Université Larbi Ben M'hidi
Dept. SM
Maths3 - L2

Interrogation

17 décembre 2025 **Durée : 1h30**

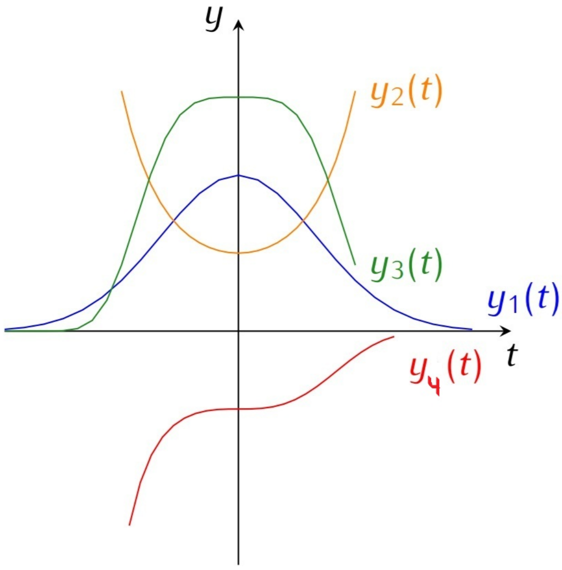
Tous les documents, autres que ceux fournis dans le sujet, sont interdits.[1]

Exercice 1

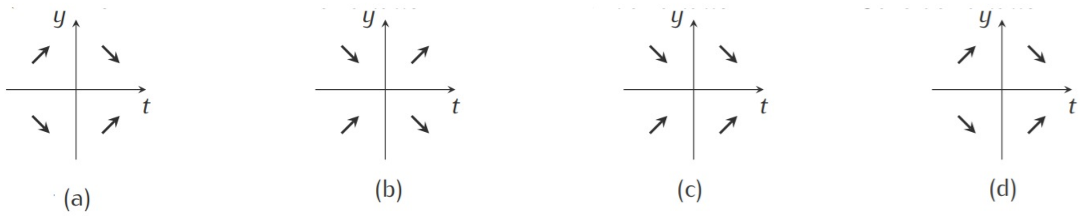
For $t \in \mathbb{R}$, consider the following four differential equations :

$$(a)\ y'(t) = -t\,y(t) \qquad (b)\ y'(t) = t\,y(t)$$
$$(c)\ y'(t) = -t^2\,y(t) \qquad (d)\ y'(t) = -t^3\,y(t)$$

The graphs of the corresponding functions are shown in the figure opposite. Without solving the differential equations, determine for each equation which of the given curves represents its solution.



Solution 1 *For each ODE, we decompose the Cartesian plane into four parts and plot the direction of variation of its solution :*



The curve y_4 (red) is the only one where the function and its derivative have opposite signs ; it can therefore only correspond to function (c).
The curve y_2 (orange) corresponds to a function that has the same sign as its derivative for $t > 0$; it is therefore the graph of (b).
For $t > 1$, we have $-t^3 < -t$, so the graph of equation (d) lies below the graph of equation (a) for all $t > 1$. We deduce that the curve y_1 (blue) represents function (d), and the curve y_3 (green) represents function (a).

Exercice 2 *Using the method of characteristics, solve the following partial differential equation :*

$$y\,u_x - x\,u_y = 0.$$

Solution 2 *The characteristic system associated with the PDE is*

$$\frac{dx}{y} = \frac{dy}{-x} = \frac{du}{0}.$$

1. Barème : 4-3-3 pts.

From the first two ratios we obtain

$$\frac{dx}{y} = \frac{dy}{-x} \quad \Rightarrow \quad x\,dx + y\,dy = 0,$$

which integrates to

$$x^2 + y^2 = constant.$$

Hence, the function

$$U(x, y) = x^2 + y^2$$

is a first integral of the system.
From the third ratio,

$$\frac{du}{0},$$

we deduce that u is constant along each characteristic curve. Thus,

$$V(x, y) = u$$

is also a first integral.
Therefore, the general solution of the PDE is an arbitrary C^1 function of the first integral :

$$u(x, y) = G(x^2 + y^2),$$

where G is an arbitrary differentiable function.

Exercice 3 *Solve the differential equation*

$$(2x + 3yx^2)\,dx + (x^3 - 3y^2)\,dy = 0.$$

Solution 3 *We first check whether the equation is exact. Let*

$$M(x, y) = 2x + 3yx^2, \qquad N(x, y) = x^3 - 3y^2.$$

Compute the mixed partial derivatives :

$$M_y = \frac{\partial}{\partial y}(2x + 3yx^2) = 3x^2, \qquad N_x = \frac{\partial}{\partial x}(x^3 - 3y^2) = 3x^2.$$

Since $M_y = N_x$, the equation is exact.
We now seek a potential function $F(x, y)$ such that

$$F_x = M(x, y) = 2x + 3yx^2, \qquad F_y = N(x, y) = x^3 - 3y^2.$$

Integrating F_x with respect to x :

$$F(x, y) = \int (2x + 3yx^2)\,dx = x^2 + x^3y + \varphi(y),$$

where $\varphi(y)$ is an arbitrary function of y.
Differentiate this with respect to y :

$$F_y(x, y) = x^3 + \varphi'(y).$$

But we require $F_y = x^3 - 3y^2$. Hence

$$\varphi'(y) = -3y^2 \quad \Rightarrow \quad \varphi(y) = -y^3 + C.$$

Therefore, the potential function is

$$F(x, y) = x^2 + x^3y - y^3 + C.$$

The general solution of the differential equation is given implicitly by

$$x^2 + x^3y - y^3 = C,$$

where C is an integration constant.