Tutorial n° 0: Vector calculus operators

Exercise 1

Calculate by integration the following quantities:

1- The circumference of a circle of radius R.
2- The area of a disk of radius R.
3- The lateral surface area of a cylinder of radius R and height H.
4- The volume of a cylinder of radius R and height H.
5- The surface area of a sphere and hemisphere of radius R.
6- The volume of a sphere of radius R.

Exercise 2

1- Calculate the gradient of the function $f(x, y, z) = x^2yz$.
2- Calculate the divergence of the vector $\vec{A} = 2xy\vec{\imath} + xy^3\vec{\jmath} - 3yz^2\vec{k}$.
3- Calculate the rotational of the vector $\vec{B} = 2xy\vec{\imath} - 3yz^2\vec{\jmath}$.

Exercise 3

1- In a Cartesian reference frame, a point M (x, y, z) is identified by the following position vector:
$\overrightarrow{\boldsymbol{OM}} = \vec{\boldsymbol{r}} = \boldsymbol{x}\,\vec{\boldsymbol{\imath}} + \boldsymbol{y}\,\vec{\boldsymbol{\jmath}} + \boldsymbol{z}\,\vec{\boldsymbol{k}}$, where: $\boldsymbol{r} = \sqrt{\boldsymbol{x^2} + \boldsymbol{y^2} + \boldsymbol{z^2}}$
Calculate: $\overrightarrow{grad}\, r, \overrightarrow{grad}\left(\frac{1}{r}\right), \overrightarrow{grad}\,(\ln r),\ div\,\vec{r}\ and\ \overrightarrow{rot}\,\vec{r}$

2- In a spherical reference frame, let $f(r, \theta, \varphi)$ be a scalar function, its gradient is given by:
$\overrightarrow{grad}\, f = \frac{\partial f}{\partial r}\overrightarrow{u_r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\overrightarrow{u_\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\overrightarrow{u_\varphi}$
Calculate $\overrightarrow{grad}\, f$ for these cases: $f_1 = r,\ f_2 = \frac{1}{r}\ and\ f_3 = k\frac{\cos\theta}{r^2}$

Table.1. Elements of length $\overrightarrow{dl}$, surface dS and volume dV in different coordinate systems

Cartesian coordinate system $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z})$	**Polar coordinate system** $(\boldsymbol{\rho}, \boldsymbol{\theta})$
$\overrightarrow{dl} = dx\,\vec{\imath} + dy\,\vec{\jmath} + dz\,\vec{k}$ $dS = dxdy$ $dV = dx\,dy\,dz$	$\overrightarrow{dl} = dr\,\overrightarrow{u_r} + r\,d\theta\,\overrightarrow{u_\theta}$ $dS = dr\,r\,d\theta$
Cylindrical coordinate system $(\boldsymbol{\rho}, \boldsymbol{\theta}, \boldsymbol{z})$	**Spherical coordinate system** $(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\varphi})$
$\overrightarrow{dl} = d\rho\,\overrightarrow{u_\rho} + \rho d\theta\,\overrightarrow{u_\theta} + dz\,\vec{k}$ $dS_\rho = \rho d\theta dz$; (lateral Surface element) $dS_z = d\rho\rho d\theta$.(Surface element of Base) $dV = d\rho\,\rho\,d\theta\,dz$	$\overrightarrow{dl} = dr\,\overrightarrow{u_r} + r\,d\theta\,\overrightarrow{u_\theta} + r\sin\theta\,d\varphi\,\overrightarrow{u_\varphi}$ $dS = r^2 \sin\theta d\theta d\varphi$ $dV = r^2\,dr\,\sin\theta\,d\theta\,d\varphi$

Table.2. Differential operators.

Operator of "Nabla"	
In the Cartesian coordinate system:	$\vec{\nabla} = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}$
In the cylindrical/polar coordinate system (ρ or r) :	$\vec{\nabla} = \frac{\partial}{\partial \rho}\overrightarrow{u_\rho} + \frac{1}{\rho}\frac{\partial}{\partial \theta}\overrightarrow{u_\theta} + \frac{\partial}{\partial z}\vec{k}$
In the spherical coordinate system:	$\vec{\nabla} = \frac{\partial}{\partial r}\overrightarrow{u_r} + \frac{1}{r}\frac{\partial}{\partial \theta}\overrightarrow{u_\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\overrightarrow{u_\varphi}$
Gradient operator	
In the Cartesian coordinate system $f\,(x, y, z)$:	$\overrightarrow{grad}\,f = \vec{\nabla}f = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial z}\vec{k}$
In the cylindrical/polar coordinate system $f\,(\rho, \theta, z)$:	$\overrightarrow{grad}\,f = \frac{\partial f}{\partial \rho}\overrightarrow{u_\rho} + \frac{1}{\rho}\frac{\partial f}{\partial \theta}\overrightarrow{u_\theta} + \frac{\partial f}{\partial z}\vec{k}$
In the spherical coordinate system $f\,(r, \theta, \varphi)$:	$\overrightarrow{grad}\,f = \frac{\partial f}{\partial r}\overrightarrow{u_r} + \frac{1}{r}\frac{\partial f}{\partial \theta}\overrightarrow{u_\theta} + \frac{1}{r\sin\theta}\frac{\partial f}{\partial \varphi}\overrightarrow{u_\varphi}$
Divergence operator	
In the Cartesian coordinate system $\vec{A}\,(A_x, A_y, A_z)$:	$div\vec{A} = \vec{\nabla}.\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$
In the cylindrical/polar coordinate system $\vec{A}\,(A_\rho, A_\theta, A_z)$	$div\vec{A} = \frac{\partial A_\rho}{\partial \rho} + \frac{1}{\rho}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$
In the spherical coordinate system $\vec{A}\,(A_r, A_\theta, A_\varphi)$:	$div\vec{A} = \frac{\partial A_r}{\partial r} + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$

Rotational operator

In the Cartesian coordinate system $\vec{A}\,(A_x, A_y, A_z)$

$$\overrightarrow{rot}\,\vec{A} = \vec{\nabla}\wedge\vec{A} = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{\imath} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{k}$$

In the cylindrical/Polar coordinate system $\vec{A}\,(A_\rho, A_\theta, A_z)$

$$\overrightarrow{rot}\,\vec{A} = \vec{\nabla}\wedge\vec{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}\right)\overrightarrow{u_\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\overrightarrow{u_\theta} + \left(\frac{\partial A_\theta}{\partial \rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \theta}\right)\vec{k}$$

In the spherical coordinate system $\vec{A}\,(A_r, A_\theta, A_\varphi)$

$$\overrightarrow{rot}\,\vec{A} = \vec{\nabla}\wedge\vec{A} = \left(\frac{1}{r}\frac{\partial A_\varphi}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial A_\theta}{\partial \varphi}\right)\overrightarrow{u_r} + \left(\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{\partial A_\varphi}{\partial r}\right)\overrightarrow{u_\theta} + \left(\frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)\overrightarrow{u_\varphi}$$

Laplacian operator	
In the Cartesian coordinate system:	$\vec{\nabla}.\vec{\nabla} = \vec{\nabla}^2 = \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ $\Delta\vec{A} = \frac{\partial^2\vec{A}}{\partial x^2} + \frac{\partial^2\vec{A}}{\partial y^2} + \frac{\partial^2\vec{A}}{\partial z^2}$ (vector field) $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$ (scalar field)
In the cylindrical/polar coordinate system:	$\Delta\vec{A} = \frac{\partial^2 A_\rho}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 A_\theta}{\partial \theta^2} + \frac{\partial^2 A_z}{\partial z^2}$
In the spherical coordinate system :	$\Delta\vec{A} = \frac{\partial^2 A_r}{\partial r^2} + \frac{1}{\rho^2}\frac{\partial^2 A_\theta}{\partial \theta^2} + \frac{1}{\rho^2\sin^2\theta}\frac{\partial^2 A_\varphi}{\partial \varphi^2}$

Properties

$\overrightarrow{grad}(f+g) = \overrightarrow{grad}\,f + \overrightarrow{grad}\,g$; $\quad \overrightarrow{grad}\,(f.g) = g.\overrightarrow{grad}\,f + f.\overrightarrow{grad}\,g$

$div\left(\overrightarrow{grad}\,f\right) = laplacien\,f = \Delta f$; $\quad div\left(f.\vec{A}\right) = \vec{A}.\overrightarrow{grad}\,f + f.div\,\vec{A}$; $\quad div\left(\overrightarrow{rot}\,\vec{A}\right) = \overrightarrow{rot}\left(\overrightarrow{grad}\,A\right) = 0$

Exercise-01

1- The perimeter of a circle (simple integral).
We have: $dl = r\,d\theta$ and $r = R$

$$C = \int_0^{2\pi} R\,d\theta = R(2\pi - 0) = 2\pi R$$

2- Area of a disk (double integral).

We have: $dS = dr\,r\,d\theta$ et $r = R = cte$

$$S_{cercle} = \iint_S dr\,r\,d\theta = \int_0^R r\,dr \int_0^{2\pi} d\theta = \left[\frac{r^2}{2}\right]_0^R [\theta]_0^{2\pi} = \pi R^2$$

3- Lateral surface of a cylinder (double integral).

We have: $dS = \rho\,d\theta\,dz$ et $\rho = R = cte$

$$V_{cyl} = \iint_S \rho\,d\theta\,dz = \rho \int_0^{2\pi} d\theta \int_0^H dz == 2\pi RH$$

4- Volume of a cylinder V (triple integral).

We have: $dV = d\rho\,\rho\,d\theta\,dz$

$$V_{cyl} = \iiint_V d\rho\,\rho\,d\theta\,dz = \int_0^R \rho\,d\rho \int_0^{2\pi} d\theta \int_0^H dz = \left[\frac{r^2}{2}\right]_0^R [\theta]_0^{2\pi}[z]_0^H = \pi R^2 H$$

5- Area of a hemisphere S (double integral).

We have: $dS = r\,d\theta\,r\,sin\,\theta\,d\varphi$ et $r = R = cte$

$$S_{sphère} = \iint_S r\,d\theta\,r\,sin\,\theta\,dz = r^2 \int_0^{\pi} sin\,\theta\,d\theta \int_0^{2\pi} d\varphi == 4\pi R^2$$

$$S_{demi-sphère} = 2\pi R^2$$

6- Volume of a sphere V (triple integral).

We have: $dV = r\,d\theta\,r\,sin\,\theta\,d\varphi$

$$V_{sphère} = \iint_V dr\,r\,d\theta\,r\,sin\,\theta\,dz = \int_0^R r^2\,dr \int_0^{\pi} sin\,\theta\,d\theta \int_0^{2\pi} d\varphi == \frac{4}{3}\pi R^3$$

Exercise-02

1- $\overrightarrow{\boldsymbol{grad}}\,\boldsymbol{f} = \vec{\nabla} f = 2xyz\,\vec{\imath} + x^2 z\,\vec{\jmath} + x^2 y\,\vec{k}$

2- $\boldsymbol{div}\vec{\boldsymbol{A}} = \vec{\nabla}.\vec{A} = 2y + 3xy^2 - 6yz$

3- $\overrightarrow{\boldsymbol{rot}}\,\vec{\boldsymbol{B}} = \vec{\nabla} \wedge \vec{B} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & -3yz^2 & 0 \end{vmatrix} = 6yz\,\vec{\imath} - 2x\,\vec{k}$

Exercise-03

We have $\begin{cases} \vec{\boldsymbol{r}} = \boldsymbol{x}\,\vec{\boldsymbol{\imath}} + \boldsymbol{y}\,\vec{\boldsymbol{\jmath}} + \boldsymbol{z}\,\vec{\boldsymbol{k}} \\ \boldsymbol{r} = \sqrt{\boldsymbol{x}^2 + \boldsymbol{y}^2 + \boldsymbol{z}^2} \end{cases}$

<u>1) Calculations</u>

$* \; \overrightarrow{grad}\,r = \frac{\partial r}{\partial x}\vec{\imath} + \frac{\partial r}{\partial y}\vec{\jmath} + \frac{\partial r}{\partial z}\vec{k}$

$$* \overrightarrow{grad}\, r = \frac{2x}{2\sqrt{x^2+y^2+z^2}}\vec{i} + \frac{2y}{2\sqrt{x^2+y^2+z^2}}\vec{j} + \frac{2z}{2\sqrt{x^2+y^2+z^2}}\vec{k} = \frac{\vec{r}}{r}$$

$$* \overrightarrow{grad}\left(\frac{1}{r}\right) = \frac{\partial\left(\frac{1}{r}\right)}{\partial x}\vec{i} + \frac{\partial\left(\frac{1}{r}\right)}{\partial y}\vec{j} + \frac{\partial\left(\frac{1}{r}\right)}{\partial z}\vec{k} = -\frac{\vec{\boldsymbol{r}}}{\boldsymbol{r}^3}$$

$$* \overrightarrow{grad}\,(\ln r) = \frac{\partial(\ln r)}{\partial x}\vec{i} + \frac{\partial(\ln r)}{\partial y}\vec{j} + \frac{\partial(\ln r)}{\partial z}\vec{k} = \frac{\vec{\boldsymbol{r}}}{\boldsymbol{r}^2}$$

$$* div\, \vec{r} = \frac{\partial r_x}{\partial x} + \frac{\partial r_y}{\partial y} + \frac{\partial r_z}{\partial z} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = \mathbf{3}$$

$$* \overrightarrow{rot}\, \vec{r} = \vec{\nabla}\wedge\vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \mathrm{x} & \mathrm{y} & \mathrm{z} \end{vmatrix} = \left(\frac{\partial \mathrm{z}}{\partial \mathrm{y}} - \frac{\partial \mathrm{y}}{\partial \mathrm{z}}\right)\vec{i} + \left(\frac{\partial \mathrm{x}}{\partial \mathrm{z}} - \frac{\partial \mathrm{z}}{\partial \mathrm{x}}\right)\vec{j} + \left(\frac{\partial \mathrm{y}}{\partial \mathrm{x}} - \frac{\partial \mathrm{x}}{\partial \mathrm{y}}\right)\vec{k} = \vec{\mathbf{0}}$$

2) Calculating: $\overrightarrow{grad}\, f_1$, $\overrightarrow{grad}\, f_2$ et $\overrightarrow{grad}\, f_3$

$$\begin{cases} \overrightarrow{grad}\, f_1 = \overrightarrow{u_r} \\ \overrightarrow{grad}\, f_2 = -\frac{1}{r^2}\overrightarrow{u_r} \\ \overrightarrow{grad}\, f_3 = -\frac{2k\cos\theta}{r^3}\overrightarrow{u_r} - \frac{k\sin\theta}{r^3}\overrightarrow{u_\theta} \end{cases}$$