fg' &= \int_a^b f(\gamma(t))g'(\gamma(t))\gamma'(t)\,dt \\
&= \int_a^b f(\gamma(t))\frac{d}{dt}(g(\gamma(t)))\,dt \\
&= f(\gamma(t))g(\gamma(t))|_{t=a}^b - \int_a^b \frac{d}{dt}(f(\gamma(t)))g(\gamma(t))\,dt \\
&= f(z_1)g(z_1) - f(z_0)g(z_0) - \int_a^b f'(\gamma(t))g(\gamma(t))\gamma'(t)\,dt \\
&= f(z_1)g(z_1) - f(z_0)g(z_0) - \int_\gamma f'g.
\end{aligned}$$

Solution 4.10
See Figure 11.2.

Solution 4.11
Note that the function $f(z) = 1/(z^2-1)$ is not differentiable at $z = \pm 1$ (because it is not defined at $z = \pm 1$). To use the Generalised Cauchy Theorem, we need a domain that excludes these points. Let D be the domain

$$D = \{z \in \mathbb{C} \mid |z| < 3, |z-1| > 1/3,\ |z+1| > 1/3\}.$$

(There are lots of choices of D that will work. The 3 may be replaced by anything larger than 2; the 1/3 by anything less than 1/2—the point is that D should contain γ_1, γ_2 and γ but not ± 1. Alternatively, one could take $D = \mathbb{C} \setminus \{\pm 1\}$.) Obviously in this case D contains γ, γ_1 and γ_2. Let $\gamma_3(t) = 2e^{-it}$, $0 \leq t \leq 2\pi$, i.e. γ_3 is γ but described in the opposite direction.

Suppose $z \notin D$. If $|z| \geq 3$ then $w(\gamma_1, z) = w(\gamma_2, z) = w(\gamma_3, z) = 0$. If $|z-1| \leq 1/3$ then z is inside the contours γ_2 and γ_3 so that $w(\gamma_1, z) = 0$, $w(\gamma_2, z) = +1$, $w(\gamma_3, z) = -1$. Similarly if $|z+1| \leq 1/3$ then $w(\gamma_1, z) = 0$, $w(\gamma_2, z) = -1$, $w(\gamma_3, z) = +1$. Hence for each

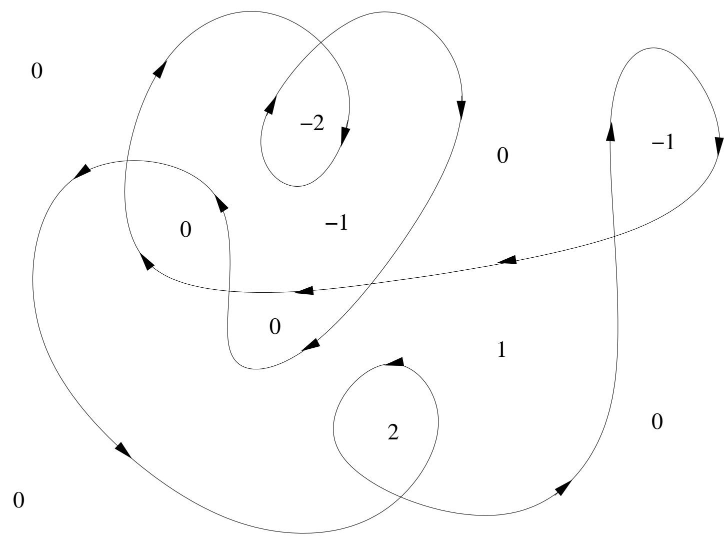

Figure 11.2: See Solution 4.7.

$z \notin D$ we have

$$w(\gamma_1, z) + w(\gamma_2, z) + w(\gamma_3, z) = 0.$$

Furthermore, since $\pm 1 \notin D$, the function f is holomorphic on D. Applying the Generalised Cauchy Theorem we have that

$$\int_{\gamma_1} f + \int_{\gamma_2} f + \int_{\gamma_3} f = 0$$

and the claim follows by noting that $\int_{\gamma_3} f = -\int_{\gamma} f$.

Solution 4.12
Let $\gamma_1(t) = e^{it}$, $0 \leq t \leq 2\pi$. Then

$$\int_{\gamma_1} f = \int_0^{2\pi} f(\gamma(t))\gamma'(t)\,dt = \int_0^{2\pi} \frac{1}{e^{it}} ie^{it}\,dt = 2\pi i.$$

Let $D = \mathbb{C} \setminus \{0\}$. We apply the Generalised Cauchy Theorem to the contours $\gamma_2, -\gamma_1$. The only point not in D is $z = 0$. Note that $w(\gamma_1, 0) = 1$ (so that $w(-\gamma_1, 0) = -1$) and $w(\gamma_2, 0) = 1$. Hence

$$w(\gamma_2, z) + w(-\gamma_1, z) = 1 - 1 = 0$$

for all $z \notin D$. By the Generalised Cauchy Theorem we have that

$$\int_{\gamma_2} f + \int_{-\gamma_1} f = 0.$$

Hence

$$\int_{\gamma_2} f = -\int_{-\gamma_1} f = \int_{\gamma_1} f = 2\pi i.$$

Solution 4.13
We apply the Generalised Cauchy Theorem (Theorem 4.5.7) to the closed contours $\gamma, -\gamma_1, \gamma_2$.

There are only two points not in D, namely z_1 and z_2. Note that

$$w(\gamma, z_1) = 1, w(-\gamma_1, z_1) = -1, w(\gamma_2, z_1) = 0$$

and

$$w(\gamma, z_2) = -1, w(-\gamma_1, z_2) = 0, w(\gamma_2, z_2) = 1.$$

Hence $w(\gamma, z) + w(-\gamma_1, z) + w(\gamma_2, z) = 0$ whenever $z \notin D$.

By the Generalised Cauchy Theorem, we have

$$\int_\gamma f + \int_{-\gamma_1} f + \int_{\gamma_2} f = 0.$$

Re-arranging this and noting that $\int_{-\gamma_1} f = -\int_{\gamma_1} f$ we have that

$$\int_\gamma f = -\int_{-\gamma_1} f - \int_{\gamma_2} f = (3 + 4i) - (5 + 6i) = -2 - 2i.$$

12. Solutions to Part 5

Solution 5.1

(i) Since $\sin^2 z = (1 - \cos 2z)/2$ and

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \tag{12.0.1}$$

we have that

$$\sin^2 z = \frac{1 - \cos 2z}{2} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4^n}{2(2n)!} z^{2n}. \tag{12.0.2}$$

As the radius of convergence for (12.0.1) is $R = \infty$, it follows that the radius of convergence for (12.0.2) is also $R = \infty$. (Alternatively, one could check this using the fact that

$$1/R = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{4^{n+1}}{2(2(n+1))!} \frac{2(2n)!}{4^n} = \lim_{n\to\infty} \frac{4}{2n(2n+1)} = 0$$

where a_n denote the coefficients in (12.0.2).)

(ii) Here

$$\frac{1}{1+2z} = 1 - 2z + 4z^2 - 8z^3 + \cdots = \sum_{n=0}^{\infty} (-2z)^n$$

(by recognising this as a sum of a geometric progression). The radius of convergence is given, using Proposition 3.2.2(ii), by

$$1/R = \lim_{n\to\infty} (2^n)^{1/n} = 2$$

so $R = 1/2$.

(iii) We have

$$e^{z^2} = \sum_{n=0}^{\infty} \frac{z^{2n}}{n!}$$

and the radius of convergence is $R = \infty$.

Solution 5.2

Let $f(z) = \mathrm{Log}(1+z)$. This is defined and holomorphic on the domain $\mathbb{C} \setminus \{z \in \mathbb{C} \mid \mathrm{Im}(z) = 0, \mathrm{Re}(z) < -1\}$. By Taylor's Theorem, we can expand f as a Taylor series at 0 valid on some disc centred at 0 as

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

where $a_n = \frac{1}{n!} f^{(n)}(0)$. Here

$$f'(z) = \frac{1}{1+z}, \; f''(z) = \frac{-1}{(1+z)^2}, \ldots$$

and in general

$$f^{(n)}(z) = \frac{(-1)^{n+1}(n-1)!}{(1+z)^n}.$$

Hence $a_n = (-1)^{n+1}(n-1)!/n! = (-1)^{n+1}/n$ for $n \neq 0$ and $a_0 = 0$.

By the ratio test, this power series has radius of convergence 1.

Solution 5.3

Let $p(z)$ be a polynomial of degree at least 1. Let $a \in \mathbb{C}$. We want to find $z_0 \in \mathbb{C}$ such that $p(z_0) = a$. Let $q(z) = p(z) - a$. Then q is a polynomial of degree at least 1. By the Fundamental Theorem of Algebra, there exists $z_0 \in \mathbb{C}$ such that $q(z_0) = 0$. Hence $p(z_0) = a$.

Solution 5.4

Since f is differentiable everywhere, for any $r > 0$ and any $m \geq 1$ we have that

$$|f^{(k+m)}(0)| \leq \frac{M_r (k+m)!}{r^{k+m}}$$

where $M_r = \sup\{|f(z)| \mid |z| = r\}$. Applying the bound on $|f(z)|$ we have that $M_r \leq K r^k$. Hence

$$|f^{(k+m)}(0)| \leq \frac{K(k+m)! r^k}{r^{k+m}} = \frac{K(k+m)!}{r^m}.$$

Since this holds for r arbitrarily large, by letting $r \to \infty$ we see that $f^{(k+m)}(0) = 0$. Substituting this into the Taylor expansion of f shows that f is a polynomial of degree at most k.

Solution 5.5

(i) Let $f(z) = |z+1|^2$ and let $z = x + iy$. Then

$$f(z) = |(x+1) + iy|^2 = (x+1)^2 + y^2.$$

Writing $f(z) = u(x, y) + iv(x, y)$ we have that $u(x, y) = (x+1)^2 + y^2$ and $v(x, y) = 0$. Hence

$$\frac{\partial u}{\partial x} = 2(x+1), \quad \frac{\partial v}{\partial y} = 0$$

and

$$\frac{\partial u}{\partial y} = 2y, \quad \frac{\partial v}{\partial x} = 0.$$

Suppose that z is a point on the unit circle γ and that $z \neq 1$; then at least one of $\partial u/\partial x \neq \partial v/\partial y$, $\partial u/\partial y \neq -\partial v/\partial x$ holds. (Note that the Cauchy-Riemann equations do hold at the point $z = -1$ and that the partial derivatives are continuous at $x = -1, y = 0$, hence by Proposition 2.5.2 f is differentiable at $z = -1$.) Hence f is not holomorphic on any domain that contains the unit circle γ.

(ii) Let z be a point on the unit circle γ. Then $\bar{z} = 1/z$. Hence

$$|z+1|^2 = (z+1)\overline{(z+1)} = (z+1)\left(\frac{1}{z}+1\right) = \frac{(z+1)^2}{z}$$

(note that this only holds on the unit circle γ). Let $g(z) = (z+1)^2/z$. Then g is holomorphic on $\mathbb{C} \setminus \{0\}$.

(iii) Let $h(z) = (z+1)^2$. Then

$$\begin{aligned}
\int_\gamma |z+1|^2 \, dz &= \int_\gamma \frac{(z+1)^2}{z} \, dz \\
&= \int_\gamma \frac{h(z)}{z-0} \, dz \\
&= 2\pi i h(0) \text{ by Cauchy's Integral Formula} \\
&= 2\pi i.
\end{aligned}$$

13. Solutions to Part 6

Solution 6.1
Let $f(z) = \sum_{m=-\infty}^{\infty} a_m z^m$ denote the Laurent series of f around $z = 0$.

(i) Since we are looking for an expansion valid for $|z| > 3$, we should look at powers of $1/z$:

$$\begin{aligned} \frac{1}{z-3} &= \frac{1/z}{1-(3/z)} \\ &= \frac{1}{z}\left(1 + \frac{3}{z} + \frac{3^2}{z^2} + \cdots\right) \\ &= \frac{1}{z} + \frac{3}{z^2} + \frac{3^2}{z^3} + \cdots + \frac{3^n}{z^{n+1}} + \cdots. \end{aligned}$$

(ii) Here

$$\frac{1}{1-z} = 1 + z + z^2 + \cdots$$

and this is valid for $|z| < 1$. Hence

$$\frac{1}{z(1-z)} = \frac{1}{z} + 1 + z + z^2 + \cdots$$

is valid for $0 < |z| < 1$.

(iii) For $z \neq 0$ we have

$$e^{1/z} = \sum_{m=0}^{\infty} \frac{1}{m! z^m} = \cdots + \frac{1}{n! z^n} + \cdots + \frac{1}{2! z^2} + \frac{1}{z} + 1.$$

Hence

$$z^3 e^{1/z} = \cdots + \frac{1}{(n+3)! z^n} + \cdots + \frac{1}{4! z} + \frac{1}{3!} + \frac{z}{2!} + z^2 + z^3.$$

(iv) Recall that

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + \frac{(-1)^n z^{2n}}{(2n)!} + \cdots.$$

For $z \neq 0$ we have

$$\cos 1/z = \cdots + \frac{(-1)^n}{(2n)! z^{2n}} + \cdots + \frac{1}{4! z^4} - \frac{1}{2! z^2} + 1.$$

Solution 6.2
Note that

$$\frac{1}{z+1} = \frac{1}{1-(-z)} = \sum_{n=0}^{\infty} (-z)^n = \sum_{n=0}^{\infty} (-1)^n z^n, \tag{13.0.1}$$

summing a geometric progression with common ratio $-z$. This expression converges for $|-z| < 1$, i.e. for $|z| < 1$.

We also have that

$$\frac{1}{z+1} = \frac{1}{z}\frac{1}{1-\left(\frac{-1}{z}\right)} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{-1}{z}\right)^n = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{z^n}, \tag{13.0.2}$$

summing a geometric progression with common ratio $-1/z$. This expression converges for $|-1/z| < 1$, i.e. for $|z| > 1$.

Similarly, we have that

$$\frac{1}{z-3} = \frac{-1}{3}\left(\frac{1}{1-\frac{z}{3}}\right) = \frac{-1}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n \tag{13.0.3}$$

and this is valid when $|z| < 3$. We also have that

$$\frac{1}{z-3} = \frac{1}{z}\frac{1}{1-\frac{3}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n = \sum_{n=1}^{\infty}\frac{3^{n-1}}{z^n}, \tag{13.0.4}$$

valid for $|z| > 3$.

Hence when $|z| < 1$ we have the Laurent expansion

$$f(z) = \sum_{n=0}^{\infty}\left((-1)^n - \frac{1}{3^{n+1}}\right)z^n.$$

For $1 < |z| < 3$ we have the Laurent expansion

$$f(z) = \cdots + (-1)^{n+1}\frac{1}{z^n} + \cdots - \frac{1}{z^2} + \frac{1}{z} - \frac{1}{3} - \frac{1}{3^2}z - \cdots - \frac{1}{3^n+1}z^n - \cdots.$$

For $|z| > 3$ we have the Laurent expansion

$$f(z) = \sum_{n=1}^{\infty}\left((-1)^{n+1} + 3^{n-1}\right)\frac{1}{z^n}.$$

Solution 6.3

(i) First note that

$$\begin{aligned}\frac{1}{z^2(z-1)} &= \frac{1}{z^2}\frac{-1}{1-z} \\ &= \frac{-1}{z^2}(1+z+z^2+\cdots+z^n+\cdots)\end{aligned}$$

and that this expansion is valid for $0 < |z| < 1$. Hence $f(z)$ has Laurent series

$$-\frac{1}{z^2} - \frac{1}{z} - 1 - z - z^2 - \cdots - z^n - \cdots$$

valid for $0 < |z| < 1$.

(ii) Let $w = z - 1$. Then $z = w + 1$ so that

$$\frac{1}{z^2(z-1)} = \frac{1}{(w+1)^2 w} = \frac{1}{w(1+w)^2}$$

Note that, using the hint,

$$\begin{aligned}\frac{1}{w(1+w)^2} &= \frac{1}{w}\sum_{n=0}^{\infty} n(-w)^{n-1} \\ &= \frac{1}{w}(1 - 2w + 3w^2 - 4w^3 + \cdots + (-1)^n n w^{n-1} + \cdots) \\ &= \frac{1}{w} - 2 + 3w - 4w^2 + \cdots + (-1)^n n w^{n-2} + \cdots\end{aligned}$$

and that this is valid provided that $0 < |w| < 1$. Substituting in for z we then have that

$$f(z) = \frac{1}{z-1} - 2 + 3(z-1) - 4(z-1)^2 + \cdots + (-1)^n n(z-1)^{n-2} + \cdots$$

and that this is valid for $0 < |z-1| < 1$.

Solution 6.4
Let $f(z) = 1/(z-1)^2$.

(i) Note that $1/(z-1)^2$ is already a Laurent series centred at 1. Hence f has Laurent series

$$f(z) = \frac{1}{(z-1)^2}$$

valid on the annulus $\{z \in \mathbb{C} \mid 0 < |z-1| < \infty\}$.

(ii) Note that $f(z) = 1/(z-1)^2$ is holomorphic on the disc $\{z \in \mathbb{C} \mid |z| < 1\}$. Therefore we can apply Taylor's theorem and expand f as a power series

$$f(z) = 1 + 2z + 3z^2 + \cdots + (n+1)z^n + \cdots$$

valid on the disc $\{z \in \mathbb{C} \mid |z| < 1\}$. (To calculate the coefficients, recall that if f has Taylor series $\sum_{n=0}^{\infty} a_n z^n$ then $a_n = f^{(n)}(0)/n!$. Here we can easily compute that $f^{(n)}(z) = (-1)^n(n+1)!(z-1)^{-n-2}$ so that $f^{(n)}(0) = (n+1)!$. Hence $a_n = n+1$. Alternatively, use the method given in Exercise 3.4.) As a Taylor series is a particular case of a Laurent series, we see that f has Laurent series

$$f(z) = 1 + 2z + 3z^2 + \cdots + (n+1)z^n + \cdots$$

valid on the disc $\{z \in \mathbb{C} \mid |z| < 1\}$.

(iii) Note that

$$\frac{1}{(z-1)^2} = \frac{1}{z^2}\frac{1}{\left(1-\frac{1}{z}\right)^2}.$$

Replacing z by $1/z$ in the first part of the computation in (ii) above, we see that

$$\frac{1}{\left(1-\frac{1}{z}\right)^2} = 1 + \frac{2}{z} + \frac{3}{z^2} + \cdots + \frac{n+1}{z^n} + \cdots$$

provided $|1/z| < 1$, i.e. provided $|z| > 1$. Multiplying by $1/z^2$ we see that

$$f(z) = \frac{1}{z^2} + \frac{2}{z^3} + \frac{3}{z^4} + \cdots + \frac{n-1}{z^n} + \cdots$$

valid on the annulus $\{z \in \mathbb{C} \mid 1 < |z| < \infty\}$.

Solution 6.5

Recall that a function $f(z)$ has a pole at z_0 if f is not differentiable at z_0 (indeed, it may not even be defined at z_0).

(i) The poles of $1/(z^2+1)$ occur when the denominator vanishes. Now $z^2+1 = (z-i)(z+i)$ so the denominator has zeros at $z = \pm i$ and both zeros are simple. Hence the poles of $1/(z^2+1)$ occur at $z = \pm i$ and both poles are simple.

(ii) The poles occur at the roots of the polynomial $z^4 + 16 = 0$. Let $z = re^{i\theta}$. Then we have

$$z^4 = r^4 e^{4i\theta} = -16 = 16e^{i\pi}.$$

Hence $r = 2$, $4\theta = \pi + 2k\pi$, $k \in \mathbb{Z}$. We get distinct values of z for $k = 0, 1, 2, 3$. Hence the poles are at

$$2e^{\frac{i\pi}{4} + \frac{ik\pi}{2}}, \ k = 0, 1, 2, 3,$$

or in algebraic form

$$\sqrt{2}(1+i), \ \sqrt{2}(1-i), \ \sqrt{2}(-1+i), \ \sqrt{2}(-1-i).$$

All the poles are simple.

(iii) The poles occur at the roots of $z^4 + 2z^2 + 1 = (z^2+1)^2 = (z+i)^2(z-i)^2$. The roots of this polynomial are at $z = \pm i$, each with multiplicity 2. Hence the poles occur at $z = \pm i$ and each pole is a pole of order 2.

(iv) The poles occur at the roots of $z^2 + z - 1$, i.e. at $z = (-1 \pm \sqrt{5})/2$, and both poles are simple.

Solution 6.6

(i) Since

$$\sin\frac{1}{z} = \sum_{m=0}^{\infty} (-1)^{2m+1} \frac{1}{(2m+1)! z^{2m+1}}$$

our function has infinitely many non-zero term in the principal part of its Laurent series. Hence we have an isolated essential singularity at $z = 0$.

(ii) By Exercise 5.1, the function $\sin^2 z$ has Taylor series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{2} \frac{2^{2n} z^{2n}}{(2n)!}.$$

Hence

$$z^{-3}\sin^2 z = \frac{1}{z} - \frac{2^4}{2 \cdot 4!} z + \frac{2^6}{2 \cdot 6!} z^3 - \cdots$$

so that $z^{-3}\sin^2 z$ has a simple pole at $z = 0$.

(iii) Since
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$
we have
$$\frac{\cos z - 1}{z^2} = \frac{-1}{2} + \frac{z^2}{4!} - \cdots$$
so that there are no terms in the principal part of the Laurent series. Hence 0 is a removable singularity.

Solution 6.7

Expand f as a Laurent series at z_0 and write

$$f(z) = \sum_{n=1}^{\infty} b_n(z - z_0)^{-n} + \sum_{n=0}^{\infty} a_n(z - z_0)^n,$$

valid in some annulus centred at z_0. Notice from Laurent's Theorem (Theorem 6.2.1) that

$$b_n = \frac{1}{2\pi i} \int_{C_r} f(z)(z - z_0)^{n-1}\, dz$$

where C_r is a circular path that lies on the domain D, centred at z_0 of radius $r > 0$, and described anticlockwise. By the Estimation Lemma we have that

$$|b_n| \leq \frac{1}{2\pi} \times 2\pi r \times Mr^{n-1} = Mr^n$$

as $|f(z)| \leq M$ at all points on C_r. As r is arbitrary and M is independent of r, we can let $r \to 0$ and conclude that $b_n = 0$ for all n. Hence there are no terms in the principal part of the Laurent series expansion of f at z_0, and so f has a removable singularity at z_0.

14. Solutions to Part 7

Solution 7.1

(i) The function $f(z) = 1/z(1-z^2)$ is differentiable except when the denominator vanishes. The denominator vanishes when $z = 0, \pm 1$ and these are all simple zeros. Hence there are simple poles at $z = 0, \pm 1$. Then by Lemma 7.4.1(i) we have

$$\begin{aligned}
\mathrm{Res}(f, 0) &= \lim_{z \to 0} z \frac{1}{z(1-z^2)} = \lim_{z \to 0} \frac{1}{1-z^2} = 1; \\
\mathrm{Res}(f, 1) &= \lim_{z \to 1} (z-1) \frac{1}{z(1-z^2)} = \lim_{z \to 1} \frac{-1}{z(1+z)} = \frac{-1}{2}; \\
\mathrm{Res}(f, -1) &= \lim_{z \to -1} (z+1) \frac{1}{z(1-z^2)} = \lim_{z \to -1} \frac{1}{z(1-z)} = \frac{-1}{2}.
\end{aligned}$$

(ii) Let $f(z) = \tan z = \sin z / \cos z$. Both $\sin z$ and $\cos z$ are differentiable on $\mathbb{C}$, so $f(z)$ is differentiable except when the denominator is 0. Hence f has poles at z where $\cos z = 0$, i.e. there are poles at $(n + 1/2)\pi$, $n \in \mathbb{Z}$. These poles are simple (as $(n + 1/2)\pi$ is a simple zero of $\cos z$). By Lemma 7.4.1(ii) we see that

$$\mathrm{Res}(f, (n+1/2)\pi) = \frac{\sin(n+1/2)\pi}{-\sin(n+1/2)\pi} = -1.$$

(iv) Let $f(z) = z/(1 + z^4)$. This has poles when the denominator vanishes, i.e. when $z^4 = -1$. To solve this equation, we work in polar coordinates. Let $z = re^{i\theta}$. Then $z^4 = -1$ implies that $r^4 e^{4i\theta} = e^{i\pi}$. Hence $r = 1$ and $4\theta = \pi + 2k\pi$. Hence the four quartic roots of -1 are:

$$e^{i\pi/4}, \; e^{3i\pi/4}, \; e^{-i\pi/4}, e^{-3i\pi/4}.$$

These are all simple zeros of $z^4 = -1$. Hence by Lemma 7.4.1(ii) we have that $\mathrm{Res}(f, z_0) = z_0/4z_0^3 = 1/4z_0^2$ so that

$$\begin{aligned}
\mathrm{Res}(f, e^{i\pi/4}) &= \frac{1}{4e^{i\pi/2}} = \frac{1}{4i} = \frac{-i}{4} \\
\mathrm{Res}(f, e^{3i\pi/4}) &= \frac{1}{4e^{3\pi/2}} = \frac{1}{-4i} = \frac{i}{4} \\
\mathrm{Res}(f, e^{-i\pi/4}) &= \frac{1}{4e^{-i\pi/2}} = \frac{1}{-4i} = \frac{i}{4} \\
\mathrm{Res}(f, e^{-3i\pi/4}) &= \frac{1}{4e^{-3i\pi/2}} = \frac{1}{4i} = \frac{-i}{4}.
\end{aligned}$$

(v) Let $f(z) = (z+1)^2/(z^2+1)^2$. Then the poles occur when the denominator is zero, i.e. when $z = \pm i$. Note that we can write

$$f(z) = \frac{(z+1)^2}{(z+i)^2(z-i)^2}.$$

Hence the poles at $z = \pm i$ are poles of order 2. By Lemma 7.4.2 (with $m = 2$) we have that

$$\begin{aligned}
\mathrm{Res}(f, i) &= \lim_{z\to i} \frac{d}{dz}(z-i)^2 f(z) \\
&= \lim_{z\to i} \frac{d}{dz}\frac{(z+1)^2}{(z+i)^2} \\
&= \lim_{z\to i} \frac{2(z+i)^2(z+1) - 2(z+1)^2(z-i)}{(z+i)^4} \\
&= \frac{2(2i)^2(i+1) - 2(i+1)^2(2i)}{(2i)^4} \\
&= \frac{-i}{2}.
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Res}(f, -i) &= \lim_{z\to -i} \frac{d}{dz}(z+i)^2 f(z) \\
&= \lim_{z\to -i} \frac{d}{dz}\frac{(z+1)^2}{(z-i)^2} \\
&= \lim_{z\to -i} \frac{2(z-i)^2(z+1) - 2(z+1)^2(z-i)}{(z-i)^4} \\
&= \frac{i}{2}.
\end{aligned}$$

Solution 7.2

(i) Let $f(z) = (\sin z)/z^2$. As $\sin z$ and z^2 are differentiable on $\mathbb{C}$, the poles occur when $z^2 = 0$. By considering the Taylor expansion of $\sin z$ around 0 we have that

$$\begin{aligned}
\frac{\sin z}{z^2} &= \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) \\
&= \frac{1}{z} - \frac{z}{3!} + \frac{z^2}{5!} - \cdots.
\end{aligned}$$

Hence $z = 0$ is a simple pole and $\mathrm{Res}(f, 0) = 1$.

(ii) Let $f(z) = (\sin^2 z)/z^4$. Recall that $\sin^2 z = (1 - \cos 2z)/2$. Hence

$$\begin{aligned}
\frac{1}{z^4}\sin^2 z &= \frac{1}{2z^4}(1 - \cos 2x) \\
&= \frac{1}{2z^4}\left(1 - 1 + \frac{(2z)^2}{2!} - \frac{(2z)^4}{4!} + \frac{(2z)^6}{6!} + \cdots\right) \\
&= \frac{1}{z^2} - \frac{2^3}{4!} + \frac{2^5}{6!}z^2 + \cdots
\end{aligned}$$

Hence f has a pole of order 2 at $z = 0$. The coefficient of the $1/z$ term in the above Laurent series for $(\sin^2 z)/z^4$ is 0. Hence $\mathrm{Res}(f, 0) = 0$.

Solution 7.3

(i) We have

$$\frac{1}{z(1-z)^2} = \frac{1}{z}(1 + 2z + 3z^2 + \cdots)$$

$$= \frac{1}{z} + 2 + 3z + 4z^4 + \cdots.$$

Hence f has a simple pole at $z = 0$ (as the most negative power appearing in the Laurent series at 0 is $1/z$). The residue $\operatorname{Res}(f, 0)$ is the coefficient of the term $1/z$ in the above Laurent series. Hence $\operatorname{Res}(f, 0) = 1$.

To calculate the Laurent series at $z = 1$ we change variables and introduce $w = z - 1$. Then $z = 1 + w$. Hence (noting that $w^2 = (-w)^2$ and summing a geometric progression)

$$\begin{aligned}
\frac{1}{z(1-z)^2} &= \frac{1}{w^2(1+w)} \\
&= \frac{1}{w^2(1-(-w))} \\
&= \frac{1}{w^2}\left(1 - w + w^2 - w^3 + \cdots\right) \\
&= \frac{1}{w^2} - \frac{1}{w} + 1 - w + w^2 - \cdots \\
&= \frac{1}{(z-1)^2} - \frac{1}{(z-1)} + 1 - (z-1) + (z-1)^2 - \cdots
\end{aligned}$$

Hence f has a pole of order 2 at $z = 1$ (as the most negative power appearing in the Laurent series at 1 is $1/(z-1)^2$). The residue $\operatorname{Res}(f, 1)$ is the coefficient of the term $(z-1)^{-1}$ in the above Laurent series. Hence $\operatorname{Res}(f, 1) = -1$.

(ii) By Lemma 7.2.2, $f(z) = 1/z(1-z)^2$ has a pole of order 1 at $z = 0$ and a pole of order 2 at $z = 1$.

By Lemma 7.4.1(i) we have

$$\operatorname{Res}(f, 0) = \lim_{z\to 0} z \times \frac{1}{z(1-z)^2} = \lim_{z\to 0} \frac{1}{(1-z)^2} = 1.$$

By Lemma 7.4.2 we have

$$\operatorname{Res}(f, 1) = \lim_{z\to 0} \frac{d}{dz}\left((z-1)^2 \times \frac{1}{z(1-z)^2}\right) = \lim_{z\to 0} \frac{d}{dz} z = \lim_{z\to 0} 1 = 1.$$

Solution 7.4

Suppose that f has a zero of order n at z_0. Then we can expand f as a Taylor series valid on a disc $\{z \in \mathbb{C} \mid |z - z_0| < r\} \subset D$:

$$f(z) = a_n(z - z_0)^n + a_{n+1}(z - z_0)^{n+1} + \cdots$$

with $a_n \neq 0$. Taking out a common factor of $(z - z_0)^n$ we can write

$$f(z) = (z - z_0)^n \phi(z)$$

where ϕ is holomorphic on $\{z \in \mathbb{C} \mid |z - z_0| < r\}$ and $\phi(z_0) = a_n \neq 0$.

Similarly, we can write $g(z) = (z - z_0)^m \psi(z)$ where ψ is holomorphic on a disc centred at z_0 and $\psi(z_0) \neq 0$.

Multiplying these together gives $f(z)g(z) = (z - z_0)^{n+m}\phi(z)\psi(z)$ where $\phi(z)\psi(z)$ is holomorphic on a disc centred at z_0 and $\phi(z_0)\psi(z_0) \neq 0$. Hence $f(z)g(z)$ has a zero of order $n + m$ at z_0.

Solution 7.5

(i) Let f denote the integrand. Note that

$$\frac{1}{z^2-5z+6} = \frac{1}{(z-2)(z-3)}$$

so that f has simple poles at $z=2, z=3$. Both of these poles are inside C_4. Hence

$$\int_{C_4} \frac{1}{z^2-5z+6}\,dz = 2\pi i\,\mathrm{Res}(f,2) + 2\pi i\,\mathrm{Res}(f,3).$$

Now by Lemma 7.4.1(i)

$$\begin{aligned}\mathrm{Res}(f,2) &= \lim_{z\to 2}\frac{(z-2)}{(z-2)(z-3)} = \lim_{z\to 2}\frac{1}{z-3} = -1\\ \mathrm{Res}(f,3) &= \lim_{z\to 3}\frac{(z-3)}{(z-2)(z-3)} = \lim_{z\to 3}\frac{1}{z-2} = 1.\end{aligned}$$

Hence

$$\int_{C_4} \frac{1}{z^2-5z+6}\,dz = 2\pi i - 2\pi i = 0.$$

(ii) Here we have the same integrand as in (i) but integrated over the smaller circle $C_{5/2}$. This time only the pole $z=2$ lies inside $C_{5/2}$. Hence

$$\int_{C_{5/2}} \frac{1}{z^2-5z+6}\,dz = 2\pi i\,\mathrm{Res}(f,2) = -2\pi i.$$

(iii) Let f denote the integrand. Note that

$$\frac{e^{az}}{1+z^2} = \frac{e^{az}}{(z+i)(z-i)}.$$

Hence f has simple poles at $z=\pm i$. Now

$$\begin{aligned}\mathrm{Res}(f,i) &= \lim_{z\to i}\frac{(z-i)e^{az}}{(z-i)(z+i)} = \lim_{z\to i}\frac{e^{az}}{z+i} = \frac{e^{ia}}{2i}\\ \mathrm{Res}(f,-i) &= \lim_{z\to -i}\frac{(z+i)e^{az}}{(z-i)(z+i)} = \lim_{z\to -i}\frac{e^{az}}{z-i} = -\frac{e^{-ia}}{2i}.\end{aligned}$$

Hence

$$\begin{aligned}\int_{C_2}\frac{e^{az}}{1+z^2}\,dz &= 2\pi i\,(\mathrm{Res}(f,i)+\mathrm{Res}(f,-i))\\ &= 2\pi i\left(\frac{e^{ia}}{2i} - \frac{e^{-ia}}{2i}\right)\\ &= 2\pi i \sin a.\end{aligned}$$

Solution 7.6

(a) (i) Note that $x^2+1 \geq x^2$. Hence $1/(x^2+1) \leq 1/x^2$. By Lemma 7.5.1, it follows that the integral is equal to its principal value.

(ii) Let $R > 1$. Let S_R denote the semi-circular path Re^{it}, $0 \leq t \leq \pi$ and let $\Gamma_R = [-R, R] + S_R$ denote the 'D-shaped' contour that travels along the real axis from $-R$ to R and then travels around the semi-circle of centre 0 and radius R lying in the top half of the complex plane from R to $-R$.
Let $f(z) = 1/(z^2 + 1)$. Then

$$\frac{1}{z^2+1} = \frac{1}{(z+i)(z-i)}$$

so f has simple poles at $z = \pm i$. Only the pole at $z = i$ lies inside Γ_R (assuming that $R > 1$). Note that

$$\mathrm{Res}(f, i) = \lim_{z \to i} \frac{(z-i)}{(z+i)(z-i)} = \lim_{z \to i} \frac{1}{z+i} = \frac{1}{2i}.$$

By Cauchy's Residue Theorem,

$$\int_{[-R,R]} f + \int_{S_R} f = \int_{\Gamma_R} f = 2\pi i \,\mathrm{Res}(f, i) = 2\pi i \frac{1}{2i} = \pi.$$

Now we show that the integral over S_R tends to zero as $R \to \infty$. On S_R we have that

$$|z^2 + 1| \geq |z^2| - 1 = R^2 - 1.$$

Hence by the Estimation Lemma,

$$\left| \int_{S_R} f \right| \leq \frac{1}{R^2 - 1} \,\mathrm{length}(S_R) = \frac{\pi R}{R^2 - 1} \to 0$$

as $R \to \infty$. Hence

$$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\, dx = \lim_{R \to \infty} \int_{[-R,R]} f = \pi.$$

(Without using complex analysis, you could have done this by noting that, in $\mathbb{R}$, $(x^2 + 1)^{-1}$ has anti-derivative $\arctan x$.)

(b) (i) Let $f(z) = e^{2iz}/(z^2 + 1)$. Note that when x is real

$$|f(x)| = \left| \frac{e^{2ix}}{x^2+1} \right| \leq \frac{1}{|x^2+1|} \leq \frac{1}{x^2}.$$

By Lemma 7.5.1, the integral is equal to its principal value.
Note that

$$f(z) = \frac{e^{2iz}}{z^2+1} = \frac{e^{2iz}}{(z-i)(z+i)}$$

so that f has simple poles at $z = \pm i$. Let Γ_R be the path as described in (a)(ii) above. Only the pole at $z = i$ lies inside this contour. See Figure 14.1. Note that

$$\mathrm{Res}(f, i) = \lim_{z \to i} \frac{(z-i)e^{2iz}}{(z-i)(z+i)} = \lim_{z \to i} \frac{e^{2iz}}{z+i} = \frac{e^{-2}}{2i}.$$

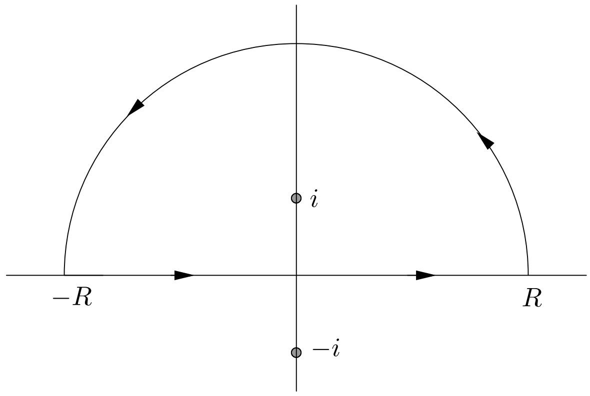

Figure 14.1: The contour Γ_R and the poles at $\pm i$.

By Cauchy's Residue Theorem,

$$\int_{[-R,R]} f + \int_{S_R} f = \int_{\Gamma_R} f = 2\pi i \operatorname{Res}(f, i) = 2\pi i \frac{e^{-2}}{2i} = \pi e^{-2}.$$

Now we show that the integral over S_R tends to zero as $R \to \infty$. On S_R we have that

$$|z^2 + 1| \geq |z^2| - 1 = |z|^2 - 1 = R^2 - 1.$$

Also, let $z = x + iy$ be a point on S_R. Then $0 \leq y \leq R$, so $-R \leq -y \leq 0$

$$|e^{2iz}| = |e^{2i(x+iy)}| = |e^{-2y+2ix}| = e^{-2y} \leq 1. \tag{14.0.1}$$

Hence $|f(z)| \leq 1/(R^2 - 1)$.
Hence by the Estimation Lemma,

$$\left| \int_{S_R} f \right| \leq \frac{1}{R^2 - 1} \operatorname{length}(S_R) = \frac{\pi R}{R^2 - 1} \to 0$$

as $R \to \infty$. Hence

$$\int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2 + 1}\, dx = \lim_{R \to \infty} \int_{[-R,R]} f = \pi e^{-2}.$$

(ii) Taking real and imaginary parts in the above we see that

$$\int_{-\infty}^{\infty} \frac{\cos 2x}{x^2 + 1}\, dx = \pi e^{-2}, \quad \int_{-\infty}^{\infty} \frac{\sin 2x}{x^2 + 1}\, dx = 0.$$

That the latter integral is equal to zero is obvious and we do not need to use complex integration to see this. Indeed, note that

$$\begin{aligned}
\int_{-\infty}^{\infty} \frac{\sin 2x}{x^2 + 1}\, dx &= \int_{-\infty}^{0} \frac{\sin 2x}{x^2 + 1}\, dx + \int_{0}^{\infty} \frac{\sin 2x}{x^2 + 1}\, dx \\
&= -\int_{0}^{\infty} \frac{\sin 2x}{x^2 + 1}\, dx + \int_{0}^{\infty} \frac{\sin 2x}{x^2 + 1}\, dx \\
&= 0
\end{aligned}$$

where we have used the substitution $x \mapsto -x$ in the first integral.

(iii) Now consider $f(z) = e^{-2iz}/(z^2+1)$. Suppose we tried to use the 'D-shaped' contour used in (ii) to calculate $\int_{-\infty}^{\infty} f(x)\,dx$. Then, with S_R as the semi-circle defined above, we would have to bound $|f(z)|$ on S_R in order to use the Estimation Lemma. However, if $z = x+iy$ is a point on S_R then, noting that $y \leq R$,

$$|e^{-2i(x+iy)}| = |e^{2y-2ix}| = |e^{2y}| \leq e^{2R}.$$

We still have the bound $1/|z^2+1| \leq 1/(R^2-1)$. So, using the Estimation Lemma,

$$\left|\int_{S_R} f\right| \leq \frac{e^{2R}}{R^2-1}\,\mathrm{length}(S_R) = \frac{e^{2R}\pi R}{R^2-1}$$

which does not tend to 0 as $R \to \infty$ (indeed, it tends to ∞).

Instead, we use a 'D-shaped' contour with the 'negative' semi-circle S'_R described by

$$Re^{-it},\ 0 \leq t \leq \pi.$$

We need to be careful about winding numbers and ensure that we travel around a contour in the correct direction to ensure that the contour is simple. Consider the contour Γ'_R which starts at $-R$ travels along the real axis to R, and then follows the negative semi-circle S'_R lying in the bottom half of the plane. If z is outside Γ'_R then $w(\Gamma'_R, z) = 0$; however, if z is inside Γ'_R then $w(\Gamma'_R, z) = -1$, so that Γ'_R is not a simple closed loop. However, $-\Gamma'_R$ is a simple closed loop and, moreover,

$$\int_{\Gamma'_R} f = -\int_{-\Gamma'_R} f.$$

See Figure 14.2.

The poles of f occur at $z = \pm i$ and both of these are simple poles. The only pole inside Γ'_R occurs at $z = -i$. Here

$$\mathrm{Res}(f,-i) = \lim_{z\to -i} \frac{(z+i)e^{-2iz}}{(z+i)(z-i)} = \lim_{z\to -i} \frac{e^{-2iz}}{z-i} = -\frac{e^{-2}}{2i}.$$

Hence

$$\int_{-\Gamma'_R} f = 2\pi i\,\mathrm{Res}(f,-i) = -\pi e^{-2}.$$

Note that if $z = x+iy$ is a point on S_R then

$$|z^2+1| \geq |z|^2 - 1 = R^2 - 1$$

and, as $-R \leq y \leq 0$

$$|e^{-2iz}| = |e^{2y-2ix}| = |e^{2y}| \leq 1.$$

Hence $|f(z)| \leq 1/(R^2-1)$ for z on Γ'_R. By the Estimation Lemma

$$\left|\int_{S'_R} f\right| \leq \frac{1}{R^2-1}\,\mathrm{length}(S'_R) = \frac{\pi R}{R^2-1} \to 0$$

as $R \to \infty$. Hence

$$-\int_{-R}^{R} f(x)\,dx - \int_{S'_R} f = \int_{-\Gamma'_R} f = -\pi e^{-2}$$

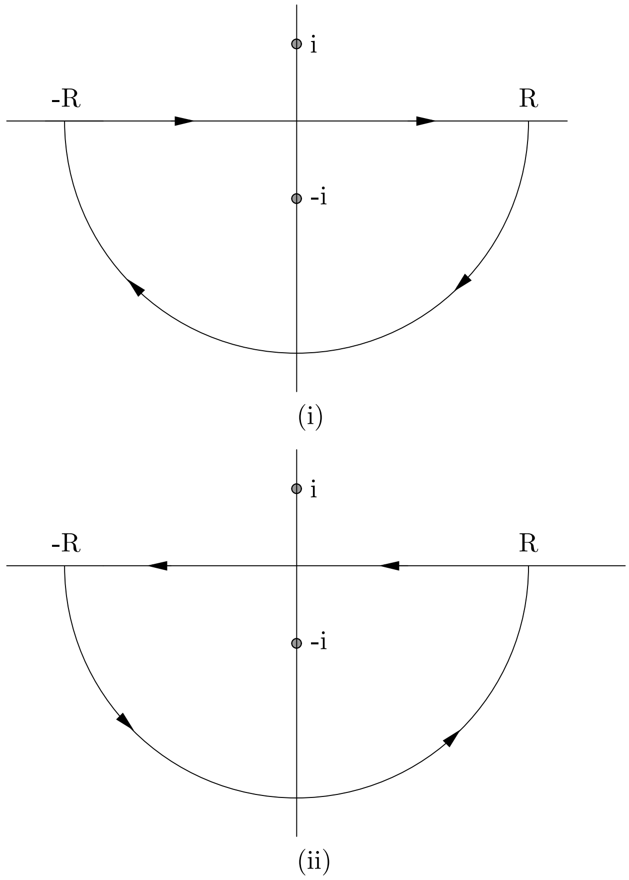

Figure 14.2: The contours (i) Γ'_R and (ii) $-\Gamma'_R$ and the poles at $\pm i$. Note that Γ'_R is not a simple closed loop but that $-\Gamma'_R$ is.

and letting $R \to \infty$ gives that

$$\int_{-\infty}^{\infty} \frac{e^{-2ix}}{x^2+1}\,dx = \pi e^{-2}.$$

Solution 7.7

We will use the same notation as in §7.5.2: S_R denotes the positive semicircle with centre 0 radius R, Γ_R denotes the contour $[-R, R] + S_R$.

(i) Let $f(z) = 1/(z^2+1)(z^2+3)$. Note that $(x^2+1)(x^2+4) \geq x^4$ so that $|f(x)| \leq 1/x^4$. Hence by Lemma 7.5.1 the integral converges and equals its principal value.

Now $f(z)$ has simple poles at $z = \pm i, \pm i\sqrt{3}$. Suppose $R > 3$. Then the poles at $z = i, i\sqrt{3}$ are contained in the 'D-shaped' contour Γ_R. Now

$$\begin{aligned}
\mathrm{Res}(f, i) &= \lim_{z \to i} \frac{z - i}{(z^2+1)(z^2+3)} \\
&= \lim_{z \to i} \frac{1}{(z+i)(z^2+3)} \\
&= \frac{1}{2i(-1+3)}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{4i} \\
\operatorname{Res}(f, i\sqrt{3}) &= \lim_{z \to i\sqrt{3}} \frac{z - i\sqrt{3}}{(z^2+1)(z^2+3)} \\
&= \lim_{z \to i\sqrt{3}} \frac{1}{(z^2+1)(z+i\sqrt{3})} \\
&= \frac{1}{(-3+1)2i\sqrt{3}} \\
&= \frac{-1}{4i\sqrt{3}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_{[-R,R]} f + \int_{S_R} f &= \int_{\Gamma_R} f \\
&= 2\pi i \left(\operatorname{Res}(f, i) + \operatorname{Res}(f, i\sqrt{3}) \right) \\
&= 2\pi i \left(\frac{1}{4i} - \frac{1}{4i\sqrt{3}} \right) \\
&= \frac{\pi}{2} \left(1 - \frac{1}{\sqrt{3}} \right).
\end{aligned}
$$

Now we show that the integral over S_R tends to 0 as $R \to \infty$. For z on S_R we have that

$$|(z^2+1)(z^2+3)| \geq (|z|^2 - 1)(|z|^2 - 3) = (R^2-1)(R^2-3).$$

Hence by the Estimation Lemma

$$\left| \int_{S_R} f \right| \leq \frac{1}{(R^2-1)(R^2-3)} \operatorname{length}(S_R) = \frac{\pi R}{(R^2-1)(R^2-3)} \to 0$$

as $R \to \infty$.

Hence

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+3)} = \lim_{R \to \infty} \int_{[-R,R]} f = \frac{\pi}{2} \left(1 - \frac{1}{\sqrt{3}} \right).$$

(ii) Note that

$$28 + 11x^2 + x^4 = (x^2+4)(x^2+7).$$

Let $f(z) = 1/(z^2+4)(z^2+7)$. Note that $(x^2+4)(x^2+7) \geq x^4$ so that $|f(x)| \leq 1/x^4$. Hence by Lemma 7.5.1 the integral converges and equals its principal value.

Now $f(z)$ has simple poles at $z = \pm 2i, \pm i\sqrt{7}$. Suppose $R > \sqrt{7}$. Then the poles at $z = 2i, i\sqrt{7}$ are contained in the 'D-shaped' contour Γ_R. Now

$$
\begin{aligned}
\operatorname{Res}(f, 2i) &= \lim_{z \to 2i} \frac{z - 2i}{(z^2+4)(z^2+7)} \\
&= \lim_{z \to 2i} \frac{1}{(z+2i)(z^2+7)} \\
&= \frac{1}{4i(-4+7)}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{12i} \\
\mathrm{Res}(f, i\sqrt{7}) &= \lim_{z \to i\sqrt{7}} \frac{z - i\sqrt{7}}{(z^2+4)(z^2+7)} \\
&= \lim_{z \to i\sqrt{7}} \frac{1}{(z^2+4)(z+i\sqrt{7})} \\
&= \frac{1}{(-7+4)2i\sqrt{7}} \\
&= \frac{-1}{6i\sqrt{7}}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\int_{[-R,R]} f + \int_{S_R} f &= \int_{\Gamma_R} f \\
&= 2\pi i \left(\mathrm{Res}(f, 2i) + \mathrm{Res}(f, i\sqrt{7}) \right) \\
&= 2\pi i \left(\frac{1}{12i} - \frac{1}{6i\sqrt{7}} \right) \\
&= \frac{\pi}{3} \left(\frac{1}{2} - \frac{1}{\sqrt{7}} \right).
\end{aligned}
$$

Now we show that the integral over S_R tends to 0 as $R \to \infty$. For z on S_R we have that

$$|(z^2+4)(z^2+7)| \geq (|z|^2-4)(|z|^2-7) = (R^2-4)(R^2-7).$$

Hence by the Estimation Lemma

$$\left| \int_{S_R} f \right| \leq \frac{1}{(R^2-4)(R^2-7)} \operatorname{length}(S_R) = \frac{\pi R}{(R^2-4)(R^2-7)} \to 0$$

as $R \to \infty$.

Hence

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+4)(x^2+7)} = \lim_{R \to \infty} \int_{[-R,R]} f = \frac{\pi}{3} \left(\frac{1}{2} - \frac{1}{\sqrt{7}} \right).$$

Solution 7.8

Let

$$f(z) = \frac{e^{iz}}{z^2+4z+5}.$$

Then $|f(x)| \leq \frac{C}{x^2}$ for some constant $C > 0$. Hence by Lemma 7.5.1 the integral $\int_{-\infty}^{\infty} f(x)\,dx$ exists and is equal to its principal value.

Now $f(z)$ has poles when $z^2+4z+5=0$, i.e. at $z = -2 \pm i$. Both of these poles are simple. Let Γ_R denote the 'D-shaped' contour $[-R, R] + S_R$. Provided R is sufficiently large, only the pole at $-2+i$ lies inside Γ_R. Now

$$
\begin{aligned}
\mathrm{Res}(f, -2+i) &= \lim_{z \to -2+i} \frac{(z-(-2+i))e^{iz}}{(z-(-2+i))(z-(-2-i))} \\
&= \lim_{z \to -2+i} \frac{e^{iz}}{z-(-2-i)}
\end{aligned}
$$

$$= \frac{e^{i(-2+i)}}{-2+i-(-2-i)}$$
$$= \frac{e^{-1-2i}}{2i}.$$

Hence
$$\int_{[-R,R]} f + \int_{S_R} f = \int_{\Gamma_R} f = 2\pi i\left(\frac{e^{-1-2i}}{2i}\right) = \pi e^{-1}\cos 2 - i\pi e^{-1}\sin 2.$$

Let $z = x+iy$ be a point on S_R. Then $0 \leq y \leq R$, so $e^{-y} \leq 1$.

$$|e^{iz}| = |e^{i(x+iy)}| = |e^{-y}| \leq 1.$$

Also, $|z^2+4z+5| \geq |z|^2 - 4|z| - 5 = R^2 - 4R - 5$. Hence on S_R

$$|f(z)| \leq \frac{1}{R^2-4R+5}.$$

By the Estimation Lemma,

$$\left|\int_{S_R} f\right| \leq \frac{1}{R^2-4R+5}\,\text{length}(S_R) = \frac{\pi R}{R^2-4R+5} \to 0$$

as $R \to \infty$. Hence
$$\lim_{R\to\infty} \int_{[-R,R]} f = \pi e^{-1}\cos 2 - i\pi e^{-1}\sin 2.$$

Taking the imaginary part we see that

$$\int_{-\infty}^{\infty} \frac{\sin x}{x^2+4x+5}\,dx = -\frac{\pi \sin 2}{e}.$$

Solution 7.9

(i) Let $z = e^{it}$ so that $\cos t = (z+z^{-1})/2$ and $dz = ie^{it}\,dt = iz\,dz$. Then

$$\int_0^{2\pi} \frac{1}{13+5\cos t}\,dt = \int_{C_1} \frac{1}{13+5\left(\frac{z+z^{-1}}{2}\right)}\frac{dz}{iz}$$
$$= \frac{2}{i}\int_{C_1} \frac{1}{5z^2+26z+5}\,dz.$$

(ii) Note that $5z^2+26z+5 = (z+5)(5z+1)$. Hence $f(z)$ has simple poles at $z=-5$ and $z=-1/5$.

We use Lemma 7.4.1(i) to calculate $\text{Res}(f,-1/5)$. Note that

$$\begin{aligned}
\text{Res}(f,-1/5) &= \lim_{z\to -1/5}(z+1/5)\frac{1}{(z+5)(5z+1)} \\
&= \lim_{z\to -1/5}\frac{5z+1}{5}\frac{1}{(z+5)(5z+1)} \\
&= \lim_{z\to -1/5}\frac{1}{5z+25} \\
&= \frac{1}{24}.
\end{aligned}$$

(iii) Note that only the pole at $z = -1/5$ lies inside C_1. Hence, by Cauchy's Residue Theorem,

$$\begin{aligned}
\int_0^{2\pi} \frac{1}{13+5\cos t}\,dt &= \frac{2}{i}\int_{C_1} \frac{1}{5z^2+26z+5}\,dz \\
&= 2\pi i \times \frac{2}{i} \times \mathrm{Res}(f, -1/5) \\
&= 2\pi i \times \frac{2}{i} \times \frac{1}{24} \\
&= \pi/6.
\end{aligned}$$

Solution 7.10
Denote by C the unit circle $C(t) = e^{it}$, $0 \leq t \leq 2\pi$.

(i) Substitute $z = e^{it}$. Then $dz = ie^{it}\,dt = iz\,dt$ so that $dt = dz/iz$ and $[0, 2\pi]$ transforms to C. Also, $\cos t = (z + z^{-1})/2$. Hence

$$\int_0^{2\pi} 2\cos^3 t + 3\cos^2 t\,dt = \int_C \left(\frac{(z+z^{-1})^3}{4} + \frac{3(z+z^{-1})^2}{4} \right) \frac{dz}{iz}.$$

Now

$$\begin{aligned}
\frac{(z+z^{-1})^3}{4} &= \frac{z^3+3z+3z^{-1}+z^{-3}}{4}, \\
\frac{(z+z^{-1})^2}{4} &= \frac{3z^2+6+3z^{-1}}{4}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_0^{2\pi} 2\cos^3 t + 3\cos^2 t\,dt \\
&= \int_C \frac{1}{i}\left(\frac{1}{4z^4} + \frac{3}{4z^3} + \frac{3}{4z^2} + \frac{3}{2z} + \frac{3}{4} + \frac{2z}{4} + \frac{z^2}{4} \right) dz.
\end{aligned}$$

Now the integrand has a pole of order 4 at $z = 0$, which is inside C, and no other poles. We can immediately read off the residue at $z = 0$ as the coefficient of $1/z$, namely $3/2i$. Hence by the Residue Theorem

$$\int_0^{2\pi} 2\cos^3 t + 3\cos^2 t\,dt = 2\pi i \frac{3}{2i} = 3\pi.$$

(ii) As before, substitute $z = e^{it}$. Then $dt = dz/iz$, $\cos t = (z + z^{-1})/2$ and $[0, 2\pi]$ transforms to C. Hence

$$\int_0^{2\pi} \frac{1}{1+\cos^2 t}\,dt = \int_C \frac{1}{1+(z+z^{-1})^2/4}\,\frac{dz}{iz} = \frac{1}{i}\int_C \frac{4z}{z^4+6z^2+1}\,dz$$

Let

$$f(z) = \frac{4z}{z^4+6z^2+1}.$$

This has poles where the denominator vanishes. The denominator is a quadratic in z^2 and we can find the zeros by the quadratic formula. Hence $f(z)$ has poles where

$$z^2 = \frac{-6 \pm \sqrt{36-4}}{2} = -3 \pm 2\sqrt{2}.$$

Now $-3-2\sqrt{2} < -1$. Hence the poles at $z = \pm i\sqrt{3+2\sqrt{2}}$ lie outside C. Note that $-1 < -3+2\sqrt{2} < 0$. Hence there are poles at $z = \pm i\sqrt{3-2\sqrt{2}}$ that lie inside C. Both of these poles are simple.

We have that

$$\begin{aligned}
&\operatorname{Res}\left(f, i\sqrt{3-2\sqrt{2}}\right) \\
&= \lim_{z\to i\sqrt{3-2\sqrt{2}}} \frac{(z-i\sqrt{3-2\sqrt{2}})4z}{(z-i\sqrt{3-2\sqrt{2}})(z+i\sqrt{3-2\sqrt{2}})(z^2-(-3-2\sqrt{2}))} \\
&= \lim_{z\to i\sqrt{3-2\sqrt{2}}} \frac{4z}{(z+i\sqrt{3-2\sqrt{2}})(z^2-(-3-2\sqrt{2}))} \\
&= \frac{4i\sqrt{3-2\sqrt{2}}}{2i\sqrt{3-2\sqrt{2}}(-3+2\sqrt{2}+3+2\sqrt{2})} \\
&= \frac{1}{2\sqrt{2}}
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{Res}\left(f, -i\sqrt{3-2\sqrt{2}}\right) \\
&= \lim_{z\to -i\sqrt{3-2\sqrt{2}}} \frac{(z+i\sqrt{3-2\sqrt{2}})4z}{(z+i\sqrt{3-2\sqrt{2}})(z-i\sqrt{3-2\sqrt{2}})(z^2-(-3-2\sqrt{2}))} \\
&= \lim_{z\to -i\sqrt{3-2\sqrt{2}}} \frac{4z}{(z-i\sqrt{3-2\sqrt{2}})(z^2-(-3-2\sqrt{2}))} \\
&= \frac{-4i\sqrt{3-2\sqrt{2}}}{-2i\sqrt{3-2\sqrt{2}}(-3+2\sqrt{2}+3+2\sqrt{2})} \\
&= \frac{1}{2\sqrt{2}}.
\end{aligned}$$

Hence

$$\begin{aligned}
&\int_0^{2\pi} \frac{1}{1+\cos^2 t}\,dt \\
&\quad = \frac{1}{i}\int_C \frac{4z}{z^4+6z^2+1}\,dz \\
&\quad = \frac{1}{i}2\pi i\left(\operatorname{Res}\left(f, i\sqrt{3-2\sqrt{2}}\right) + \operatorname{Res}\left(f, -i\sqrt{3-2\sqrt{2}}\right)\right) \\
&\quad = 2\pi\left(\frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\right) \\
&\quad = \sqrt{2}\pi.
\end{aligned}$$

Solution 7.11
Let

$$f(z) = \frac{1}{z^4}\cot \pi z = \frac{\cos \pi z}{z^4 \sin \pi z}$$

Then f has poles when the denominator vanishes, i.e. poles at $z = n$, $n \in \mathbb{Z}$. The pole at $z = n$, $n \neq 0$, is simple and by Lemma 7.4.1(ii) we have that

$$\mathrm{Res}(f, n) = \frac{\cos \pi n}{4n^3 \sin \pi n + n^4 \pi \cos \pi n} = \frac{1}{\pi n^4}.$$

When $z = 0$, we use the expansion for $\cot z$:

$$\cot z = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \cdots.$$

Hence

$$\frac{\cot \pi z}{z^4} = \frac{1}{\pi z^5} - \frac{\pi}{3z^3} - \frac{\pi^3}{45z} - \frac{2\pi^5 z}{945} - \cdots$$

so that $z = 0$ is a pole of order 5 and we can read off the residue as the coefficient of $1/z$. Hence $\mathrm{Res}(f, 0) = -\pi^3/45$.

Consider the square contour C_N described in §7.5.4. The poles at $z = -N, \ldots, 0, \ldots, N$ lie inside C_N. Hence, by Cauchy's Residue Theorem

$$\begin{aligned}
\int_{C_N} f &= 2\pi i \sum_{n=-N}^{N} \mathrm{Res}(f, n) \\
&= 2\pi i \left(\sum_{n=-N}^{-1} \frac{1}{\pi n^4} - \frac{\pi^3}{45} + \sum_{n=1}^{N} \frac{1}{\pi n^4} \right) \\
&= 2\pi i \left(\frac{2}{\pi} \sum_{n=1}^{N} \frac{1}{n^4} - \frac{\pi^3}{45} \right).
\end{aligned}$$

By Lemma 7.5.2, we have for z on C_N

$$|f(z)| \leq \frac{M}{|z|^4} \leq \frac{M}{N^4}.$$

Also, $\mathrm{length}(C_N) = 4(2N + 1)$. By the Estimation Lemma,

$$\left| \int_{C_N} f \right| \leq \frac{4M(2N + 1)}{N^4} \to 0$$

as $N \to \infty$. Hence

$$\lim_{N \to \infty} \frac{2}{\pi} \sum_{n=1}^{N} \frac{1}{n^4} - \frac{\pi^3}{45} = 0$$

and rearranging this gives

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(This method doesn't work for $\sum_{n=1}^{\infty} 1/n^3$. If we write $f(z) = z^{-3} \cot \pi z$ then $\mathrm{Res}(f, n) = 1/\pi n^3$ for $n \neq 0$ and $\mathrm{Res}(f, 0) = 0$. Summing over the residues we get

$$\sum_{n=-N}^{N} \frac{1}{\pi n^3} = \sum_{n=-N}^{-1} \frac{1}{\pi n^3} + \sum_{n=1}^{N} \frac{1}{\pi n^3} + 0$$

and the first two terms cancel, as $(-n)^3 = n^3$. So the residues on the negative integers cancel with the residues at the positive integers. Suppose we took a square contour that just enclosed the poles at the positive integers, say a square contour with corners at

$$\frac{1}{2} + iN, \frac{1}{2} - iN, N + \frac{1}{2} + iN, N + \frac{1}{2} - iN$$

(draw a picture!) then we cannot bound $f(z)$ on the edge from $\frac{1}{2} + iN, \frac{1}{2} - iN$ in such a way that the Estimation Lemma will then ensure that $|\int f|$ tends to zero. In fact, there is no known closed formula for $\sum_{n=1}^{\infty} 1/n^3$. See `http://en.wikipedia.org/wiki/Apery's_constant`.)

Solution 7.12

Suppose f has Laurent series $\sum_{n=-\infty}^{\infty} a_n(z - z_0)^n$ valid on the annulus $\{z \in \mathbb{C} \mid R_1 < |z - z_0| < R_2\}$. By Theorem 6.2.1 the coefficients a_n are given by

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

where C_r is a circular path described anticlockwise centred at z_0 and with radius r, where r is chosen such that $R_1 < r < R_2$.

(i) We calculate that Laurent series of $f(z) = 1/z(z - 1)$ valid on the annulus $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$. Here $z_0 = 0$. Choose $r \in (0, 1)$. We have that

$$a_n = \frac{1}{2\pi i} \int_{C_r} \frac{f(z)}{z^{n+1}} \, dz = \frac{1}{2\pi i} \int_{C_r} \frac{1}{z^{n+2}(z - 1)} \, dz$$

where C_r is the circular path with centre 0 and radius $r \in (0, 1)$, described once anticlockwise.

It is straightforward to locate the singularities of the integrand. For all $n \in \mathbb{Z}$ the integrand has a simple pole at 1. When $n \geq -1$, the integrand also has a pole of order $n + 2$ at 0.

For $n = -2, -3, \ldots$ the integrand has no poles inside C_r when $r < 1$. Hence, by Cauchy's Residue Theorem, $a_n = 0$ for $n = -2, -3, \ldots$. For $n \geq -1$, the pole at 0 lies inside C_r. We can calculate the residue of the integrand at 0 by using, for example, Lemma 7.4.2. Here

$$\begin{aligned}
\text{Res}\left(\frac{1}{z^{n+2}(z-1)}, 0\right) &= \lim_{z \to 0} \frac{1}{(n+1)!} \frac{d^{n+1}}{dz^{n+1}} \left(z^{n+2} \frac{1}{z^{n+2}(z-1)}\right) \\
&= \lim_{z \to 0} \frac{1}{(n+1)!} \frac{d^{n+1}}{dz^{n+1}} \left(\frac{1}{z-1}\right) \\
&= \lim_{z \to 0} \frac{1}{(n+1)!} (-1)^n (n+1)! \frac{1}{(z-1)^{n+1}} = -1.
\end{aligned}$$

Hence, by Cauchy's Residue Theorem, $a_n = -1$ for $n = -1, 0, 1, 2, \ldots$. Hence f has Laurent series

$$f(z) = -\frac{1}{z} - 1 - z - z^2 - z^3 - \cdots$$

valid on the annulus $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$.

We can check this directly by noting that

$$\begin{aligned}\frac{1}{z(z-1)} &= \frac{-1}{z(1-z)}\\ &= \frac{-1}{z}(1+z+z^2+z^3+\cdots)\\ &= \frac{-1}{z}-1-z-z^2-z^3-\cdots\end{aligned}$$

valid for $0<|z|<1$ (where we have used the sum to infinity of the geometric progression $1+z+z^2+\cdots = 1/(1-z)$).

(ii) We calculate the Laurent series of f valid on the annulus $\{z\in\mathbb{C} \mid 1<|z|<\infty\}$. Here $z_0=0$ and a_n is given by

$$\frac{1}{2\pi i}\int_{C_r}\frac{1}{z^{n+2}(z-1)}\,dz$$

where C_r is the circular path with centre 0 and radius r where r is now chosen such that $r\in(1,\infty)$. The integrand has, for all $n\in\mathbb{Z}$, a simple pole at $z=1$ and, for $n\geq -1$, a pole of order $n+2$ at 0. Both of these poles lie inside C_r.

We have already calculated, for $n\geq -1$, the residue of the pole at 0. Indeed,

$$\mathrm{Res}\left(\frac{1}{z^{n+2}(z-1)},0\right)=-1$$

for $n\geq -1$. The residue of the pole at 1 is given by

$$\mathrm{Res}\left(\frac{1}{z^{n+2}(z-1)},1\right)=\lim_{z\to 1}(z-1)\frac{1}{z^{n+2}(z-1)}=1.$$

Hence, by Cauchy's Residue Theorem,

$$a_n=\begin{cases}\mathrm{Res}\left(\frac{1}{z^{n+2}(z-1)},0\right)+\mathrm{Res}\left(\frac{1}{z^{n+2}(z-1)},1\right)=0, & \text{for } n\geq -1,\\ \mathrm{Res}\left(\frac{1}{z^{n+2}(z-1)},1\right)=1, & \text{for } n=-2,-3,\ldots.\end{cases}$$

Hence f has Laurent series

$$f(z)=\frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots$$

valid on the annulus $\{z\in\mathbb{C}\mid 1<|z|<\infty\}$.

To check this directly, first observe that

$$\left(1-\frac{1}{z}\right)^{-1}=1+\frac{1}{z}+\frac{1}{z^2}+\cdots$$

provided that $|z|>1$, by summing the geometric progression. Hence

$$\begin{aligned}\frac{1}{z(z-1)} &= \frac{1}{z^2\left(1-\frac{1}{z}\right)}=\frac{1}{z^2}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)\\ &= \frac{1}{z^2}+\frac{1}{z^3}+\frac{1}{z^4}+\cdots\end{aligned}$$

valid on the annulus $\{z\in\mathbb{C}\mid 1<|z|<\infty\}$.

(iii) We calculate the Laurent series of f valid on the annulus $\{z \in \mathbb{C} \mid 0 < |z-1| < 1\}$. Here $z_0 = 1$ and a_n is given by

$$a_n = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{(z-1)^{n+1}}\,dz = \frac{1}{2\pi i}\int_{C_r} \frac{1}{z(z-1)^{n+2}}\,dz$$

where C_r is the circular path with centre 1 and radius r, described once anticlockwise, where r is chosen such that $r \in (0,1)$. The integrand has, for all $n \in \mathbb{Z}$, a simple pole at 0 and, for $n \geq -1$, a pole of order $n+2$ at 1. As < 1, only the pole at 1 lies inside C_r. Hence, by Cauchy's Residue Theorem, $a_n = 0$ for $n = -2, -3, -4, \ldots$ and

$$a_n = \mathrm{Res}\left(\frac{1}{z(z-1)^{n+2}}, 1\right)$$

for $n \geq -1$. Using Lemma 7.4.2 we have that

$$\begin{aligned}
\mathrm{Res}\left(\frac{1}{z(z-1)^{n+2}}, 1\right) &= \lim_{z\to 1} \frac{1}{(n+1)!}\frac{d^{n+1}}{dz^{n+1}}\left((z-1)^{n+2}\frac{1}{z(z-1)^{n+2}}\right) \\
&= \lim_{z\to 1} \frac{1}{(n+1)!}\frac{d^{n+1}}{dz^{n+1}}\left(\frac{1}{z}\right) \\
&= \lim_{z\to 1} \frac{1}{(n+1)!}(n+1)!\frac{1}{z^{n+2}}(-1)^{n+1} = (-1)^{n+1}.
\end{aligned}$$

Hence f has Laurent series

$$f(z) = \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + (z-1)^3 - \cdots.$$

To check this directly, it is convenient to change variables and let $w = z-1$. Then

$$\frac{1}{z(z-1)} = \frac{1}{w(w+1)} = \frac{1}{w}\left(1 - w + w^2 - w^3 + \cdots\right)$$

where we have used the fact that

$$\frac{1}{1+w} = \frac{1}{1-(-w)} = 1 - w + w^2 - w^3 + \cdots,$$

summing the geometric progression. Hence

$$f(z) = \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + (z-1)^3 - \cdots.$$

valid on the annulus $\{z \in \mathbb{C} \mid 0 < |z-1| < 1\}$.

(iv) We calculate the Laurent series of f valid on the annulus $\{z \in \mathbb{C} \mid 1 < |z-1| < \infty\}$. Hence $z_0 = 1$ and a_n is given by

$$a_n = \frac{1}{2\pi i}\int_{C_r} \frac{f(z)}{(z-1)^{n+1}}\,dz = \frac{1}{2\pi i}\int_{C_r} \frac{1}{z(z-1)^{n+2}}\,dz$$

where C_r is the circular path with centre 1 and radius r, described once anticlockwise, where r is chosen such that $r \in (1,\infty)$. The integrand has, for all $n \in \mathbb{Z}$, a simple pole at 0 and, for $n \geq -1$, a pole of order $n+2$ at 1. Both of these poles lie inside C_r. We have already calculated that, for $n \geq 1$,

$$\mathrm{Res}\left(\frac{1}{z(z-1)^{n+2}}, 1\right) = (-1)^{n+1}.$$

The residue of the pole at 0 is given by

$$\mathrm{Res}\left(\frac{1}{z(z-1)^{n+2}}, 0\right) = \lim_{z\to 0} z\frac{1}{z(z-1)^{n+2}} = (-1)^{n+2} = (-1)^n.$$

We can now calculate the coefficients a_n by using Cauchy's Residue Theorem. When $n \geq -1$ we have that

$$a_n = \mathrm{Res}\left(\frac{1}{z(z-1)^{n+2}}, 0\right) + \mathrm{Res}\left(\frac{1}{z(z-1)^{n+2}}, 1\right) = (-1)^n + (-1)^{n+1} = 0.$$

When $n = -2, -3, \ldots$ we have that

$$a_n = \mathrm{Res}\left(\frac{1}{z(z-1)^{n+2}}, 1\right) = (-1)^n.$$

Hence f has Laurent series

$$f(z) = \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \frac{1}{(z-1)^4} - \cdots$$

valid on the annulus $\{z \in \mathbb{C} \mid 1 < |z-1| < \infty\}$.

To see this directly we again change variables and let $w = z - 1$. Hence

$$\begin{aligned}
\frac{1}{z(z-1)} &= \frac{1}{w(w+1)} \\
&= \frac{1}{w^2\left(1+\frac{1}{w}\right)} \\
&= \frac{1}{w^2}\left(1 - \frac{1}{w} + \frac{1}{w^2} - \frac{1}{w^3} + \cdots\right) \\
&= \frac{1}{w^2} - \frac{1}{w^3} + \frac{1}{w^4} - \cdots
\end{aligned}$$

for $|w| > 1$, by summing the geometric progression. Hence

$$f(z) = \frac{1}{(z-1)^2} - \frac{1}{(z-1)^3} + \frac{1}{(z-1)^4} - \cdots$$

valid on the annulus $\{z \in \mathbb{C} \mid 1 < |z-1| < \infty\}$.

Solution 7.13

We use the same notation as in §7.5.2: S_R denotes the positive semi-circle with centre 0 and radius R, Γ_R denotes the contour $[-R, R] + S_R$.

We will actually integrate

$$f(z) = \frac{ze^{iz}}{(z^2+a^2)(z^2+b^2)}.$$

Note that $|f(x)| \leq C/|x|^3$ for some constant $C > 0$. Hence by Lemma 7.5.1 the integral $\int_{-\infty}^{\infty} f(x)\,dx$ exists and is equal to its principal value.

This has poles where the denominator vanishes, i.e. at $z = \pm ia, \pm ib$, and all of these poles are simple. If R is taken to be larger than b then the poles inside Γ_R occur at $z = ia, ib$. We can calculate

$$\begin{aligned}
\mathrm{Res}(f, ia) &= \lim_{z \to ia} \frac{(z-ia)ze^{iz}}{(z-ia)(z+ia)(z^2+b^2)} \\
&= \lim_{z \to ia} \frac{ze^{iz}}{(z+ia)(z^2+b^2)} \\
&= \frac{iae^{-a}}{2ia(b^2-a^2)} \\
&= \frac{e^{-a}}{2(b^2-a^2)}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathrm{Res}(f, ib) &= \lim_{z \to ib} \frac{(z-ib)ze^{iz}}{(z-ib)(z+ib)(z^2+a^2)} \\
&= \lim_{z \to ib} \frac{ze^{iz}}{(z+ib)(z^2+a^2)} \\
&= \frac{ibe^{-b}}{2ib(-b^2+a^2)} \\
&= \frac{-e^{-b}}{2(b^2-a^2)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\int_{[-R,R]} f\,dz + \int_{S_R} f\,dz &= \int_{\Gamma_R} f\,dz \\
&= 2\pi i\left(\mathrm{Res}(f, ia) + Res(f, ib)\right) \\
&= \frac{2\pi i}{2(b^2-a^2)}(e^{-a} - e^{-b})
\end{aligned}$$

provided that $R > b$.

Now if z is a point on S_R then $|z| > R$. Hence

$$|(z^2+a^2)(z^2+b^2)| \geq (|z|^2 - a^2)(|z|^2 - b^2) = (R^2-a^2)(R^2-b^2).$$

Also, writing $z = x + iy$ so that $0 \leq y \leq R$, we have that $|e^{iz}| = |e^{i(x+iy)}| = |e^{-y+ix}| = |e^{-y}| \leq 1$. Hence

$$|f(z)| \leq \frac{R}{(R^2-a^2)(R^2-b^2)}.$$

By the Estimation Lemma,

$$\int_{S_R} f(z)\,dz \leq \frac{R}{(R^2-a^2)(R^2-b^2)}\,\mathrm{length}(S_R) = \frac{\pi R^2}{(R^2-a^2)(R^2-b^2)}$$

which tends to zero as $R \to \infty$. Hence

$$\int_{[-R,R]} f\,dz = \frac{\pi i}{(b^2-a^2)}(e^{-a} - e^{-b}).$$

By taking the imaginary part, we see that

$$\int_{-R}^{R} \frac{x \sin x}{(x^2+a^2)(x^2+b^2)}\,dx = \frac{\pi}{(b^2-a^2)}(e^{-a}-e^{-b}).$$

(As a check to see if we have made a mistake, note that the real part is zero. Hence

$$\int_{-\infty}^{\infty} \frac{x \cos x}{(x^2+a^2)(x^2+b^2)}\,dx = 0.$$

This is obvious as the integrand is an even function, and so must integrate (from $-\infty$ to ∞) to zero.)

Solution 7.14
Let

$$f(z) = \frac{\cot \pi z}{z^2+a^2},\ a \neq 0.$$

As

$$f(z) = \frac{\cos \pi z}{(z^2+a^2)\sin \pi z}$$

this has poles where the denominator vanishes, i.e. poles at $z = \pm ia$ and at $z = n$, $n \in \mathbb{Z}$. These poles are all simple. We can calculate

$$\begin{aligned}
\mathrm{Res}(f, ia) &= \lim_{z\to ia} \frac{(z-ia)\cos \pi z}{(z-ia)(z+ia)\sin \pi z} \\
&= \lim_{z\to ia} \frac{\cos \pi z}{(z+ia)\sin \pi z} \\
&= \frac{\cos i\pi a}{2ia \sin i\pi a} \\
&= \frac{\cosh \pi a}{-2a \sinh \pi a} \\
&= \frac{-\coth \pi a}{2a}
\end{aligned}$$

using the facts that $\cos iz = \cosh z$, $\sin iz = i \sinh z$. Similarly,

$$\begin{aligned}
\mathrm{Res}(f, -ia) &= \lim_{z\to -ia} \frac{(z+ia)\cos \pi z}{(z+ia)(z-ia)\sin \pi z} \\
&= \lim_{z\to -ia} \frac{\cos \pi z}{(z-ia)\sin \pi z} \\
&= \frac{\cos(-i\pi a)}{-2ia \sin(-i\pi a)} \\
&= \frac{\cosh \pi a}{-2a \sinh \pi a} \\
&= \frac{-\coth \pi a}{2a}.
\end{aligned}$$

For $z = n$, we use Lemma 7.4.1(ii) to see that

$$\mathrm{Res}(f, n) = \frac{\cos \pi n}{2n \sin \pi n + (n^2+a^2)\pi \cos \pi n} = \frac{1}{\pi(n^2+a^2)}.$$

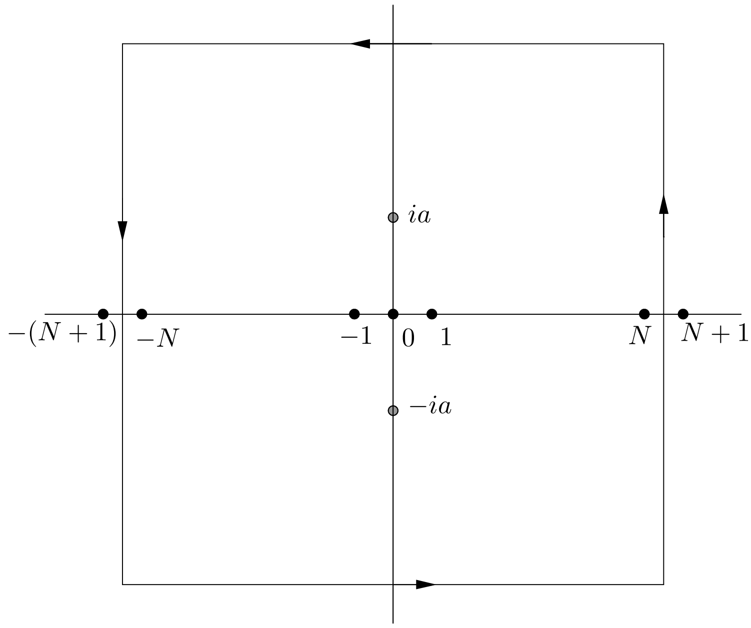

Figure 14.3: The contour C_N encloses the poles at $-N, \ldots, -1, 0, 1, \ldots, N$ and at $-ia, ia$ (if $N > |a|$).

(Note that unlike in the previous question this is valid for $z = 0$ as well. This is because $f(z)$ does not have a pole at $z = 0$ and so we have a simple pole at $z = 0$ for $f(z) \cot \pi z$.)

Let C_N denote the square contour as described in §7.5.4; see Figure 14.3. If $N > |a|$ then C_N encloses the poles at $z = -N, \ldots, 0, \ldots, N$ and $z = \pm ia$. Hence by Cauchy's Residue Theorem

$$\begin{aligned}
\int_{C_N} f &= 2\pi i \left(\sum_{n=-N}^{N} \mathrm{Res}(f, n) + \mathrm{Res}(f, ia) + \mathrm{Res}(f, -ia) \right) \\
&= 2\pi i \left(\sum_{n=-N}^{N} \frac{1}{\pi(n^2 + a^2)} - \frac{\coth \pi a}{2a} - \frac{\coth \pi a}{2a} \right) \\
&= 2\pi i \left(\sum_{n=-N}^{-1} \frac{1}{\pi(n^2 + a^2)} + \frac{1}{\pi a^2} + \sum_{n=1}^{N} \frac{1}{\pi(n^2 + a^2)} - \frac{1}{a} \coth \pi a \right) \\
&= 2\pi i \left(2 \sum_{n=1}^{N} \frac{1}{\pi(n^2 + a^2)} + \frac{1}{\pi a^2} - \frac{1}{a} \coth \pi a \right)
\end{aligned}$$

Note that if z is on C_N then $|z^2 + a^2| \geq |z|^2 - a^2 \geq N^2 - a^2$. Hence, by the bound on $\cot \pi z$ from Lemma 7.5.2, and the Estimation Lemma we have that

$$\left| \int_{C_N} f \right| \leq \frac{4M(2N+1)}{N^2 - a^2}$$

(as $\mathrm{length}(C_N) = 4(2N+1)$), which tends to zero as $N \to \infty$. Hence

$$\lim_{N \to \infty} 2\pi i \left(2 \sum_{n=1}^{N} \frac{1}{\pi(n^2 + a^2)} + \frac{1}{\pi a^2} - \frac{1}{a} \coth \pi a \right) = 0$$

and rearranging this gives

$$\sum_{n=1}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{2a} \coth \pi a - \frac{1}{2a^2}.$$

Solution 7.15

(i) Let $f(z) = e^z/z$. Note that e^z is holomorphic and non-zero on $\mathbb{C}$ and that $1/z$ is holomorphic on $\mathbb{C}$ except at $z = 0$ where it has a simple pole. Hence $f(z)$ has a simple pole at $z = 0$.

By Lemma 7.4.1(i), we have

$$\operatorname{Res}(f, 0) = \lim_{z \to 0} z \frac{e^z}{z} = \lim_{z \to 0} e^z = 1.$$

Noting that 0 lies inside C_1, Cauchy's Residue Theorem tells us that

$$\int_{C_1} \frac{e^z}{z}\, dz = 2\pi i \operatorname{Res}(f, 0) = 2\pi i.$$

(ii) Let $z = e^{it}$. Then $dz = ie^{it}\, dt = iz\, dt$. As z moves along C_1, we have that t varies between 0 and 2π. Hence, noting that $z = e^{it} = \cos t + i \sin t$,

$$\begin{aligned}
2\pi i &= \int_{C_1} \frac{e^z}{z}\, dz \\
&= \int_0^{2\pi} \frac{e^{\cos t + i \sin t}}{e^{it}} ie^{it}\, dt \\
&= i \int_0^{2\pi} e^{\cos t + i \sin t}\, dt \\
&= i \int_0^{2\pi} e^{\cos t} e^{i \sin t}\, dt \\
&= i \int_0^{2\pi} e^{\cos t} \left(\cos(\sin t) + i \sin(\sin t)\right)\, dt \\
&= -\int_0^{2\pi} e^{\cos t} \sin(\sin t)\, dt + i \int_0^{2\pi} e^{\cos t} \cos(\sin t)\, dt
\end{aligned}$$

Comparing real and imaginary parts gives the claimed integrals.

Solution 7.16

(i) Note that $e^x < 1 + e^x$ so that $1/(1 + e^x) < 1/e^x$. Hence

$$\frac{e^{ax}}{1+e^x} \leq e^{(a-1)x}.$$

As $a \in (0, 1)$, we have that $a - 1 < 0$. Note that, if $x > 1$, we can choose $C > 0$ such that $x^2 \leq Ce^{(1-a)x}$. Hence $e^{(a-1)x} \leq C/x^2$, provided $x > 1$. Hence the hypotheses of Lemma 7.5.1 hold so that $\int_{-\infty}^{\infty} f(x)\, dx$ exists and is equal to the principal value $\lim_{R\to\infty} \int_{-R}^{R} f(x)\, dx$.

(ii) Let $f(z) = e^{az}/(1+e^z)$. Then f is holomorphic except when the denominator vanishes. Let $z = x + iy$. The denominator vanishes precisely when $e^{x+iy} = -1$. Taking the modulus we have that $e^x = 1$, so $x = 0$. The solutions of $e^{iy} = -1$ are precisely $y = (2k+1)\pi$, $k \in \mathbb{Z}$. Hence f has singularities at $z = (2k+1)\pi i$, $k \in \mathbb{Z}$.

Write $f(z) = p(z)/q(z)$ with $p(z) = e^{az}$, $q(z) = 1 + e^z$. As $q'(z) = e^z$, we have $q'((2k+1)\pi i) = -1 \neq 0$. So $(2k+1)\pi i$ is a simple zero of q. As $p((2k+1)\pi i) \neq 0$, it follows that $(2k+1)\pi i$ is a simple pole, for each $k \in \mathbb{Z}$.

From Lemma 7.4.1(ii), the residue at πi is $p(\pi i)/q'(\pi i) = -e^{a\pi i}$.

The locations of the poles are illustrated in Figure 14.4.

(iii) The contour Γ_R is illustrated in Figure 14.4. The contour Γ_R winds once around the

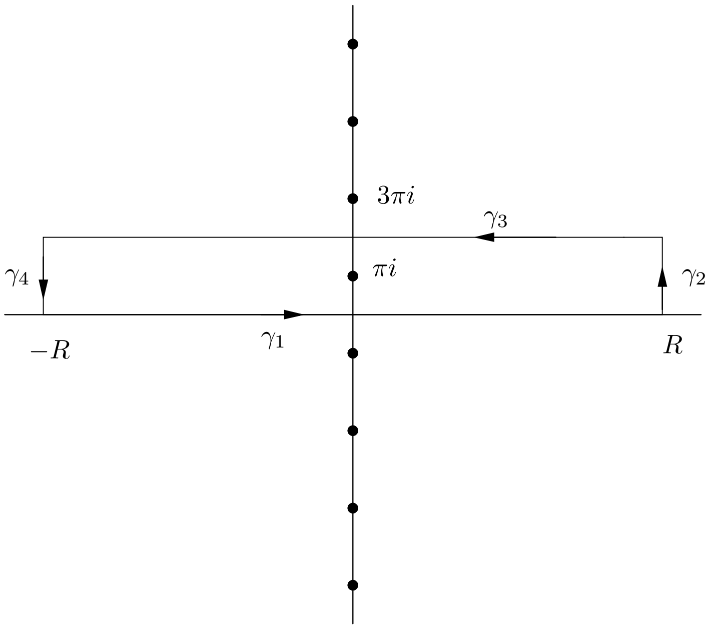

Figure 14.4: The poles of $f(z) = e^{az}/(1 + e^z)$ and the contour Γ_R.

pole at πi but not around any other pole. Hence, by Cauchy's Residue Theorem,

$$\int_{\Gamma_R} f = 2\pi i \operatorname{Res}(f, \pi i) = -2\pi i e^{-a\pi i}.$$

(iv) Choose the parametrisations

$$\gamma_{1,R}(t) = t, \ -R \leq t \leq R$$

and

$$\gamma_{2,R}(t) = -t + 2\pi i, \ -R \leq t \leq R$$

(note that $\gamma_{2,R}(t)$ starts at $R + 2\pi i$ and ends at $-R + 2\pi i$). Then

$$\int_{\gamma_{1,R}} f = \int_{-R}^{R} \frac{e^{at}}{1 + e^t} \, dt$$

and

$$\begin{aligned}
\int_{\gamma_{2,R}} f &= \int_{-R}^{R} \frac{e^{a(-t+2\pi i)}}{1+e^{-t+2\pi i}}(-1)\,dt \\
&= -\int_{-R}^{R} \frac{e^{a(s+2\pi i)}}{1+e^{s+2\pi i}}\,ds \text{ (substituting } s=-t) \\
&= -e^{2\pi i a}\int_{-R}^{R} \frac{e^{as}}{1+e^{s}}\,ds \\
&= -e^{2\pi i a}\int_{\gamma_{1,R}} f.
\end{aligned}$$

(v) First note that length$(\gamma_{2,R}) = 2\pi$. If z is a point on $\gamma_{R,2}$ then $z = R + it$ for some $0 \leq t \leq 2\pi$. Then, for z on $\gamma_{R,2}$ we have

$$|f(z)| \leq \sup_{t\in[0,2\pi]} \frac{|e^{a(R+it)}|}{|1+e^{R+it}|} \leq \frac{e^{aR}}{e^{R}-1}$$

(where we have used the reverse triangle inequality to bound the denominator). Hence, by the Estimation Lemma,

$$\left|\int_{\gamma_{2,R}} f\right| \leq \frac{2\pi e^{aR}}{e^{R}-1} \to 0$$

as $R \to \infty$, as $0 < a < 1$.

The case of $\gamma_{R,4}$ is similar. Again, length$(\gamma_{4,R}) = 2\pi$. If z is a point on $\gamma_{R,4}$ then $z = -R + it$ for some $0 \leq t \leq 2\pi$. Then, for z on $\gamma_{R,4}$ we have

$$|f(z)| \leq \sup_{t\in[0,2\pi]} \frac{|e^{a(-R+it)}|}{|1+e^{-R+it}|} \leq \frac{e^{-aR}}{1-e^{-R}}.$$

Hence, by the Estimation Lemma,

$$\left|\int_{\gamma_{4,R}} f\right| \leq \frac{2\pi e^{-aR}}{1-e^{-R}} \to 0$$

as $R \to \infty$, as $0 < a < 1$.

(vi) By parts (iii) and (iv) we know that

$$\begin{aligned}
-2\pi i e^{a\pi i} &= \int_{\Gamma_R} f \\
&= \int_{\gamma_{1,R}} f + \int_{\gamma_{2,R}} f + \int_{\gamma_{3,R}} f + \int_{\gamma_{4,R}} f \\
&= (1-e^{2\pi i a})\int_{-R}^{R} \frac{e^{ax}}{1+e^{x}}\,dx + \int_{\gamma_{2,R}} f + \int_{\gamma_{4,R}} f.
\end{aligned}$$

Letting $R \to 0$ and using part (v) we see that

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{ax}}{1+e^{x}}\,dx = -\frac{2\pi i e^{a\pi i}}{1-e^{2\pi i a}} = \frac{\pi}{\sin \pi a}.$$