برنامج التحليل 2

1) تكامل ريمان و الدوال الأصلية.

2) المعادلات التفاضلية العادية من الدرجة الأولى.

3) صيغ تايلر و النشر المحدود.

Analysis 2 program

1) Riemann integral and primitive functions.(or antiderivative).

2) First order ordinary differential equations.

3) Tyler's Formulas and Limited developments.

Riemann integral and primitives

1.1 Riemann integral

Definition.1.1 (partition)

A *partition* P of $[a, b]$ is a finite set of numbers $\{x_0, x_1, x_2, \ldots \ldots \ldots \ldots, x_{n-1}, x_n\}$ such that $a = x_0 < x_1 < x_2 < \ldots \ldots \ldots \ldots < x_{n-1} < x_n = b$

We write $\Delta x_i = x_i - x_{i-1}$.

We define the norm of *partition* P is the positive number $\|P\| = \max_{1\le i\le n}(x_i - x_{i-1})$.

Remak:

When the n subintervals have equal length $\Delta x_i = \frac{b-a}{n}$

The i^{th} term of the partition is $x_i = a + i\frac{b-a}{n}$ (This makes $x_n = b$.)

Definition 1.2 (Darboux sums)

Suppose $f: [a, b] \to \mathbb{R}$ is bounded and P is a partition of $[a, b]$. Define

$m_i = \inf\{f(x): x_{i-1} < x < x_i\}$ $\qquad M_i = \sup\{f(x): x_{i-1} < x < x_i\}$

$s(P, f) = \sum_{i=1}^{n} m_i \Delta x_i$ $\qquad S(P, f) = \sum_{i=1}^{n} M_i \Delta x_i$.

We call $s(P, f)$ the *lower Darboux sum* and $S(P, f)$ the *upper Darboux sum*.

Lemma.1.1. *Let P and Q be two partitions of* [a, b] *such that $P \subset Q$ Then*

$$s(P, f) \le s(Q, f)$$
$$S(P, f) \ge S(Q, f).$$

(The partition Q is called a refinement of P.)

Proof

First let us consider a particular case. Let P' be a partition formed from P by adding one extra point, say $c \in [x_{i-1}, x_i]$. Let $m_i' = \sup_{x_{i-1}\le x\le c} f(x)$, $m_i'' = \sup_{c\le x\le x_i} f(x)$.

Then $m_i' \ge m_i$, $m_i'' \ge m_i$, and we have

$$s(P', f) = \sum_{i=1}^{i-1} m_i \Delta x_{i-1} + m_i'(\mathrm{c} - x_{i-1}) + m_i''(x_i - \mathrm{c}) + \sum_{i=i+1}^{n} m_i \Delta x_{i+1}$$
$$\ge \sum_{i=1}^{i-1} m_i \Delta x_{i-1} + m_i(x_i - x_{i-1}) + \sum_{i=i+1}^{n} m_i \Delta x_{i+1} = s(P, f).$$

Similarly one obtains that

$S(P', f) \leq S(P, f)$.

Now to prove the assertion one has to P consequetly a finite number of points in order to form Q.

Lemma.1.2

Let P and Q be arbitrary partitions of [*a*, *b*]. *Then*

$$s(P, f) \leq S(Q, f).$$

Proof

Consider the partition $P \cup Q$. By Lemma 1 1 we have

$s(P, f) \leq s(P \cup Q, f) \leq S(P \cup Q, f) \leq S(Q, f)$.

Proposition 1.1

Let $f: [a, b] \to \mathbb{R}$ *be a bounded function. Let* $m, M \in \mathbb{R}$ *be such that for all* $x \in [a, b]$, *we have* $m < x < M$. *Then for every partition* P *of* $[a, b]$,

$$m(b - a) \leq s(P, f) \leq S(P, f) \leq M(b - a)$$

Proof

Let P be a partition of [*a*, *b*]. Note that $m \leq m_i \leq M_i \leq M$ for all i and $\sum_{i=1}^{n} \Delta x_i = (b - a)$. Therefore,

$$m(b - a) = \sum_{i=1}^{n} m \, \Delta x_i \leq \sum_{i=1}^{n} m_i \, \Delta x_i \leq \sum_{i=1}^{n} M_i \, \Delta x_i \leq \sum_{i=1}^{n} M \, \Delta x_i = M(b - a)$$

Definition 1.3

As the sets of lower and upper Darboux sums are bounded, we define

Lower Darboux integral $\underline{\int_a^b} f = \sup s(P, f): P \text{ a partition of } [a, b]$.

Upper Darboux integral $\overline{\int_a^b} f = \inf S(P, f) : P \text{ a partition of } [a, b]$.

Lemma.1.3

$$\underline{\int_a^b} f \leq \overline{\int_a^b} f.$$

Proof

Fix a partition Q. *Then by Lemma 1 2*

$$\forall P: s(P, f) \leq S(Q, f).$$

Therefore

$$\underline{\int_a^b} f = \sup_P s(P, f) \leq S(Q, f).$$

And from the above

$$\forall Q: \underline{\int_a^b} f \leq S(Q, f).$$

Hence

$$\underline{\int_a^b} f \leq \inf_Q S(Q, f) = \overline{\int_a^b} f.$$

Proposition 1.2. *Let* $f : [a, b] \to \mathbb{R}$ *be a bounded function. Let* $m, M \in \mathbb{R}$ *be such that for all* $x \in [a, b]$, *we have* $m \leq f(x) \leq M$. *Then*

$$m(b - a) \le \underline{\int_a^b} f \le \overline{\int_a^b} f \le M(b - a)$$

Proof. *By Proposition 1.1 , for every partition P,*

$$m(b-a) \le s(P,f) \le S(P,f) \le M(b-a)$$

The inequality $m(b-a) \le s(P,f)$ *implies* $m(b - a) \le \underline{\int_a^b} f$*.The inequality* $S(P,f) \le M(b-a)$ *implies* $\overline{\int_a^b} f \le M(b - a)$.

Definition 1.4.A function $f:[a,b] \to \mathbb{R}$ is called Riemann integrable if

$$\underline{\int_a^b} f = \overline{\int_a^b} f.$$

The common value is called integral of f over $[a,b]$ and is denoted by $\int_a^b f(x)\,dx$.

Proposition 1.3. Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function. Let $m, M \in \mathbb{R}$ be such that $m \le f(x) \le M$ for all $x \in [a,b]$. Then

$$m(b - a) \le \int_a^b f(x)\,dx \le M(b - a).$$

Proof Is a direct consequence of Proposition 1.2.

Example 1 1

We integrate constant functions. If $f(x) = c$ for some constant c, then we *take* $m = M = c$. In Proposition 1.3. Thus f is integrable on $[a,b]$ and

$$\int_a^b f(x)\,dx = c(b - a).$$

Theorem 1.1

A function $f:[a,b] \to \mathbb{R}$ is Riemann integrable if and only if for any $\varepsilon > 0$ there exists a partition P of [a, b] *such that $S(P,f) - s(P,f) < \varepsilon$.*

Proof

1 Necessity: Let $\underline{\int_a^b} f = \overline{\int_a^b} f$, i.e. let us assume that f is integrable.

$\exists P_1, P_2 : s(P_1,f) > \underline{\int_a^b} f - \frac{\varepsilon}{2}$ and $S(P_2,f) < \overline{\int_a^b} f + \frac{\varepsilon}{2}$.

Let $Q = P_1 \cup P_2$. Then

$$\underline{\int_a^b} f - \frac{\varepsilon}{2} < s(P_1,f) \le S(P_1 \cup P_2,f) \le S(P_1 \cup P_2,f) \le S(P_2,f) < \overline{\int_a^b} f + \frac{\varepsilon}{2}.$$

Therefore (since $\underline{\int_a^b} f = \overline{\int_a^b} f$)

$$S(Q,f) - s(Q,f) < \varepsilon.$$

2 sufficiency: Fix $\varepsilon > 0$. Let P be a partition such that $S(P,f) - s(P,f) < \varepsilon$.

Note that

$$\overline{\int_a^b} f - \underline{\int_a^b} f = S(P,f) - s(P,f) < \varepsilon.$$

Therefore it follows that

$$\forall \varepsilon > 0: \overline{\int_a^b} f - \underline{\int_a^b} f < \varepsilon.$$

This implies that

$$\overline{\int_a^b} f = \underline{\int_a^b} f.$$

Example 1.2

Let us show $f(x) = x^2$ is integrable on $[a, b]$ for all $b > a > 0$. We will see later that continuous functions are integrable, but let us demonstrate how we do it directly.

Let ε be given. Take $n \in \mathbb{N}$ and let $x_i = a + i\frac{b-a}{n}$ form the partition $P = \{x_0, x_1, x_2, \dots\dots\dots\dots, x_{n-1}, x_n\}$ of $[a, b]$. Then $\Delta x_i = \frac{b-a}{n}$ for all i. As f is increasing, for every subinterval $[x_{i-1}, x_i]$,

$$m_i = \inf\{f(x): x_{i-1} < x < x_i\} = \left(a + (i-1)\frac{b-a}{n}\right)^2$$

$$M_i = \sup\{f(x): x_{i-1} < x < x_i\} = \left(a + i\frac{b-a}{n}\right)^2$$

Then

$$\begin{aligned} S(P,f) - s(P,f) &= \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i \\ &= \frac{b-a}{n}\sum_{i=1}^{n}\left(\left(a + i\frac{b-a}{n}\right)^2 - \left(a + (i-1)\frac{b-a}{n}\right)^2\right) \\ &= \frac{b-a}{n}\sum_{i=1}^{n}\left(\left(a + n\frac{b-a}{n}\right)^2 - \left(a + 0\frac{b-a}{n}\right)^2\right) = \frac{\mathrm{b}^3}{n} \\ &= \frac{b-a}{n}(b^2 - a^2). \end{aligned}$$

Picking n to be such that, $\frac{b-a}{n}(b^2 - a^2) < \varepsilon$ the proposition is satisfied, and the function is integrable.

On the other hand, as we know from algebra (or can be proven by induction):

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad \text{and} \quad \sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}$$

So

$$\begin{aligned} S(P,f) &= \frac{b-a}{n}\sum_{i=1}^{n}\left(a + i\frac{b-a}{n}\right)^2 \\ &= \frac{b-a}{n}\left[\sum_{i=1}^{n} a^2 + 2a\frac{b-a}{n}i + \left(\frac{b-a}{n}\right)^2 i^2\right] \\ &= \frac{b-a}{n}\left[a^2 n + 2a\frac{b-a}{n}\sum_{i=1}^{n} i + \left(\frac{b-a}{n}\right)^2 \sum_{i=1}^{n} i^2\right] \\ &= \frac{b-a}{n}\left[a^2 n + 2a\frac{b-a}{n}\frac{n(n+1)}{2} + \left(\frac{b-a}{n}\right)^2 \frac{n(2n+1)(n+1)}{6}\right] \end{aligned}$$

$$= (b-a)\left[a^2 + a(b-a)\frac{(n+1)}{n} + (b-a)^2\frac{(2n+1)(n+1)}{6n^2}\right].$$

Similarly one obtains that

$$s(P,f) = (b-a)\left[a^2 + a(b-a)\frac{(n-1)}{n} + (b-a)^2\frac{(2n-1)(n-1)}{6n^2}\right].$$

So

$$\lim_{n\to\infty} S(P,f) = \lim_{n\to\infty} s(P,f) = \frac{(b-a)}{3}(b^2+ab+a^2) = \frac{1}{3}(b^2-a^2).$$

Finally we obtain

$$\int_a^b x^2 dx = \frac{1}{3}(b^2-a^2).$$

Definition 1.5 (Riemann sums)
Let $f: [a,b] \to \mathbb{R}$ be defined on the interval $[a,b]$ and let $P = \{x_0, x_1, x_2, \dots\dots\dots\dots, x_{n-1}, x_n\}$ be a partition of $[a,b]$.

Let $C = \{c_1, c_2, \dots\dots c_{i-1}, c_i, \dots. \quad c_{n-1}, c_n\}$ where c_i denote any value in the i^{th} subinterval ($c_i \in [x_{i-1}, x_i]$). The Riemann sum of a function f on $[a,b]$ that corresponds to P and the point system C is

$$R(P,f,\mathrm{C}) = \sum_{i=1}^{n} f(c_i)\,\Delta x_i.$$

Theorem 1.2
A function f is Riemann integrable on $[a,b]$ if there is a number L such that for each $\varepsilon > 0$ there is $\delta > 0$ such that if P is any partition of $[a,b]$ with $\|P\| < \delta$ then $|R(P,f,\mathrm{C}) - L| < \varepsilon$. (In other words $\lim_{\|P\|\to 0} R(P,f,\mathrm{C}) = L$).And we have $L = \int_a^b f(x)dx$.
The set of all Riemann integrable functions in $[a,b]$ is denoted by $\mathcal{R}([a,b])$.
Proof
1 Necessity: using $|R(P,f,\mathrm{C}) - L| \leq S(P,f) - s(P,f)$.

2 sufficiency: To do this, we will first show that

$$S(P,f) = \sup_{\mathrm{C}} R(P,f,\mathrm{C}), \quad s(P,f) \inf_{\mathrm{C}} R(P,f,\mathrm{C}).$$

Remark
If the function f is Riemann integrable on $[a,b]$ then the number $\int_a^b f(x)dx$ is the common limit of the two sequences $u_n = \frac{b-a}{n}\sum_{i=0}^{n-1} f(a + \frac{b-a}{n} i)$ and $v_n = \frac{b-a}{n}\sum_{i=1}^{n} f(a + \frac{b-a}{n} i)$.
Example

Calculate the limit of the sum $v_n = \sum_{i=0}^{n} \frac{n}{(n+i)^2}$.
We have

$$v_n = \sum_{i=0}^{n-1} \frac{n}{(n+i)^2} + \frac{1}{4n} = \frac{1}{n}\sum_{i=0}^{n-1} \frac{1}{\left(1+\frac{i}{n}\right)^2} + \frac{1}{4n}$$

by putting $[a,b] = [1,2]$; $f(x) = \frac{1}{x^2}$ then

$$v_n = \frac{b-a}{n}\sum_{i=0}^{n-1} f(a + \frac{b-a}{n} i\,) + \frac{1}{4n}.$$

So

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty}\left(\frac{1}{n}\sum_{i=0}^{n} f(1+\frac{1}{n} i\,)\right) + 0 = \int_a^b f(x)dx = \int_1^2 \frac{1}{x^2}dx = \frac{1}{2}$$

1.2 Integrable functions

Theorem 1.3

Let f f: $[\mathrm{a,b}] \to \mathbb{R}$ be monotone. Then f is Riemann integrable.

Proof

Suppose that f *is increasing so that* $f(a) \le f(b)$.

If $f(a) = f(b)$ then f is constant, so f *is Riemann integrable and* $\int_a^b f(x)dx = f(a)(b-a)$.

If $f(a) < f(b)$ let $\varepsilon > 0$ and P a partition of $[a,b]$ such that $\|P\| < \delta = \frac{\varepsilon}{f(b)-f(a)}$.

For this partition we obtain

$$S(P,f) - s(P,f) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\,\Delta x_i$$

$$< \delta \sum_{i=1}^{n} (f(x_i) - f(x_{i-1})) = \delta\big(f(b) - f(a)\big) = \varepsilon.$$

Theorem 1.4

Let f: $[a,b] \to \mathbb{R}$ be continuous. Then f is Riemann integrable.

Proof

Let $P = \{x_0, x_1, x_2, \dots\dots\dots\dots, x_{n-1}, x_n\}$ be a partition of $[a,b]$ and
$m_i = \inf\limits_{x_{i-1}\le x\le x_i} f(x)$; $M_i = \sup\limits_{x_{i-1}\le x\le x_i} f(x)$.
Let $\varepsilon > 0$. A continuous function on closed interval $[x_{i-1}, x_i]$ is uniformly continuous and reaches its upper and lower bounds at least once, so there exists $\delta > 0$ such that

$\forall x_{i-1}, x_i \in [a,b]: |x_i - x_{i-1}| < \delta \Longrightarrow |f(x_i) - f(x_{i-1})| < \frac{\varepsilon}{b-a}$ and there is at least x_i' ; x_i'' are from the subinterval $[x_{i-1}, x_i]$ where $m_i = f(x_i')$; $M_i = f(x_i'')$.
Choose a partition P such that $\|P\| < \delta$ so

$$S(P,f) - s(P,f) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i = \sum_{i=1}^{n} (f(x_i) - f(x_{i-1}))\,\Delta x_i$$

$$< \frac{\varepsilon}{b-a} \sum_{i=1}^{n} \Delta x_i = \frac{\varepsilon}{b-a}(b-a) = \varepsilon.$$

Theorem 1.5

If $f, g: [a,b] \to \mathbb{R}$ are integrable, then $fg: [a,b] \to \mathbb{R}$ is integrable. If, in addition, $g \neq 0$ and $\frac{1}{g}$ is bounded, then $\frac{f}{g}: [a,b] \to \mathbb{R}$ is integrable.

1.3. Properties of the Riemann integral

Let $f, g: [a,b] \to \mathbb{R}$ *Riemann integrable functions on* $[a,b]$. The integral has the following three basic properties.

1) Linearity:

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b f(x)dx \quad , \quad \int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx$$

2) Monotonicity:

If $\forall x \in [a,b]: f(x) \leq g(x), \text{then } \int_a^b f(x)dx \leq \int_a^b g(x)dx.$

3) Additivity:If $\alpha, \beta, \gamma \in [a,b]$, then

a) $\int_\alpha^\beta f(x)dx = \int_\alpha^\gamma f(x)dx + \int_\gamma^\beta f(x)dx.$

b). $\int_\alpha^\alpha f(x)dx = 0.$

c) $\int_\alpha^\beta f(x)dx = -\int_\beta^\alpha f(x)dx.$

4) If f is continuous on $[a,b]$ and $\forall x \in [a,b]: f(x) \geq 0$ then

$$\left(\int_a^b f(x)dx = 0\right) \Longrightarrow (\forall x \in [a,b]: f(x) = 0).$$

5) $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx.$

6) If f is continuous on $[a,b]$, then there exists $c \in [a,b]$, where

$$\int_a^b f(x)dx = f(c)(b-a).$$

1.4 Integrals and primitives

Definition 1.6

Let $f: [a,b] \to \mathbb{R}$ function, we say that function F is a primitive function of f over $[a,b]$ if and only if F is differentiable over $[a,b]$ and $\forall x \in [a,b]: F'(x) = f(x)$.

Proposition 1.4

If F_1 and F_2 are primitive functions of f on $[a,b]$ then$\forall x \in [a,b]: F_1(x) - F_2(x) = C$ where C is a real constant.

Example

The function $F(x) = \frac{1}{3}x^3$ is a primitive of the function $f(x) = x^2$ over $\mathbb{R}$ because

$$\forall x \in \mathbb{R}: F'(x) = \left(\frac{1}{3}x^3\right)' = x^2 = f(x).$$

Theorem 1.6 (The fundamental theorem of calculus 1)

if $F: [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable in $]a,b[$ with $F' = f$ where $f: [a,b] \to \mathbb{R}$ is Riemann integrable, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof

Let $P = \{x_0, x_1, x_2, \dots \dots \dots \dots, x_{n-1}, x_n\}$ be a partition of $[a,b]$.

The function F is continuous on the closed interval $[x_{i-1}, x_i]$ and differentiable in the open interval $]x_{i-1}, x_i[$ with $F' = f$. By the mean value theorem, there exists

$c_i \in]x_{i-1}, x_i[$ such that

$$F(x_i) - F(x_{i-1}) = F'(c_i)(x_i - x_{i-1})$$
$$= f(c_i)(x_i - x_{i-1})$$

Since f is Riemann integrable, it is bounded and it follows that

$$m_i(x_i - x_{i-1}) \le f(c_i)(x_i - x_{i-1}) \le M_i(x_i - x_{i-1})$$

or

$$m_i(x_i - x_{i-1}) \le F(x_i) - F(x_{i-1}) \le M_i(x_i - x_{i-1})$$

where

$$M_i = \sup_{x_{i-1} \le x \le x_i} f(x) \quad \text{and} \quad m_i = \inf_{x_{i-1} \le x \le x_i} f(x).$$

So

$$\sum_{i=1}^{n} m_i \, \Delta x_i \le \sum_{i=1}^{n} \big(F(x_i) - F(x_{i-1})\big) \, \Delta x_i \le \sum_{i=1}^{n} M_i \, \Delta x_i$$

Hence $s(P, f) \le F(b) - F(a) \le S(P, f)$ of every partition of $[a, b]$ wich implies that $\underline{\int_a^b} f \le F(b) - F(a) \le \overline{\int_a^b} f$. Since f is integrable i.e. $\underline{\int_a^b} f = \overline{\int_a^b} f$ we obtain

$$F(b) - F(a) = \int_a^b f(x) \, dx.$$

Theorem 1.7 (The fundamental theorem of calculus 2)

Suppose that $f: [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and $F: [a, b] \to \mathbb{R}$ is defined by

$\forall x \in [a, b]$: $F(x) = \int_a^x f(t) \, dt$.Then F is differentiable over $[a, b]$ and

$\forall x \in [a, b]$: $F'(x) = f(x)$ (that is, F is a primitive function of f over $[a, b]$).

Proof

Let $x, h \in [a, b]$ and $h > 0$. Then

$$\frac{F(x+h) - F(x)}{h} = \frac{\int_a^{x+h} f(t) \, dt - \int_a^x f(t) \, dt}{h} = \frac{1}{h} \int_x^{x+h} f(t) \, dt.$$

Let $\varepsilon > 0$. Since f is continuous at x there exists $\delta > 0$ such that

$$|f(t) - f(x)| < \varepsilon \quad \text{for} \quad |t - x| < \delta.$$

It follows that if $0 < h < \delta$ then

$$\left| \frac{F(x+h) - F(x)}{h} - f(x) \right| = \left| \frac{1}{h} \int_x^{x+h} f(t) \, dt - f(x) \right|$$

$$= \left| \frac{1}{h} \int_x^{x+h} \big(f(t) - f(x)\big) dt \right|$$

$$\le \frac{1}{h} \int_x^{x+h} |f(t) - f(x)| \, dt$$

$$\le \frac{1}{h} \sup_{x < t < x+h} |f(t) - f(x)| \left| \int_x^{x+h} \right| dt$$

$$\leq \frac{1}{h}\varepsilon h = \varepsilon.$$

So

$$\lim_{\substack{> \\ h\to 0}} \frac{F(x+h)-F(x)}{h} = f(x).$$

In the same way, we obtain

$$\lim_{\substack{< \\ h\to 0}} \frac{F(x+h)-F(x)}{h} = f(x).$$

Which proves the result.

Corollary 1.1

Let $f: [a,b] \to \mathbb{R}$ be continuous in $[a,b]$ and F is a primitive function of f over $[a,b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof Proof is a direct consequence of Theorem 1.7.

Example

Since $F(x) = \frac{1}{3}x^3$ is primitive function of $f(x) = x^2$ over $\mathbb{R}$. Then

$$\forall a,b \in \mathbb{R}: \int_a^b f(x)\,dx = F(b) - F(a) = \frac{1}{3}b^3 - \frac{1}{3}a^3.$$

Theorem 1.8 (Change of variables)

Let $\varphi : [a,b] \to$ ☒ *be a continuously differentiable function, let* f *be continuous over* $\varphi([a,b])$*, Then* $\int_a^b f(x)dx = \int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$*, where* $b = \varphi(\beta)$; $a = \varphi(\alpha)$ *and* $x = \varphi(t)$; $dx = \varphi'(t)dt$.

Proof

The function $f(\varphi)\varphi'$ *is continuous and therefore integrable.* Let F be a primitive of f and then $F(\varphi)$ is a primitive of $f(\varphi(t))\varphi'(t)$. So according to the Corollary 1.1,

$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt = F\big(\varphi(\beta)\big) - F\big(\varphi(\alpha)\big) = F(b) - F(a) = \int_a^b f(x)dx.$$

Example 1

Calculate the integral $J = \int_0^1 \sqrt{1-x^2}\,dx$. (Put $x = \varphi(t) = \sin t$).

$$x = \varphi(t) = \sin t \Longrightarrow dx = \cos t\,dt$$

$\sin\alpha = 0 \iff \alpha = 0, \pi, -\pi, 2\pi, -2\pi, \ldots$ (The value of α can be chosen from among the values $0, \pi, -\pi, 2\pi, -2\pi \ldots$).

$\sin\beta = 1 \iff \beta = \frac{\pi}{2}, \frac{-3\pi}{2}, \frac{5\pi}{2} \ldots$ (The value of β can be chosen from among the values $\frac{\pi}{2}, \frac{-3\pi}{2}, \frac{5\pi}{2} \ldots$).

So

$$J = \int_0^{\frac{\pi}{2}} \sqrt{1 - (\sin t)^2} \cos t \, dt = \int_0^{\frac{\pi}{2}} \sqrt{(\cos t)^2} \cos t \, dt.$$

Since $\forall x \in \left[0, \frac{\pi}{2}\right] : \cos t \geq 0$. Then

$$J = \int_0^{\frac{\pi}{2}} (\cos t)^2 \, dt = \frac{1}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2t) \, dt$$

$$= \frac{1}{2}\left[t + \frac{1}{2}\sin 2t\right]_0^{\frac{\pi}{2}} = \frac{\pi}{4}.$$

Example 2

Calculate the integral $K = \int_0^4 \frac{\sqrt{x}}{\sqrt{x}+1} dx$. (Put $x = \varphi(t) = t^2$).

$$x = \varphi(t) = t^2 \Longrightarrow dx = 2tdt.$$

$\varphi(\alpha) = a \quad \iff \alpha^2 = 0 \iff \alpha = 0$ (The value of α can be chosen from among the values

$\varphi\ \varphi(\beta) = b \iff \beta^2 = 4 \iff \beta = -2, \beta = 2$ (The value of β can be chosen from among the values $-2\,, 2$). So

$$K = \int_0^{-2} \frac{\sqrt{t^2}}{\sqrt{t^2}+1} 2tdt$$

Since $\forall t \in [-2,0]: t \leq 0$. Then

$$K = 2\int_{-2}^{0} \frac{t^2}{-t+1} dt = 2\int_{-2}^{0} (-t - 1 - \frac{1}{t-1}) \, dt$$

$$= 2\left[-\frac{1}{2}t^2 - t - ln|t-1|\right]_{-2}^{0} = 2ln3.$$

Theorem 1.9 (Integration by parts)*. Suppose that $u, v : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable in (a,b), and u', v' are integrable on $[a,b]$. Then*

$$\int_a^b uv'\,dx = [uv]_a^b - \int_a^b u'v\,dx.$$

Proof. *The function $u\,v$ is continuous on $[a,b]$ and, by the product rule, differentiable in (a,b) with derivative $(uv)' = u'v + uv'$. Since u, v, u' and v' are integrable on $[a,b]$. Theorem 1.4 implies that $u'v,\ uv'$ and $(uv)'$, are integrable. From Theorem 1.5, we get that $\int_a^b (uv' + u'v)\,dx = \int_a^b uv'\,dx + \int_a^b u'v\,dx = [uv]_a^b$, which proves the result.*

Example 1

calculate the integral $I = \int_0^1 Arc\,tan\,x\,dx$.

$$\begin{cases} v' = 1 \\ u = Arc\,tan\,x \end{cases} \Rightarrow \begin{cases} v = x \\ u' = \dfrac{1}{x^2+1} \end{cases}$$

$$I = [uv]_0^1 - \int_0^1 u'v\,dx = [xArc\,tan\,x]_0^1 - \int_0^1 \frac{1}{x^2+1}x\,dx$$

$$I = \left[xArc\,tan\,x - \frac{1}{2}ln(x^2+1)\right]_0^1 = \frac{\pi}{4} - \frac{1}{2}ln2.$$

Example 2

calculate the integral $I = \int_1^2 x\ln\frac{x}{x+1}\,dx$.

$$\begin{cases} v' = x \\ u = \ln\dfrac{x}{x+1} \end{cases} \Rightarrow \begin{cases} v = \dfrac{1}{2}x^2 \\ u' = \dfrac{1}{x(x+1)} \end{cases}$$

$$I = [uv]_1^2 - \int_1^2 u'v\,dx$$

$$I = \left[\frac{1}{2}x^2\ln\frac{x}{x+1}\right]_1^2 - \int_1^2 \frac{1}{2}x^2\frac{1}{x(x+1)}\,dx$$

$$I = \left[\frac{1}{2}x^2\ln\frac{x}{x+1}\right]_1^2 - \int_1^2 \frac{1}{2}x^2\frac{1}{x(x+1)}\,dx$$

$$I = \frac{5}{2}\ln2 - 2\ln3 - \int_1^2 \frac{1}{2}\frac{x}{(x+1)}\,dx$$

$$I = \frac{5}{2}\ln2 - 2\ln3 - \frac{1}{2}\int_1^2 1 - \frac{1}{(x+1)}\,dx$$

$$I=\frac{5}{2}\ln 2-2\ln 3-\frac{1}{2}[x-\ln(x+1)]_1^2$$

$$I=\frac{5}{2}\ln 2-2\ln 3-\frac{1}{2}[1+\ln 2-\ln 3]=2\ln 2-\frac{3}{2}\ln 3-\frac{1}{2}.$$

Definition 1.7.(The Indefinite Integral)
The set of all primitive functions of f is the ***indefinite integral*** of f with respect to x and denoted by $\int f(x)\,dx$ where
$\int f(x)\,dx$ is read " the integral of f w.r.t x ".
Note: The above definition says that if a function F is an primitive of f, then

$$\int f(x)\,dx=F(x)+C \quad \text{where C is a real constant.}$$

Example

$$\int x^3\,dx=\frac{1}{4}x^4+C$$

primitives of usual functions

$\int f(x)\,dx$	f
$\frac{1}{\alpha+1}x^{\alpha+1}+C$	$\left(\alpha\in\mathbb{R}^*-\{-1\}\text{حيث}\right)x^{\alpha}$
$ln\lvert x\rvert+C$	$\frac{1}{x}$
e^x+C	e^x
$-\cos x+C$	$\sin x$
$\sin x+C$	$\cos x$
$-ln\lvert\cos x\rvert+C$	$\tan x$
$\tan x+C$	$\frac{1}{\cos^2 x}$
$-cotan\,x+C$	$\frac{1}{\sin^2 x}$
$\cosh x+C$	$\sinh x$
$\sinh x+C$	$\cosh x$
$\frac{1}{a}Arctan\,\frac{x}{a}+C$	$\frac{1}{x^2+a^2}$
$Arcsin\,\frac{x}{a}+C$	$\frac{1}{\sqrt{a^2-x^2}}$
$\frac{1}{2a}ln\left\lvert\frac{x+a}{x-a}\right\rvert+C$	$\frac{1}{x^2-a^2}$
$\frac{1}{\alpha+1}\left(u(x)\right)^{\alpha+1}+C$	$\left(u(x)\right)^{\alpha}u'(x)$ $\left(u\in C^1(I)\text{ و }\alpha\in\mathbb{R}^*-\{-1\}\text{حيث}\right)$
$ln\lvert u(x)\rvert+C$	$\frac{u'(x)}{u(x)}$ $\left(u\in C^1(I)\text{ و }\forall x\in I: u(x)\neq 0\text{ حيث}\right)$

$e^{u(x)}+C$	$u'(x)e^{u(x)}$ $(u\in C^1(I)$حيث$)$
$G\big(u(x)\big)u'(x)+C$ G is a primitive of g over I	$g\big(u(x)\big)u'(x)$ Where $u \in C^1(I)$ and g *is continuous over* $u(I)$

Theorem 1.10 (change the variable)

Let $h: I \longrightarrow J$ C^1-diffeomorphism. We put $x=h(t)$ and $dx=h'(t)dt$ then

$$\int f(x)\,dx = \int f\big(h(t)\big)\,h'(t)dt \quad \text{and } t=h^{-1}(x).$$

Note: A function $h: I \longrightarrow J$ is called C^1-diffeomorphism if

a) h is a bejiction of I on J;

b) h and h^{-1} admit derivatives of order 1, continuous, respectively on I and J.

Example 1

Calculate $I=\int\sqrt{1-x^2}\,dx$.

We put $x=h(t)=\sin t$ where $h: \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\longrightarrow]-1,1[$ (h is C^1-diffeomorphism), and $dx = \cos t\,dt$.

So

$$I=\int\sqrt{1-sin^2 t}\cos t\,dt=\int\sqrt{\cos^2 t}\cos t\,dt.$$

Since $\forall t \in \left]-\frac{\pi}{2},\frac{\pi}{2}\right[: \cos t \geq 0$ we get

$$I=\int cos^2 t\,dt=\frac{1}{2}\int(1+\cos 2t)\,dt$$

$$=\frac{1}{2}\left(t+\frac{1}{2}\sin 2t\right)+C=\frac{1}{2}t+\frac{1}{2}\cos t\sin t+C$$

$$=\frac{1}{2}t+\frac{1}{2}\sqrt{1-sin^2 t}\sin t+C.$$

Substituting $t=h^{-1}(x)=Arcsinx$ we get the following result:

$$I=\frac{1}{2}Arcsinx+\frac{1}{2}x\sqrt{1-x^2}+C.$$

Example 2

Calculate $J=\int\frac{x}{\sqrt{x}+1}dx$.

We put $x = h(t) = t^2$ where $h:]0, +\infty[\longrightarrow]0, +\infty[$ (h is C^1-diffeomorphism), and $dx = 2tdt$.

So

$$J = \int \frac{t^2}{\sqrt{t^2} + 1} 2tdt$$

Since $\forall t \in \ ]0, +\infty[: t > 0$ we get

$$J = \int \frac{t^2}{\sqrt{t^2} + 1} 2tdt$$

$$= 2 \int \frac{t^3}{\mathrm{t} + 1} dt$$

$$= 2 \int \left(t^2 - \frac{1}{\mathrm{t} + 1} - \mathrm{t} + 1 \right) dt$$

$$= 2 \left(\frac{1}{3} t^3 - \ln(\mathrm{t} + 1) - \frac{1}{2} t^2 + \mathrm{t} \right)$$

$$J = \frac{2}{3} t^3 - 2\ln(\mathrm{t} + 1) - t^2 + 2\mathrm{t} + C.$$

Substituting $t = h^{-1}(x) = \sqrt{x}$ we get the following result:

$$J = \frac{2}{3} x\sqrt{x} - x + 2\sqrt{x} - 2\ln(\sqrt{x} + 1) + C.$$

Theorem 1.11 (Integration by parts)*.*

Let I be a interval for $\mathbb{R}$ and v, u are functions of class C^1 on the interval I then

$$\int uv' \, dx = uv - \int u'v \, dx.$$

Example 1

Calculate $I = \int xe^{2x} \, dx$. By putting:

$$\begin{cases} v' = e^{2x} \\ u = x \end{cases} \Rightarrow \begin{cases} v = \frac{1}{2} e^{2x} \\ u' = 1 \end{cases}$$

$$I = \int xe^{2x} \, dx = uv - \int u'v \, dx$$

$$= x \frac{1}{2} e^{2x} - \int \frac{1}{2} e^{2x} \, dx$$

$$= \left(\frac{1}{2} x - \frac{1}{4} \right) e^{2x} + C.$$

Example 2

Calculate $J = \int e^x \sin x \, dx$. By putting:

$$\begin{cases} v' = \sin x \\ u = e^x \end{cases} \Longrightarrow \begin{cases} v = -\cos x \\ u' = e^x \end{cases}.$$

We get

$$J = -e^x \cos x + \int e^x \cos x \, dx.$$

Again we put

$$\begin{cases} v' = \cos x \\ u = e^x \end{cases} \Longrightarrow \begin{cases} v = \sin x \\ u' = e^x \end{cases},$$

so

$$J = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

$$J = -e^x \cos x + e^x \sin x - J$$

$$2J = -e^x \cos x + e^x \sin x$$

we obtain

$$= -e^x \cos x + e^x \sin x + C.$$

Example 3*

Calculate $J = \int x\sqrt{x} \, dx$. By putting:

$$\begin{cases} v' = x \\ u = \sqrt{x} \end{cases} \Longrightarrow \begin{cases} v = \dfrac{1}{2}x^2 \\ u' = \dfrac{1}{2\sqrt{x}} \end{cases}.$$

We get

$$J = \frac{1}{2}x^2\sqrt{x} - \int \frac{1}{2\sqrt{x}}\frac{1}{2}x^2 \, dx$$

$$= \frac{1}{2}x^2\sqrt{x} - \frac{1}{4}\int x\sqrt{x}$$

$$= \frac{1}{2}x^2\sqrt{x} - \frac{1}{4}J$$

$$J + \frac{1}{4}J = \frac{1}{2}x^2\sqrt{x}$$

we obtain

$$J = \frac{2}{5}x^2\sqrt{x} + C.$$

1.5 Special integration methods:

1.5.1 Integration of a rational function

Definition 1.8

Let P, Q be two real polynomials, $Q(x) \neq 0$. Function $x \to \frac{P(x)}{Q(x)}$ is called rational function or rational fraction

Definition 1.9

The functions $x \to \frac{A}{(x-a)^k}$, $x \to \frac{Mx+N}{(x^2+px+q)^k}$ where $k \in \mathbb{N}^*, a, A, M, N \in \mathbb{R}, p^2 - 4q < 0$, are called simple elements, of the first and second species respectively.

Theorem 1.12

Any rational fraction $\frac{P}{Q}$ is represented in the unique form $\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}$

the polynomials S, R being respectively the quotient and the remainder of the division of P by Q.

Theorem 1.13

Let $\deg P < \deg Q$,

$$Q(x) = (x-a_1)^{m_1}(x-a_2)^{m_2} \dots \dots (x-a_k)^{m_k}.(x^2+p_1x+q_1)^{n_1}$$

$$(x^2+p_2x+q_2)^{n_2} \dots \dots (x^2+p_sx+q_s)^{n_s},\ p_j^2 - 4q_j < 0, \forall\ 1 \leq j \leq s.$$

Then the fraction $\frac{P}{Q}$ is represented in the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{(x-a_1)^2} + \dots + \frac{A_{m_1}}{(x-a_1)^{m_1}}$$

$$+\frac{B_1}{x-a_2} + \frac{B_2}{(x-a_2)^2} + \dots + \frac{B_{m_2}}{(x-a_2)^{m_2}}$$

$$+ \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$$

$$+\frac{C_1}{x-a_k} + \frac{C_2}{(x-a_k)^2} + \dots + \frac{C_k}{(x-a_k)^{m_k}}$$

$$+\frac{M_1^1x+N_1^1}{x^2+p_1x+q_1} + \frac{M_2^1x+N_2^1}{(x^2+p_1x+q_1)^2} + \dots + \frac{M_{n_1}^1x+N_{n_1}^1}{(x^2+p_1x+q_1)^{n_1}}$$

$$+\frac{M_1^2x+N_1^2}{x^2+p_2x+q_2}+\frac{M_2^2x+N_2^2}{(x^2+p_2x+q_2)^2}+\cdots+\frac{M_{n_2}^2x+N_{n_2}^2}{(x^2+p_2x+q_2)^{n_2}}$$

$$+\cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\frac{M_1^sx+N_1^s}{x^2+p_sx+q_s}+\frac{M_2^sx+N_2^s}{(x^2+p_sx+q_s)^2}+\cdots+\frac{M_{n_s}^sx+N_{n_s}^s}{(x^2+p_sx+q_s)^{n_2}}.$$

where A, B, C, M, N, p, q, a, are real canstants.

Examples

1) $\frac{4x^4-4x^3-3x^2-12x+13}{(2x-1)^2(x-2)}=x+2+\frac{1}{x-2}+\frac{4}{2x-1}+\frac{-4}{(2x-1)^2}$.

2) $\frac{5x^7-x^6+6x^5+11x^4+29x^3+66x^2+29x+27}{(x-1)^3(x+2)^2(x^2+x+1)^2}=\frac{1}{x-1}+\frac{-1}{(x-1)^2}+\frac{2}{(x-1)^3}+\frac{-2}{x+2}+\frac{3}{(x+2)^2}+\frac{x-1}{x^2+x+1}+\frac{2x+1}{(x^2+x+1)^2}$.

Integration of a rational fraction

To calculate the integral of a fraction $\frac{P(x)}{Q(x)}$, we first write this fraction as the sum of a polynomial and a finite number of rational fractions in the form $\frac{A}{(x-a)^k}$ or $\frac{Mx+N}{((x-\alpha)^2+\beta^2)^k}$ where k is a natural number other than zero and $\beta, \alpha, N, M, A, a$ are real numbers, so the integral rational fractions returns to calculate integrals of the type $\int \frac{A}{(x-a)^k}dx$ and $\int \frac{Mx+N}{((x-\alpha)^2+\beta^2)^k}dx$.

Calculate the integral $\int \frac{A}{(x-a)^k}dx$

$$\int \frac{1}{x-a}dx = ln|x-a|+C$$

$$\forall\, k>1:\ \int \frac{1}{(x-a)^k}dx=\frac{-1}{(k-1)(x-a)^{k-1}}+C.$$

Calculate the integral $\int \frac{Mx+N}{((x-\alpha)^2+\beta^2)^k}dx$.

Calculating this integral after changing the variable $x=\alpha+\beta t$ leads to calculating integrals of two types: $I_k=\int \frac{t}{(1+t^2)^k}dt$ And $J_k=\int \frac{1}{(1+t^2)^k}dt$, where we have:

$$I_1=\frac{1}{2}ln(1+t^2)+C \ \text{ and } \ \forall\, k>1:\ I_k=\frac{-1}{2(k-1)(1+t^2)^{k-1}}+C.$$

As for the $J_k = \int \frac{1}{(1+t^2)^k} dt$ integral, we use integration by parts and obtain the following recurrence relation:

$$J_1 = Arctanx + C \text{ and } \forall\, k \geq 1:\ 2kJ_{k+1} = (2k-1)J_k + \frac{t}{(1+t^2)^k} \ldots\ldots (*)$$

Example 1

Calculate the integral $I = \int \frac{2x^4 - x^3 + 2x^2 - 1}{(x^2+1)(x-1)} dx$.

By Euclidean division we get:

$$\frac{2x^4 - x^3 + 2x^2 - 1}{x^3 - x^2 + x - 1} = 2x + 1 + \frac{x^2 + x}{(x^2+1)(x-1)}.$$

We put $\frac{x^2+x}{(x^2+1)(x-1)} = \frac{ax+b}{x^2+1} + \frac{c}{x-1}$ we get $a = 0, b = 1, c = 1$.

So

$$I = \int \left(2x + 1 + \frac{1}{x^2+1} + \frac{1}{x-1}\right) dx$$

$$= x^2 + x + Arctanx + ln|x-1| + C.$$

Example 2

Calculate the integral $J = \int \frac{2x^2 - 6x + 11}{(x+1)(x-2)^2} dx$.

We put $\frac{2x^2-6x+11}{(x+1)(x-2)^2} = \frac{a}{x+1} + \frac{c}{x-2} + \frac{c}{(x-2)^2}$ we get $a = 2, b = -1, c = 1$.

So

$$J = \int \left(\frac{2}{x+1} - \frac{1}{x-2} + \frac{1}{(x-2)^2}\right) dx$$

$$= 2\ln(x+1) - \ln(x-2) - \frac{1}{x-2} + C.$$

Example 3*

Calculate the integral $J = \int \frac{8x^6 - 8x^5 + 2x^4 + 23x^3 - 15x^2 + 7x + 2}{(x+1)^2(2x^2-2x+1)^2} dx$

By Euclidean division we get:

$$\frac{8x^6 - 8x^5 + 2x^4 + 23x^3 - 15x^2 + 7x + 2}{(x+1)^2(2x^2-2x+1)^2} = 2 + \frac{-8x^5 + 10x^4 + 15x^3 - 17x^2 + 11x}{(x+1)^2(2x^2-2x+1)^2}.$$

We put

$$\frac{-8x^5+10x^4+15x^3-17x^2+11x}{(x+1)^2(2x^2-2x+1)^2} = \frac{a}{x+1}+\frac{b}{(x+1)^2}+\frac{cx+d}{2x^2-2x+1}+\frac{ex+f}{(2x^2-2x+1)^2}$$

we get:

$$a=-2, b=-1, c=0, d=3, e=1, f=0.$$

So

$$I=\int\left(2+\frac{-2}{x+1}+\frac{-1}{(x+1)^2}+\frac{3}{2x^2-2x+1}+\frac{x}{(2x^2-2x+1)^2}\right)dx$$

$$I=2x-2ln|x+1|+\frac{1}{x+1}+\int\left(\frac{3}{2x^2-2x+1}+\frac{x}{(2x^2-2x+1)^2}\right)dx.$$

Since $2x^2-2x+1=2\left(\left(x-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2\right)$, to calculate the integral on the second side we put $x=\frac{1}{2}+\frac{1}{2}t$.

So

$$\int\left(\frac{3}{2x^2-2x+1}+\frac{x}{(2x^2-2x+1)^2}\right)dx=3\int\frac{1}{t^2+1}dx+\int\frac{t+1}{(t^2+1)^2}dx$$
$$=3Arc\,tan\,x+\int\frac{t}{(t^2+1)^2}dx+\int\frac{1}{(t^2+1)^2}dx$$

$$\int\left(\frac{3}{2x^2-2x+1}+\frac{x}{(2x^2-2x+1)^2}\right)dx=3Arc\,tan\,t-\frac{1}{2}\frac{1}{t^2+1}+\underbrace{\int\frac{1}{(t^2+1)^2}dx}_{J_2}$$

Substituting $k=1$ in the regressive relationship $(*)$ we get: $2J_2=J_1+\frac{t}{(1+t^2)^1}=Arc\,tan\,x+\frac{t}{1+t^2}$ and from there $J_2=\frac{1}{2}Arc\,tan\,t+\frac{t}{2(t^2+1)}$.

So

$$\int\left(\frac{3}{2x^2-2x+1}+\frac{x}{(2x^2-2x+1)^2}\right)dx=\frac{7}{2}Arc\,tan\,t+\frac{t-1}{2(t^2+1)}.$$

Substituting $t=2x-1$ we get

$$\int\left(\frac{3}{2x^2-2x+1}+\frac{x}{(2x^2-2x+1)^2}\right)dx=\frac{7}{2}Arc\,tan(2x-1)+\frac{x-1}{2(2x^2-2x+1)}.$$

So

$$I = 2x - 2ln|x+1| + \frac{1}{x+1} + \frac{7}{2}Arctan(2x-1) + \frac{x-1}{2(2x^2-2x+1)} + C.$$