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Département SM – Maths3 (L2)
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# Solutions to Interro 1 in Maths3

## Contents



## 1 Exercise One

**Exercice 1.** *Compute the value of the triple integral*

$$\iiint_D z\,dx\,dy\,dz,$$

*where the domain $D \subset \mathbb{R}^3$ is defined by*

$$D = \Big\{(x, y, z) \in \mathbb{R}^3 : 0 \le \sqrt{x} + \sqrt{y} + \sqrt{z} \le 1\Big\}.$$

**Solution 1.**

**▷ First Method.**

*The condition $\sqrt{x} + \sqrt{y} + \sqrt{z} \le 1$ implies $x, y, z \ge 0$. Put*

$$u = \sqrt{x}, \qquad v = \sqrt{y}, \qquad w = \sqrt{z},$$

*so $x = u^2,\ y = v^2,\ z = w^2$ and*

$$dx\,dy\,dz = (2u)(2v)(2w)\,du\,dv\,dw = 8uvw\,du\,dv\,dw,$$

*while the domain becomes the simplex*

$$S = \{(u, v, w) : u \ge 0,\ v \ge 0,\ w \ge 0,\ u + v + w \le 1\}.$$

*Thus*

$$I = \iiint_D z\,dx\,dy\,dz = 8\int_S uvw^3\,du\,dv\,dw.$$

*Integrating in the order $u, v, w$ yields*

$$I = 8\int_{w=0}^{1}\int_{v=0}^{1-w}\int_{u=0}^{1-w-v} uvw^3\,du\,dv\,dw = 4\int_0^1 w^3\Big(\int_0^{1-w} v(1-w-v)^2\,dv\Big)\,dw.$$

*Compute the inner integral by expansion:*

$$\int_0^{1-w} v(1-w-v)^2\,dv = \int_0^{1-w} v\Big((1-w)^2 - 2(1-w)v + v^2\Big)\,dv = \frac{(1-w)^4}{12}.$$

*Hence*

$$I = \frac{1}{3}\int_0^1 w^3(1-w)^4\,dw.$$

*Evaluate the remaining integral by expanding the polynomial:*

$$w^3(1-w)^4 = w^3 - 4w^4 + 6w^5 - 4w^6 + w^7,$$

*so*

$$\int_0^1 w^3(1-w)^4\,dw = \left[\frac{w^4}{4} - \frac{4w^5}{5} + \frac{6w^6}{6} - \frac{4w^7}{7} + \frac{w^8}{8}\right]_0^1 = \frac{1}{4} - \frac{4}{5} + 1 - \frac{4}{7} + \frac{1}{8}.$$

*Putting over the common denominator* 280 *gives*

$$\frac{70 - 224 + 280 - 160 + 35}{280} = \frac{1}{280}.$$

*Therefore*

$$I = \frac{1}{3}\cdot\frac{1}{280} = \frac{1}{840},$$

*and finally*

$$\boxed{\iiint_D z\,dx\,dy\,dz = \frac{1}{840}}$$

▷ **Second Method.**

## 2 Exercise Two

**Exercice 2.** *Determine whether the following improper integrals are convergent:*

*(1)* $\displaystyle\int_0^3 \frac{1}{t-2}\,dt$

*(2)* $\displaystyle\int_0^{+\infty} \sin^2\left(\frac{1}{t}\right)\,dt$

**Solution 2.**

### (1) Analyze $\int_0^3 \frac{1}{t-2}\,dt$

*The integrand* $\frac{1}{t-2}$ *has a vertical asymptote at* $t = 2$*, which lies within the interval of integration. We must split the integral at this singularity:*

$$\int_0^3 \frac{1}{t-2}\,dt = \lim_{a\to 2^-}\int_0^a \frac{1}{t-2}\,dt + \lim_{b\to 2^+}\int_b^3 \frac{1}{t-2}\,dt.$$

*Evaluating each part:*

$$\lim_{a\to 2^-}\int_0^a \frac{1}{t-2}\,dt = \lim_{a\to 2^-}\left[\ln|t-2|\right]_0^a = \lim_{a\to 2^-}\left(\ln|a-2| - \ln 2\right) = -\infty,$$

$$\lim_{b\to 2^+}\int_b^3 \frac{1}{t-2}\,dt = \lim_{b\to 2^+}\left[\ln|t-2|\right]_b^3 = \lim_{b\to 2^+}\left(\ln 1 - \ln|b-2|\right) = +\infty.$$

*Because one of the limits diverges, the improper integral* ***diverges****.*

**(2) Analyze $\int_0^{+\infty} \sin^2\left(\frac{1}{t}\right)\, dt$**

*Consider the integral*

$$\int_0^{+\infty} \sin^2\left(\frac{1}{t}\right)\, dt.$$

*We analyze the behavior at the critical points $t = 0^+$ and $t \to +\infty$.*

- ***As $t \to +\infty$:*** *Since $\frac{1}{t} \to 0$, we use the approximation $\sin x \sim x$ as $x \to 0$. Thus,*

$$\sin^2\left(\frac{1}{t}\right) \sim \left(\frac{1}{t}\right)^2 = \frac{1}{t^2}.$$

  *Since $\int_1^{\infty} \frac{1}{t^2}\, dt$ converges and $0 \leq \sin^2\left(\frac{1}{t}\right) \leq \frac{1}{t^2}$ for sufficiently large $t$, the integral converges at infinity by comparison.*

- ***As $t \to 0^+$:*** *The function $\sin^2\left(\frac{1}{t}\right)$ oscillates between $0$ and $1$, but remains bounded:*

$$0 \leq \sin^2\left(\frac{1}{t}\right) \leq 1.$$

  *Therefore,*

$$0 \leq \int_0^1 \sin^2\left(\frac{1}{t}\right)\, dt \leq \int_0^1 1\, dt = 1 < \infty.$$

  *Thus, the integral converges near zero by comparison with the constant function $1$.*

  *Since the integral converges both near zero and at infinity, we conclude that the improper integral* ***converges****.*

## 3 Exercise Three

**Exercice 3.** *Determine whether each of the following improper integrals converges, and compute its value whenever it does:*

*(1)* $\int_0^{+\infty} \frac{\ln t}{\sqrt{t}}\, dt$

*(2)* $\int_0^{+\infty} e^{-x} \sin^2(x)\, dx$

**Solution 3.**

**(1) Analyze $\int_0^{+\infty} \frac{\ln t}{\sqrt{t}}\, dt$**

**▷ First Method**

*The integrand has potential singularities at $t = 0$ (where $\ln t \to -\infty$) and as $t \to +\infty$ (where the behavior needs careful examination). Split the integral:*

$$I = \int_0^1 \frac{\ln t}{\sqrt{t}}\, dt + \int_1^{\infty} \frac{\ln t}{\sqrt{t}}\, dt = I_1 + I_2.$$

○ ***Analysis of $I_2 = \int_1^{\infty} \frac{\ln t}{\sqrt{t}}\, dt$:***

*For large $t$, we can use the limit comparison test. Consider:*

$$\lim_{t\to\infty} \frac{\frac{\ln t}{\sqrt{t}}}{\frac{1}{t^{1/4}}} = \lim_{t\to\infty} \frac{\ln t}{t^{1/4}} = 0,$$

*since logarithmic growth is slower than any positive power. However, a better comparison is with $\frac{1}{t^p}$ where $p < 1$.*
*Actually, for $t \geq e$, we have $\ln t \geq 1$, so:*

$$\frac{\ln t}{\sqrt{t}} \geq \frac{1}{\sqrt{t}}.$$

*Since $\int_1^\infty \frac{1}{\sqrt{t}}\,dt$ diverges (it's a p-integral with $p = \frac{1}{2} < 1$), by the comparison test, $I_2$ diverges.*
*Since $I_2$ diverges, it follows that the whole integral $I$ also diverges; hence, testing $I_1$ is unnecessary.*

### ▷ Second Method (Bertrand Criterion)

*The integral*

$$\int_1^\infty \frac{\ln t}{\sqrt{t}}\,dt$$

*is of the Bertrand form $\int_1^\infty \frac{(\ln t)^\beta}{t^\alpha}\,dt$ with $\alpha = \frac{1}{2}$, $\beta = 1$. Since $\alpha < 1$, the integral diverges. Hence,*

$$\int_0^\infty \frac{\ln t}{\sqrt{t}}\,dt \text{ diverges.}$$

## (2) Analyze $J = \int_0^\infty e^{-x}\sin^2(x)\,dx$

*Since $0 \leq \sin^2(x) \leq 1$ for all $x$, we have:*

$$0 \leq e^{-x}\sin^2(x) \leq e^{-x}.$$

*The integral $\int_0^\infty e^{-x}\,dx = 1$ converges. By the comparison test, $J$ converges absolutely.*

***Computation of $J$:***
*Use the trigonometric identity $\sin^2(x) = \frac{1-\cos(2x)}{2}$:*

$$\begin{aligned} J &= \int_0^\infty e^{-x} \cdot \frac{1-\cos(2x)}{2}\,dx \\ &= \frac{1}{2}\int_0^\infty e^{-x}\,dx - \frac{1}{2}\int_0^\infty e^{-x}\cos(2x)\,dx \\ &= \frac{1}{2}\cdot 1 - \frac{1}{2}K \\ &= \frac{1}{2} - \frac{1}{2}K, \end{aligned}$$

*where $K = \int_0^\infty e^{-x}\cos(2x)\,dx$.*
*Now, use integration by parts twice for $K$.*

*First integration by parts: Let* $u = \cos(2x)$*,* $\mathrm{d}v = e^{-x}\,dx$*, so* $\mathrm{d}u = -2\sin(2x)\,dx$*,* $v = -e^{-x}$*.*

$$\begin{aligned} K &= \Big[-e^{-x}\cos(2x)\Big]_0^\infty - \int_0^\infty (-e^{-x})(-2\sin(2x))\ dx \\ &= (0-(-1)) - 2\int_0^\infty e^{-x}\sin(2x)\ dx \\ &= 1 - 2L, \end{aligned}$$

*where* $L = \int_0^\infty e^{-x}\sin(2x)\ dx$*.*
*Second integration by parts for* $L$*: Let* $u = \sin(2x)$*,* $\mathrm{d}v = e^{-x}\,dx$*, so* $\mathrm{d}u = 2\cos(2x)\,dx$*,* $v = -e^{-x}$*.*

$$\begin{aligned} L &= \Big[-e^{-x}\sin(2x)\Big]_0^\infty - \int_0^\infty (-e^{-x})(2\cos(2x))\ dx \\ &= (0-0) + 2\int_0^\infty e^{-x}\cos(2x)\ dx \\ &= 2K. \end{aligned}$$

*Substituting back:*

$$K = 1 - 2L = 1 - 2(2K) = 1 - 4K,$$

*so* $5K = 1$*, hence* $K = \frac{1}{5}$*.*
*Now substitute back into* $J$*:*

$$J = \frac{1}{2} - \frac{1}{2}\cdot\frac{1}{5} = \frac{1}{2} - \frac{1}{10} = \frac{5}{10} - \frac{1}{10} = \frac{4}{10} = \frac{2}{5}.$$

*Therefore, the integral converges and its value is:*

$$\boxed{J = \frac{2}{5}}$$