University of Oum El Bouaghi — Academic year: 2025/2026
Introduction to topology — License 2 - Mathematics

Sheet of exercises N°2

Exercise 1 *We equip $X = \{a, b, c, d\}$ with the topology $\tau = \{\varnothing, X, \{a\}, \{d\}, \{a, d\}\}$.*

1. *What are the closed sets of the topological space (X, τ)?*
2. *Give the set of neighborhoods of each of the points a, b and c.*
3. *Determine the closure, the interior and the boundary of the subset $A = \{a, b, c\}$.*
4. *Specify the accumulation points and the isolated points of A.*
5. *Determine the topology τ_A induced on A.*

Exercise 2 *On the interval $X = [0, 1[$, we introduce the family τ of sets of the form $[0, a[$ with $0 \leq a \leq 1$.*

$$\tau = \{[0, a[\ ;\ 0 \leq a \leq 1\}.$$

1. *Show that τ is a topology on X.*
2. *What are the closed sets of (X, τ)?*
3. *Determine the closure and the interior of the interval $I = \left[\frac{1}{4}, \frac{3}{4}\right]$ in (X, τ).*
4. *Verify that (X, τ) is not separated.*
5. *Let $(x_n)_n$ be a sequence of numbers in X that converges in the usual sense to $\frac{1}{2}$, i.e.*

$$\forall \varepsilon > 0,\ \exists N \in \mathbb{N},\ \forall n \geq N : \quad \left| x_n - \frac{1}{2} \right| < \varepsilon.$$

   Show that in (X, τ), $(x_n)_n$ converges to any number $\ell \in \left[\frac{1}{2}, 1\right[$.

6. *Show that $\mathbb{Q} \cap X$ is dense in (X, τ). Deduce that (X, τ) is separable.*

Exercise 3 (Cofinite topology)

1. *Let X be a nonempty set.*
   - **a)** *Show that the family $\tau = \{\varnothing\} \cup \{\mathcal{O} \in \mathcal{P}(X)\ ;\ \mathcal{O}^{\complement} \text{ is finite}\}$ is a topology on X (called Cofinite topology on X).*
   - **b)** *Show that if X is an infinite set, then the topological space (X, τ) is not separated.*
   - **c)** *Show that if X is an infinite set, then any infinite subset of X is everywhere dense. Deduce that (X, τ) is separable.*

2. *We equip $\mathbb{R}$ with the cofinite topology τ.*

   a) *Determine the interior, the closure and the boundary of each of the following parts:*

$$\mathbb{N}, \quad \mathbb{N}^* = \mathbb{N} \setminus \{0\} \quad \text{and} \quad \mathbb{Z}^* = \mathbb{Z} \setminus \{0\}.$$

   b) *Show that $\mathbb{N}$ does not admit any isolated point. Deduce the set of accumulation points of $\mathbb{N}$.*

   c) *Compare the topology τ with the usual topology τ_u of $\mathbb{R}$.*

3. *Let f be the map of $(\mathbb{R}, \tau)$ in $(\mathbb{R}, \tau_u)$ defined by*

$$f(x) = \begin{cases} 0, & x \leq 0, \\ x^2, & x > 0. \end{cases}$$

   a) *Draw the representative curve of f.*

   b) *Show that f is not continuous at any point of $\mathbb{R}$.*

Exercise 4 *For every real number a, we set $J_a =]-\infty, a]$, and denote by τ the collection consisting of the empty set $\varnothing$ and all unions of intervals J_a.*

1. *Show that:*

$$\bigcup_{a \in \mathbb{R}} J_a = \mathbb{R} \qquad \text{and} \qquad \bigcup_{a \in \mathbb{R}^*_-} J_a =]-\infty, 0[.$$

2. *Prove that the family τ is a topology on $\mathbb{R}$.*

3. *Describe the forms of open sets in the space $(\mathbb{R}, \tau)$, then deduce the forms of closed sets in the space $(\mathbb{R}, \tau)$.*

4. *Compare the topology τ and the cofinite topology σ on $\mathbb{R}$.*

5. *Is $(\mathbb{R},\tau)$ a separated (Hausdorff) space? Justify your answer.*

6. *Determine the collection of neighborhoods of an arbitrary point a in $(\mathbb{R}, \tau)$, then find a basis $\mathcal{B}(a)$ for the neighborhoods of a.*

7. *Determine the interior, closure, and derived set of each of the following subsets:*

$$A = \{-5, 1\}, \quad B =]-5, 1[, \quad C = \mathbb{N}.$$

8. *Determine the closure of the set $\mathbb{Z}$. Is $(\mathbb{R}, \tau)$ a separable space? Justify your answer.*

9. *Prove that $1/2$ is the limit in $(\mathbb{R}, \tau)$ for the sequence with general term $x_n = 1/n$ as n tends to $+\infty$.*

Exercise 5 *Show that in a separated space X, a point $x \in X$ is an accumulation point of a part A if and only if every neighborhood of x intersects A in an infinite number of points.*

Exercise 6 *We propose to show that a subspace of a separable topological space is not necessarily separable.*

Let X be an **uncountable infinite set** *and a be a fixed point of X.*

1. *Consider the family $\tau = \{\varnothing\} \cup \{B \in \mathcal{P}(X) \; ; \; a \in B\}$.*

   a) *Show that τ is a topology on X (called Particular point topology).*

   b) *Describe the closed sets of the space (X, τ).*

   c) *Deduce that the space (X, τ) is separable.*

2. *Let the subset $A = X \setminus \{a\}$. Show that the topology τ_A induced on A by τ is the discrete topology on A and explain why (A, τ_A) is not separable.*

Exercise 7 *Let (X, τ) be a topological space, A a topological subspace of X and B a subset of A.*

i) *Show that the closure of B in A is equal to $A \cap \overline{B}$.*

ii) *Show that the set of accumulation points of B in A is equal to $A \cap B'$.*

iii) *Show that the interior of B in A contains the interior of B in X (no equality in general).*

iv) *Show that if A is open in X, then the interior of B in A coincides with the interior of B in X.*

Exercise 8 *Let X be a topological space. Show that an application f from X to $\mathbb{R}$ (equipped with the usual topology) is continuous if and only if $f^{-1}(]a, +\infty[)$ and $f^{-1}(]-\infty, a[)$ are open whatever $a \in \mathbb{R}$.*

Exercise 9 *Let f, g be two continuous applications from X to Y, topological spaces, Y being separated.*

1. *Show that the set $\{x \in X \; ; \; f(x) = g(x)\}$ is closed in X.*

2. *Deduce that if f and g coincide on an everywhere dense subset of X, then $f = g$.*

Exercise 10 (Pasting Lemma) *Let X, Y be topological spaces and A, B be non-empty subsets of X such that $X = A \cup B$. Let $f_1 : A \to Y$ and $f_2 : B \to Y$ be two continuous applications such that for all $x \in A \cap B$, we have $f_1(x) = f_2(x)$. We set:*

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in A, \\ f_2(x) & \text{if } x \in B. \end{cases}$$

1. *Let U be a subset of Y. Determine $f^{-1}(U)$ in terms of $f_1^{-1}(U)$ and $f_2^{-1}(U)$.*
2. *Deduce that if A and B are open (resp. closed), then f is continuous.*

Exercise 11 *We take $X =]0,1[\cup\{2\}$, $Y =]0,1]$ (each equipped with the topology induced by the usual topology of $\mathbb{R}$) and we consider the application $f : X \to Y$ given by*

$$f(x) = \begin{cases} x & \text{if } 0 < x < 1, \\ 1 & \text{if } x = 2. \end{cases}$$

1. *Verify that f is a bijection and determine its reciprocal f^{-1}.*
2. *Verify that f is continuous on X.*
3. *Show that f^{-1} is not continuous at 1 (We can consider the real sequence with general term $x_n = 1 - \frac{1}{n}$). Conclude.*

Exercise 12 *Let X and Y be two homeomorphic topological spaces.*

1. *Show that X is separated if and only if Y is separated.*
2. *Show that X is separable if and only if Y is separable.*

Exercise 13 *For every real number a, set $D_a = \{(x,y) \in \mathbb{R}^2 \ ; \ y = x + a\}$.*

1. *Describe D_2 and sketch it in an orthonormal coordinate system.*
2. *Show that $\mathbb{R}^2 = \bigcup_{a\in\mathbb{R}} D_a$.*
3. *Show that the family τ consisting of the empty set $\varnothing$ and all unions of the sets D_a defines a topology on $\mathbb{R}^2$.*
4. *Show that D_a is both open and closed.*
5. *Determine the boundary $\partial(\{(1,1)\})$.*
6. *Show that the space $(\mathbb{R}^2, \tau)$ is not Hausdorff.*
7. *Determine the topology τ_A induced on $A = \{(1,0),(0,1)\}$.*
8. *Show that the $x-$axis (axis of abscissas) is everywhere dense.*
9. *Determine a fundamental system of neighborhoods for a point (b,c) in $(\mathbb{R}^2, \tau)$.*
10. *Let (u_n) be the sequence in $\mathbb{R}^2$ defined by $u_n = (n,n)$.*

    i) *Show that $(1,1)$ is a limit of (u_n).*

    ii) *Determine all the other limits.*