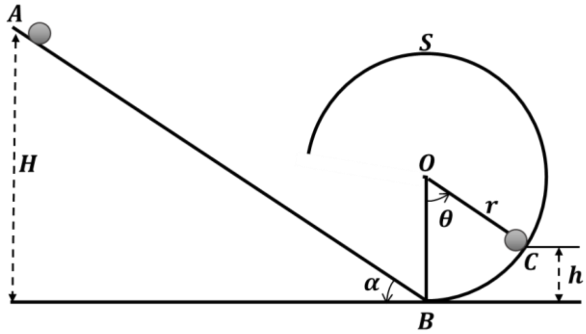
Tutorial n°6: Work and energy

Exercise 1 (Worked on in the tutorial and lecture sessions)/ Pour le Cours: 11 et 15/12/2025

A particle of mass (m) is released at point A without initial velocity (**Figure below**). We want to know what height (H) is necessary for the particle to reach the top point of the gutter (S). the frictions are neglected.



1- Applying the kinetic energy theorem, calculate the velocity V_B at point B.
2- Express the height h as a function of r and θ.
3- Applying the mechanical energy theorem, calculate the velocity V_C at point C as a function of h , g and V_B.
4- Applying the fundamental principle of dynamics, deduce the expression of the contact force N as a function of: m, r, θ, V_B *and* g.
5- Show that the minimum velocity that the particle must acquire at point B to reach point S is $\boldsymbol{V_{B,min}} = \boldsymbol{2\sqrt{gr}}$. What is the corresponding value of H?

Exercise 2

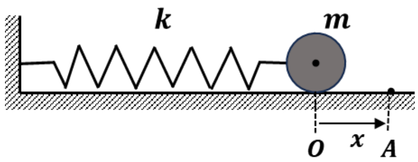
A particle M is subjected to a force field defined in the Cartesian coordinate system as follows:

$$\vec{F} = (x - \lambda y)\vec{\imath} + (2y - 5x)\vec{\jmath}$$

1- What is the value of λ so that $\vec{F}$ derives from a potential U.
2- Find the expression of this potential energy $U(x, y)$, knowing that $U(0,0) = 2$.

Exercise 3

we consider a system (block-spring), placed on a horizontal rough plane with a friction coefficient $\boldsymbol{\mu_c}$ (see **Figure below**). The block is displaced from its equilibrium position $(\boldsymbol{O})$ by a distance $\boldsymbol{x}$ and then released from point **(A)** without initial velocity.



1- Represent the forces acting on the block at the point $(\boldsymbol{A})$.
2- Using kinetic energy theorem, determine the expression of the velocity $\boldsymbol{V_O}$ when it passes through its equilibrium position, as a function of $\boldsymbol{m}, \boldsymbol{g}, \boldsymbol{x}, \boldsymbol{k}$, and $\boldsymbol{\mu_c}$.
3- Determine the minimum distance $\boldsymbol{x}$ for the block to reach the equilibrium position.

Exercise 4 (Optionally/For students)

We consider two force fields: $\overrightarrow{F_1} = 3x^2y\vec{\imath} + (x^3 + x\,y^2)\vec{\jmath}$ and $\overrightarrow{F_2} = (y^3 + xy^2)\vec{\jmath}$.

1- Calculate the work done by $\overrightarrow{F_1}$, $\overrightarrow{F_2}$ and $\overrightarrow{F_3} = \overrightarrow{F_1} - \overrightarrow{F_2}$ to go from the origin $\boldsymbol{O}$ to $\boldsymbol{A(1,1)}$ in following paths: the path $\boldsymbol{T_1}$ defined by the straight line, $\boldsymbol{y = x}$, then the path $\boldsymbol{T_2}$ defined by the parabola $\boldsymbol{y = x^2}$.
2- Do these forces derive from potential energy?

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Exercise-04

$\vec{F_1} = 3x^2y\vec{i} + (x^3 + x\,y^2)\vec{j}$

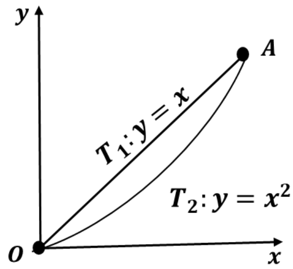
$\vec{F_2} = (y^3 + xy^2)\vec{j}$

$\vec{F_3} = 3x^2y\vec{i} + (x^3 - y^3)\vec{j}$

1) Calculate the works:

$T_1 : y = x\,; straight\ ligne$

$T_2 : y = x^2\,; parabolic$



For trajectory T_1

$$T_1 \begin{cases} y = x \\ dy = dx \end{cases}$$

$$W(\vec{F}) = \int dW = \int \vec{F}.\vec{dl}$$

$$and \begin{cases} \vec{F} = Fx\,\vec{\imath} + Fy\,\vec{\jmath} \\ \vec{dl} = dx\,\vec{\imath} + dy\,\vec{\jmath} \end{cases}$$

$$\Rightarrow W(\vec{F}) = Fx\,dx + Fy\,dy$$

$$W_{T_1}(\vec{F_1}) = \int \vec{F_1}.\vec{dl} = \int_0^1 5\,x^3\,dx = \frac{5}{4}J$$

$$W_{T_1}(\vec{F_2}) = \int \vec{F_2}.\vec{dl} = \int_0^1 2\,x^3\,dx = \frac{1}{2}J$$

$$W_{T_1}(\vec{F_3}) = \int \vec{F_3}.\vec{dl} = W_{T_1}(\vec{F_1}) - W_{T_1}(\vec{F_2}) = \frac{3}{4}J$$

For trajectory T_2

$$T_2 \begin{cases} y = x^2 \\ dy = 2x\,dx \end{cases}$$

$$W(\vec{F}) = \int dW = \int \vec{F}.\vec{dl}$$

$$\Rightarrow W(\vec{F}) = Fx\,dx + Fy\,2x\,dx$$

$$W_{T_2}(\vec{F_1}) = \int \vec{F_1}.\vec{dl} = \frac{9}{7}J$$

$$W_{T_2}(\vec{F_2}) = \int \vec{F_2}.\vec{dl} = \frac{15}{28}J$$

$$W_{T_2}(\vec{F_3}) = W_{T_1}(\vec{F_1}) - W_{T_1}(\vec{F_2}) = \frac{3}{4}J$$

2) Forces that derive from potential energy

If the force is conservative, i.e., does not depend on the followed path, it derives from a potential energy.

We observe that:

$W_{T_1}(\vec{F_1}) \neq W_{T_2}(\vec{F_1})$: $\vec{F_1}$ $non - conservative$

$W_{T_1}(\vec{F_2}) \neq W_{T_2}(\vec{F_2})$: $\vec{F_2}$ $non - conservative$

$W_{T_1}(\vec{F_3}) = W_{T_2}(\vec{F_3})$: $\vec{F_3}$ $conservative$

Therefore, $\vec{F_3}$ is conservative

To verify that $\vec{F_3}$ is derived from a potential energy U, we mut verify the relationship: $\vec{F_3} = -\overrightarrow{grad}\,U$

$$\Rightarrow \vec{F} = -\frac{\partial U}{\partial x}\vec{\imath} - \frac{\partial U}{\partial y}\vec{\jmath}$$

$$\Rightarrow \begin{cases} F_x = -\dfrac{\partial U}{\partial x} = 3\,x^2\,y \\ F_y = -\dfrac{\partial U}{\partial y} = x^3 - y^3 \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{\partial F_x}{\partial y} = -\dfrac{\partial^2 U}{\partial x \partial y} = 3\,x^2 \\ \dfrac{\partial F_y}{\partial x} = -\dfrac{\partial^2 U}{\partial y \partial x} = 3\,x^2 \end{cases}$$

As long as: $-\dfrac{\partial^2 U}{\partial x \partial y} = -\dfrac{\partial^2 U}{\partial y \partial x} \Rightarrow \vec{F_3} = -\overrightarrow{grad}\,U$

Therefore, $\vec{F_3}$ derives from a potential energy U.

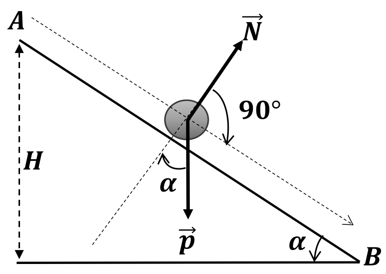
Exercise-01

1/ Calculate the velocity V_B

By applying the kinetic energy theorem between two positions A and B, we obtain:

$$\Delta E_c = \sum W_{AB}(\vec{F})$$

$$\Rightarrow E_c(B) - E_c(A) = W_{AB}(\vec{P}) + W_{AB}(\vec{N})$$



where :

$$\begin{cases} W_{AB}(\vec{P}) = \int dW = \int \vec{P}.\vec{dl} = \int_A^B P\,dl\cos(\frac{\pi}{2} - \alpha) \\ W_{AB}(\vec{N}) = \int dW = \int \vec{N}.\vec{dl} = \int_A^B N\,dl\cos 90 \end{cases}$$

Knowing that: $\cos(\frac{\pi}{2} - \alpha) = \sin\alpha$

$$\Rightarrow \begin{cases} W_{AB}(\vec{P}) = P\,AB\sin\alpha = mgH \\ W_{AB}(\vec{N}) = 0,\ car\ \vec{N} \perp \overrightarrow{AB} \end{cases}$$

$$\Rightarrow \frac{1}{2}m\,V_B^2 - \frac{1}{2}m\,V_A^2 = mgH + 0$$

$$\Rightarrow \frac{1}{2}m\,V_B^2 = mgH$$

$$\Rightarrow \boldsymbol{V_B = \sqrt{2gH}}$$

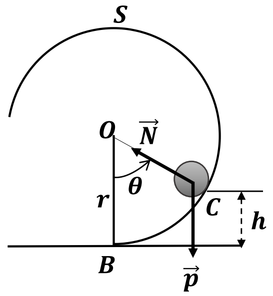
2/ <u>Expression of the height h as function of r, θ</u>

According to the figure: $h = r - r\cos\theta$

$$\Rightarrow \boldsymbol{h = r\,(1-\cos\theta)}$$

3/ <u>The velocity V_C as function of h and V_B.</u>

by applying the mechanical energy theorem between two positions B and C, we obtain:



$$E_m(C) = E_m(B)$$

$$\Rightarrow E_c(C) + E_P(C) = E_c(B) + E_P(B)$$

$$\Rightarrow E_c(C) - E_c(B) = E_P(B) - E_P(C)$$

$$\Rightarrow \frac{1}{2}m\,V_c^2 - \frac{1}{2}m\,V_B^2 = 0 - mgh$$

$$\Rightarrow V_c^2 = V_B^2 - 2gh$$

$$\Rightarrow \boldsymbol{V_c = \sqrt{V_B^2 - 2gh}}$$

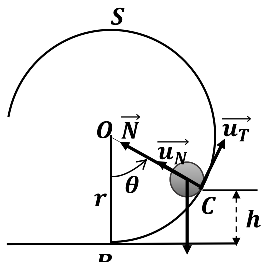
4/ <u>The contact force N as function of : m, r, θ, V_B and g .</u>

By applying the fundamental principle of dynamics (Newton's 2nd law) at C:

$$\sum \vec{F}_{ext} = m\,\vec{\gamma}$$

$$\vec{P} + \vec{N} = m\,\vec{\gamma}$$

By projecting the forces onto the normal and the tangential axes, we deduce the contact force:



$$\begin{cases} -P\sin\theta + 0 = m\,\gamma_T \\ -P\cos\theta + N = m\,\gamma_N \end{cases}$$

$$\Rightarrow -mg\cos\theta + N = m\,\frac{V_C^2}{r}$$

$$\Rightarrow N = \frac{m}{r}\,(V_B^2 - 2gh) + mg\cos\theta$$

$$\Rightarrow N = \frac{m}{r}\,V_B^2 - 2mg(1-\cos\theta) + mg\cos\theta$$

$$\Rightarrow \boldsymbol{N = \frac{m}{r}\,V_B^2 - 2mg + 3mg\cos\theta}$$

5/ <u>The minimum velocity to reach point S</u>

For the particle to move from point B with minimum velocity, it must reach point S with zero velocity.

By applying the mechanical energy theorem between two positions B and S, we obtain:

$$E_m(S) = E_m(B)$$

$$\Rightarrow E_c(S) - E_c(B) = E_P(B) - E_P(S)$$

$$\Rightarrow \frac{1}{2}m\,V_S^2 - \frac{1}{2}m\,V_B^2 = 0 - mgh$$

Knowing that: $\begin{cases} V_s = 0 \\ h = 2r \end{cases} \Rightarrow \frac{1}{2}m\,V_B^2 = 2mgr$

$$\Rightarrow \boldsymbol{V_{B,min} = 2\sqrt{gr}}$$

***The height (H) for the particle to reach point S is:**

$$V_{B,min} = 2\sqrt{gr} = \sqrt{2gH}$$

$$\boldsymbol{H = 2r}$$

<u>Exercise 2</u>

<u>1/ The value of λ</u>

$\vec{F}$ derives from a potential, i.e.: $\vec{F} = -\overrightarrow{grad}\,U$

$$\Rightarrow \vec{F} = -\frac{\partial U}{\partial x}\vec{\imath} - \frac{\partial U}{\partial y}\vec{\jmath}$$

and: $\vec{F} = F_x\,\vec{\imath} + F_y\,\vec{\jmath}$

therfore: $\Rightarrow \begin{cases} F_x = -\dfrac{\partial U}{\partial x} = x - \lambda y \ldots\ldots\ldots (1) \\ F_y = -\dfrac{\partial U}{\partial y} = 2y - 5x \ldots\ldots..(2) \end{cases}$

To get rid of the potential U we derive eq (1) with respect to ∂y and eq (2) with respect to ∂x:

$$\Rightarrow \begin{cases} \dfrac{\partial F_x}{\partial y} = -\dfrac{\partial^2 U}{\partial x \partial y} = -\lambda \ldots\ldots\ldots (1) \\ \dfrac{\partial F_y}{\partial x} = -\dfrac{\partial^2 U}{\partial y \partial x} = -5 \ldots\ldots..(2) \end{cases}$$

As long as: $-\dfrac{\partial^2 U}{\partial x \partial y} = -\dfrac{\partial^2 U}{\partial y \partial x} \Rightarrow \boldsymbol{\lambda = 5}$

<u>2/ Expression of the potential energy $U(x, y)$</u>

We have:

$$\Rightarrow \begin{cases} -\dfrac{\partial U}{\partial x} = x - 5y \dots\dots\dots (1) \\ -\dfrac{\partial U}{\partial y} = 2y - 5x \dots\dots (2) \end{cases}$$

By integrating the eq(1) : $\dfrac{\partial U}{\partial x} = 5y - x$

$\Rightarrow dU = (5y - x)\, dx \Rightarrow \int dU = \int (5y - x)\, dx$

$\Rightarrow U = \left(5yx - \dfrac{x^2}{2}\right) + C(y) \dots\dots\dots .eq(*)$

To determine the integral constant C(y), we derive the potential obtained with respect to ∂y:

$\Rightarrow \dfrac{\partial U}{\partial y} = 5x + \dfrac{dC(y)}{dy}$

knowing that : $\dfrac{\partial U}{\partial y} = 5x - 2y$

$\Rightarrow 5x + \dfrac{dC(y)}{dy} = 5x - 2y \Rightarrow \dfrac{dC(y)}{dy} = -2y$

By integrating $C(y)$

$\Rightarrow \int dC(y) = \int -2y\, dy \Rightarrow C(y) = -y^2 + c$

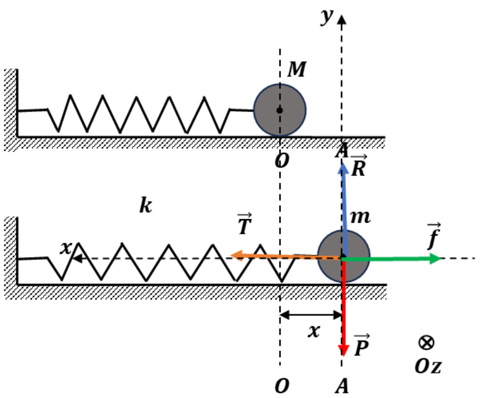
$\Rightarrow U = 5yx - \dfrac{x^2}{2} - y^2 + c$

By determining the constant c : $U(0,0) = c = 2$

$$\Rightarrow \boldsymbol{U = 5yx - \frac{x^2}{2} - y^2 + 2}$$

Exercise 3

1/ Forces acting on the block at (A)



2/ Calculate the velocity V_O

By applying the kinetic energy theorem between two positions A and O, we obtain:

$$\Delta E_c = \sum W_{AO}(\vec{F})$$

$$\boldsymbol{E_c(O) - E_c(A) = W_{AO}(\vec{P}) + W_{AO}(\vec{R}) + W_{AO}(\vec{f})}$$

where :

$$\begin{cases} W_{AO}(\vec{P}) = \vec{P}.\overrightarrow{AO} = P\ x \cos(-90°) = 0 \\ W_{AO}(\vec{R}) = \vec{R}.\overrightarrow{AO} = R\ x \cos(+90°) = 0 \\ W_{AO}(\vec{f}) = \vec{f}.\overrightarrow{AO} = f\ x \cos(180°) = -f\ x \\ W_{AO}(\vec{T}) = \frac{1}{2}kx^2 \end{cases}$$

Knowing that:

$$\begin{cases} R = P \quad (Newton's\ laws) \\ f = \mu c\ R \Rightarrow \boldsymbol{f = \mu_c\, mg} \\ \boldsymbol{V_A = 0} \end{cases}$$

$$\Rightarrow \frac{1}{2} m\, V_O^2 - \frac{1}{2} m\, V_A^2 = \frac{1}{2}kx^2 - \mu_c\, m\, g\, x$$

$$\Rightarrow \frac{1}{2} m\, V_O^2 = \frac{1}{2}kx^2 - \mu_c\, m\, g\, x$$

$$\Rightarrow \boldsymbol{V_O = \sqrt{\frac{k}{m}x^2 - 2\,\mu_c\, g\, x}}$$

3/ Minimum distance $\boldsymbol{x}$

The minimum distance $\boldsymbol{x}$ for the block to reach the equilibrium position O (i.e., not stop before it),

it should be $V_O \geq 0$

$$\Rightarrow \frac{k}{m}x^2 - 2\,\mu_c\, g\, x \geq 0$$

$$\Rightarrow x(\frac{k}{m}x - 2\,\mu_c\, g\,) \geq 0$$

$$\Rightarrow \boldsymbol{x \geq \frac{2\,\mu_c\, m\, g}{k}}$$