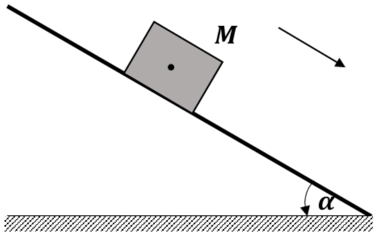
Tutorial n°5: Dynamics

Rectilinear and Rotational Motions

Exercise 1

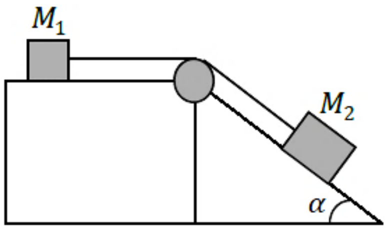
The **figure below** represents a body with a mass m of $800\ g$, moving over a rough plane inclined by α. The kinetic friction coefficient μ_c is 0.40. We take $g = 10\ ms^{-2}$.

1- Represent the forces acting on the mass m at the point M.
2- Calculate the angle of inclination so that the body slides with a constant velocity.
3- Calculate the normal reaction force N for $\alpha = 35°$.
4- Calculate the friction force f for $\alpha = 35°$.
5- Calculate the acceleration γ for $\alpha = 35°$.



Exercise 2

A body M_1 of mass $m_1 = 5\ kg$, placed on a horizontal plane, is related to another body M_2 of mass m_2 placed on an inclined plane making an angle $\alpha = 30°$ with the horizontal, via an inextensible wire of negligible mass and passing through the groove of a pulley of negligible mass (**Figure opposite**). The contact between the two bodies and the two supports (horizontal and inclined) is characterized by static friction coefficient $\mu_s = 0.3$ and kinetic friction coefficient $\mu_c = 0.2$. We will take $g = 10\ m.s^{-2}$.

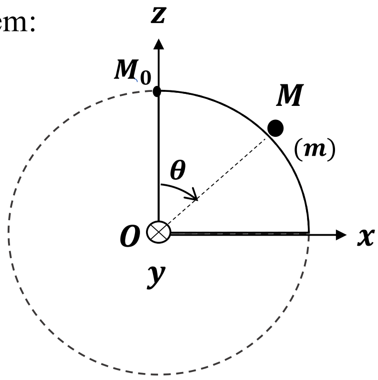


1- Determine the maximum value of the mass $\boldsymbol{m_{2max}}$ necessary to keep the system in equilibrium.
2- We take $m_2 = 7\ kg$, calculate the **acceleration** of the two bodies and the **tension** of the wire.

Exercise 3

We consider a material point of mass (m) sliding without frictions on a surface formed by a sphere of radius (R) and center O (**Figure below**). At time $t = 0\ s$, the mass is abandoned without initial velocity from the point M_0, at a point M. In the intrinsic coordinate system:

1- Represent the forces acting on the mass m at the point M .
2- Establish the equations of movement using the fundamental principle of dynamics (**Newton's 2nd law**)
3- Demonstrate that the velocity acquired at point M is given by the expression: $\boldsymbol{V(M) = \sqrt{2Rg\,(1 - cos\theta)}}$.
4- Determine the expression of the contact force N at the point M.
5- Calculate the angle θ_l at which the particle leaves the surface of the sphere and deduce its corresponding velocity V_l.

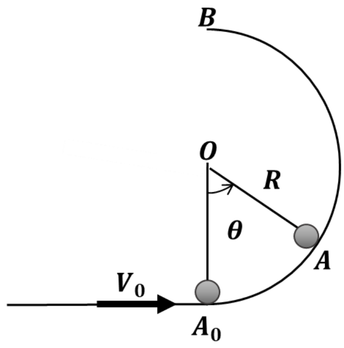


Exercise 4

A mobile M of mass $(\boldsymbol{m})$ is launched with velocity $\boldsymbol{V_0}$ from point $\boldsymbol{A_0}$ located at the bottom of a semicircular trajectory of radius $(\boldsymbol{R})$ (**see figure**). All friction forces are neglected.

1- Using **2nd law of Newton** at point M, give the equations of the movement in the polar coor-sys.
2- Determine the expression of the angular velocity $\boldsymbol{\omega_M}$.

3- Deduce the expression of linear velocity $\boldsymbol{V_M}$.
4- Determine the contact force N exerted by the track on A.
5- Find the condition on $\boldsymbol{V_0}$ for the particle to arrive at point B.

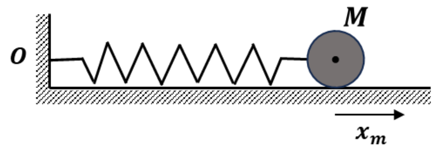


Exercise 5

A point object M of mass $(\boldsymbol{m})$, attached to an elastic spring of stiffness constant $(\boldsymbol{k})$ and empty length $(\boldsymbol{L_0})$, moves along a plan. The mass of the spring is assumed to be null and the friction forces are neglected. At equilibrium (t=0s), we move this mass away from its equilibrium position by a quantity $\boldsymbol{x_m}$ and then release it without initial velocity.

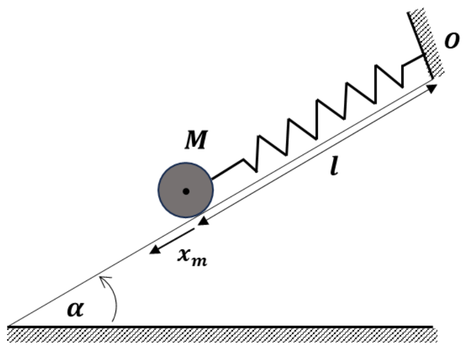
a. For horizontal plane

1- Represent the forces acting on the mass m.
2- Determine the differential equation of movement by using Newton's 2nd law, in the Cartesian coordinate system (Oxyz).
3- Deduce the natural oscillation frequency $\boldsymbol{\omega_0}$.
4- Let $\boldsymbol{x(t) = A\,sin\,(\omega_0 t) + B\,cos\,(\omega_0 t)}$ be a solution for the differential equation, find the two constants $\boldsymbol{A}$ and $\boldsymbol{B}$.
5- Deduce the natural oscillation period $\boldsymbol{T_0}$ for a system.



b. For inclined plane (α)

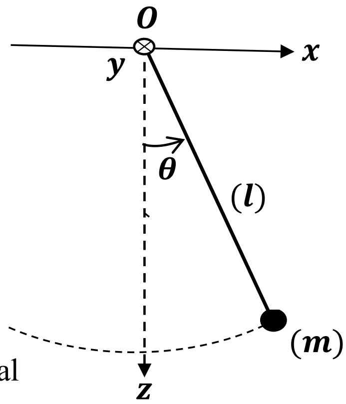
1- Represent the forces acting on the mass m and give their components in the Cartesian coordinate system (Oxyz).
2- Establish the equations of movement using Newton's 2nd law.
3- Find the static equilibrium position $\boldsymbol{l}$.
4- Deduce the differential equation of movement.



Exercise 6

We consider a simple pendulum consisting of a material point M of mass m, attached to an inextensible wire of length l and negligible mass. We pull the pendulum at an angle $(\theta = \theta_0)$ from its equilibrium position $(\theta = 0)$ and we let it go without initial velocity. The friction forces are neglected.

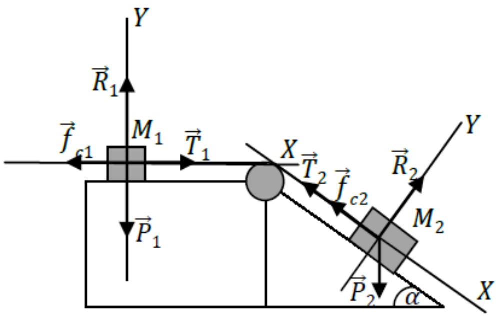
1- Represent the forces acting on the point M and give their components in the polar coordinate system.
2- Determine the differential equation of movement in the polar coordinate system using:
   a. The fundamental principle of dynamics.
   b. The kinetic moment theorem.
   c. The mechanical energy theorem.
3- Let $\boldsymbol{\theta(t) = A\sin(\omega t) + B\,cos\,(\omega t)}$ be a solution for the differential equation, find the two constants A and **B** in the case of weak oscillations.



Exercise-02 :

1/ The maximum value of the mass m_{2max} necessary to keep the system in equilibrium:

In equilibrium, we use Newton's 1st law (PI):



$$\sum \vec{F}_{ext} = \vec{0} \implies \begin{cases} (m_1): \ \vec{P}_1 + \vec{R}_1 + \vec{T}_1 + \vec{f}_1 = \vec{0} \\ (m_2): \ \vec{P}_2 + \vec{R}_2 + \vec{T}_2 + \vec{f}_2 = \vec{0} \end{cases}$$

Projections: $(m_1): \begin{cases} (Ox): T_1 - f_1 = 0 \\ (Oy): R_1 - P_1 = 0 \end{cases} \Rightarrow \begin{cases} T_1 = f_1 \\ R_1 = P_1 \end{cases}$

$(m_2): \begin{cases} (Ox): P_{2x} - T_2 - f_2 = 0 \\ (Oy): R_2 - P_{2y} = 0 \end{cases} \Rightarrow \begin{cases} T_2 = P_{2x} - f_2 \\ R_2 = P_{2y} \end{cases}$

Static friction forces:

$$\begin{cases} f_1 = \mu_S R_1 \\ f_2 = \mu_S R_2 \end{cases} \Rightarrow \begin{cases} f_1 = \mu_S P_1 = \mu_S m_1 g \\ f_2 = \mu_S P_{2y} = \mu_S m_{2max} g \cos\alpha \end{cases}$$

The expression of tensions are : $\begin{cases} T_1 = f_1 \\ T_2 = P_{2x} - f_2 \end{cases} \Rightarrow \begin{cases} T_1 = \mu_S m_1 g_1 \\ T_2 = m_{2max} g \sin\alpha - \mu_S m_{2max} g \cos\alpha \end{cases}$

An inextensible wire with negligeable mass means that: $T_1 = T_2 = \mathrm{T}$

Which allows us to obtain the expression of m_{2max} : $\boldsymbol{m_{2max}} = \frac{\mu_S m_1}{\sin\alpha - \mu_S \cos\alpha} = \mathbf{6.25\ Kg}$

2/ The acceleration of the two bodies and the tension of the wire if $m_2 = 7kg$:

In motion, we use Newton's 2nd law (FPD):

$$\sum \vec{F}_{ext} = \mathrm{m}\,\vec{\gamma} \implies \begin{cases} (m_1): \ \vec{P}_1 + \vec{R}_1 + \vec{T}_1 + \vec{f}_1 = \mathrm{m}_1\vec{\gamma} \\ (m_2): \ \vec{P}_2 + \vec{R}_2 + \vec{T}_2 + \vec{f}_2 = \mathrm{m}_2\vec{\gamma} \end{cases}$$

Projections: $(m_1): \begin{cases} (Ox): T_1 - f_1 = \mathrm{m}_1\gamma \ \ldots\ldots\ldots\ldots (1) \\ (Oy): R_1 - P_1 = 0 \end{cases}$

$(m_2): \begin{cases} (Ox): P_{2x} - T_2 - f_2 = \mathrm{m}_2\gamma \ \ldots\ldots (2) \\ (Oy): R_2 - P_{2y} = 0 \end{cases}$

Kinetic friction forces :

$$\begin{cases} f_1 = \mu_c R_1 \\ f_2 = \mu_c R_2 \end{cases} \Rightarrow \begin{cases} f_1 = \mu_c P_1 = \mu_c m_1 g \\ f_2 = \mu_c P_{2y} = \mu_c m_2 g \cos\alpha \end{cases}$$

The acceleration γ will be determined from eq (1) and (2): $\begin{cases} eq(3): \mathrm{m}_1\gamma = T_1 - f_1 \\ eq(4): \mathrm{m}_2\gamma = P_{2x} - T_2 - f_2 \end{cases}$

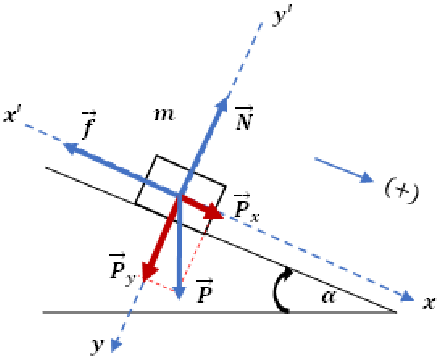
$$\Rightarrow \begin{cases} \mathrm{m}_1\gamma = T - \mu_c m_1 g \\ \mathrm{m}_2\gamma = m_2 g \sin\alpha - T - \mu_c m_2 g \cos\alpha \end{cases} \Rightarrow \mathrm{m}_1\gamma + \mathrm{m}_2\gamma = m_2 g \sin\alpha - \mu_c m_2 g \cos\alpha - \mu_c m_1 g$$

$$\gamma = \frac{\boldsymbol{m_2(g \sin\alpha - \mu_c\, g \cos\alpha) - \mu_c\, m_1 g}}{\mathbf{m_1 + m_2}} = \mathbf{1{,}07\ m.s^{-2}}$$

According to eq (1), the tension of the wire is: $\boldsymbol{T} = \mathbf{m_1}\boldsymbol{\gamma} + \boldsymbol{\mu_c\, m_1 g} = \mathbf{15.35\ N}$

Exercise-01

1/ Representation of forces



3/ Angle of inclination α

The body moves at constant velocity, this means that the sum of the forces must be zero (principle of inertia, Newton's 1st law):

$$V = C^{te} \Rightarrow \sum \vec{F}_{ext} = \vec{0}$$

$$\vec{P} + \vec{f} + \vec{N} = \vec{0}$$

By projection on the two axes, it comes:

$$\begin{cases} P_x - f = 0 \\ p_y - N = 0 \end{cases} \Rightarrow \begin{cases} P_x = f \\ p_y = N \end{cases} \Rightarrow \begin{cases} p \sin\alpha = \mu N \\ p \cos\alpha = N \end{cases}$$

$$\Rightarrow \frac{p \sin\alpha}{p \cos\alpha} = \frac{\mu N}{N} \Rightarrow \tan\alpha = \mu \Rightarrow \quad \boldsymbol{\alpha = 21.8°}$$

4/ The normal reaction force N for $\alpha = 35°$

$$p \cos\alpha = N \Rightarrow N = mg\cos\alpha = 8\cos 35$$

$$\boldsymbol{N = 6.55\ N}$$

3/ The friction force f for $\alpha = 35°$

$$f = \mu N \Rightarrow \boldsymbol{f = 2.62\ N}$$

5/ Acceleration for α= 35°

Angle of inclination 35° >21.8° (case of constant Velocity), this means that the Velocity is variable (fundamental principle of dynamics, Newton's 2nd law):

$$PFD \Rightarrow \sum \vec{F}_{ext} = m\vec{\gamma}$$

By projection on the two axes, it comes:

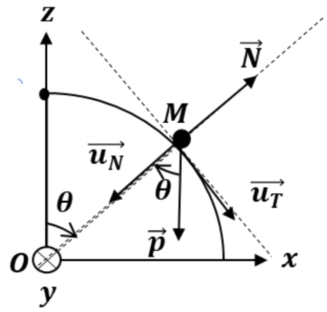
$$\begin{cases} P_x - f = m\gamma \\ p_y - N = 0 \end{cases} \Rightarrow p \sin\alpha - f = m\gamma$$

$$\Rightarrow \gamma = \frac{mg \sin\alpha - f}{m} \Rightarrow \boldsymbol{\gamma = 2.46\ m/s^2}$$

Exercise-03

1/ The forces acting on the mass m at the point M

The forces acting on point M are: $\vec{P}$ (the weight force) and $\vec{N}$(the contact force).



2/ Equations of movement

Using the fundamental principle of dynamics (Newton's 2nd law (FPD)), in the intrinsic coordinate system:

$$\sum \vec{F}_{ext} = m\vec{\gamma}$$

$$\Rightarrow \vec{P} + \vec{N} = m\vec{\gamma}$$

By projecting the forces onto the tangential (T) and the normal (N) axes, we determine their component:

$$\vec{P}\begin{pmatrix} mg\sin\theta \\ mg\cos\theta \end{pmatrix} ; \vec{N}\begin{pmatrix} 0 \\ -N \end{pmatrix}$$

$$\text{So: } \vec{P}\begin{cases} mg\sin\theta \\ mg\cos\theta \end{cases} + \vec{N}\begin{cases} 0 \\ -N \end{cases} = m\vec{\gamma}\begin{cases} \gamma_T \\ \gamma_N \end{cases}$$

$$\text{knowing that : } \begin{cases} \gamma_T = \dfrac{dV}{dt} \\ \gamma_N = \dfrac{V^2}{R} \end{cases}$$

The equations of movement will be as follows:

$$\Rightarrow \begin{cases} \boldsymbol{mg\sin\theta + 0 = m\dfrac{dV}{dt}} \ldots\ldots\ldots (1) \\ \boldsymbol{mg\cos\theta - N = m\dfrac{V^2}{R}} \ldots\ldots.. (2) \end{cases}$$

3/ Show that $V(M) = \sqrt{2Rg\,(1 - \cos\theta)}$.

According to eq (1), we have:

$$mg\sin\theta = m\frac{dV}{dt}$$

$$\Rightarrow g\sin\theta = \frac{dV}{d\theta}\frac{d\theta}{dt} = \frac{dV}{d\theta}\frac{V}{R}$$

$$\text{and } \begin{cases} \omega = \dfrac{V}{R} : angular\ velocity,\ because \\ V = \dfrac{ds}{dt} = R\dfrac{d\theta}{dt} = R\omega \\ \omega = \dfrac{d\theta}{dt} \end{cases}$$

$$\Rightarrow \boldsymbol{\frac{d\theta}{dt} = \frac{V}{R}}$$

$$\Rightarrow V.dV = g\,R\sin\theta\ d\theta$$

$$\Rightarrow \int_0^V V.dV = g\,R\int_0^\theta \sin\theta\ d\theta$$

$$\Rightarrow \boldsymbol{V(M) = \sqrt{2Rg\,(1 - \cos\theta)}}$$

4/ **Expression of the contact force N**

According to eq (2), we have:

$$mg\cos\theta - N = m\frac{V^2}{R}$$

$$\Rightarrow N = mg\cos\theta - m\frac{V^2}{R}$$

$$\Rightarrow N = mg\cos\theta - 2mg(1-\cos\theta)$$

$$\boldsymbol{N = mg\,(3\cos\theta - 2)}$$

5/ **Angle θ_L and velocity V_L at which the particle leaves the surface of the sphere**

When the particle leaves the surface of the sphere: the contact force N is canceled, we write:

$$\Rightarrow N = mg(3\cos\theta - 2) = 0$$

$$\Rightarrow \cos\theta = \frac{2}{3}$$

$$\Rightarrow \boldsymbol{\theta_L = arccos\left(\frac{2}{3}\right) = 48°}$$

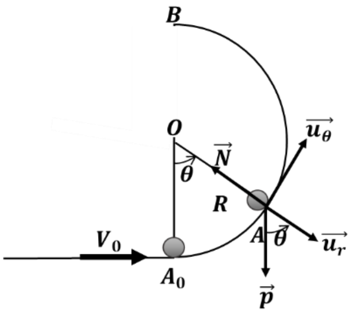
The corresponding velocity V_L is:

$$V = \sqrt{2Rg\,(1-\cos(48°)} = \sqrt{2Rg\,(1-2/3)}$$

$$\Rightarrow \boldsymbol{V_L = \sqrt{\frac{2}{3}Rg}}$$

Exercise-04

1/ **Equations of movement**



According to the Newton's 2nd law (PDF):

$$\sum \vec{F}ext = m\vec{\gamma} \Longrightarrow \vec{P} + \vec{N} = m\vec{\gamma}$$

By projection on the radial and angular axes:

$$\vec{P}\begin{cases} mg\cos\theta \\ -mg\sin\theta \end{cases} + \vec{N}\begin{cases} -N \\ 0 \end{cases} = m\vec{\gamma}\begin{cases} \boldsymbol{\gamma_r} \\ \boldsymbol{\gamma_\theta} \end{cases}$$

$$\begin{cases} \gamma_r = \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2 \\ \gamma_\theta = 2\dfrac{dr}{dt}\dfrac{d\theta}{dt} + r\dfrac{d^2\theta}{dt^2} \end{cases} \xRightarrow{r=R=cst} \begin{cases} \gamma_r = -R\left(\dfrac{d\theta}{dt}\right)^2 \\ \gamma_\theta = R\dfrac{d^2\theta}{dt^2} \end{cases}$$

The equations of movement will be as follows:

$$\Rightarrow \begin{cases} \boldsymbol{mg\cos\theta - N = -mR\left(\dfrac{d\theta}{dt}\right)^2} \;\dots\dots(1) \\ \boldsymbol{-mg\sin\theta = mR\dfrac{d^2\theta}{dt^2}} \;\dots\dots\dots\dots\dots(2) \end{cases}$$

2/ Angular velocity ω_M

According to eq (2), we have:

$$-mg\sin\theta = mR\frac{d}{dt}\left(\frac{d\theta}{dt}\right); \qquad \omega = \frac{d\theta}{dt}$$

$$\Rightarrow -mg\sin\theta = mR\frac{d\omega}{dt}\frac{d\theta}{d\theta}$$

$$\Rightarrow \int_{\omega 0}^{\omega} \omega\, d\omega = -\frac{g}{R}\int_0^\theta \sin\theta\, d\theta$$

$$\Rightarrow \boldsymbol{\omega_M = \sqrt{{\omega_0}^2 - \frac{2g}{R}(1-\cos\theta)}}$$

3/ Linear velocity V_M

We have: $\omega = \frac{V}{R} \Rightarrow \boldsymbol{V_M = \sqrt{{V_0}^2 - 2gR(1-\cos\theta)}}$

4/ Contact force N

According to eq (1), we have:

$$mg\cos\theta - N = -mR\left(\frac{d\theta}{dt}\right)^2$$

$$\boldsymbol{N = mg\,(2 - \cos\theta - {\omega_0}^2)}$$

5/ Velocity V_0

For mobile M to arrive at point B, it must:

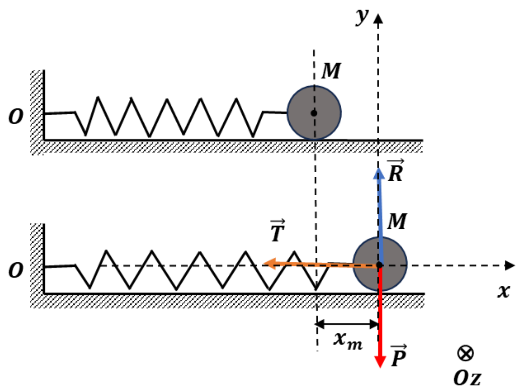
$$V_M > 0 \; and \; \theta = \pi \qquad \Rightarrow {V_0}^2 - 2gR(1-\cos\theta) > 0$$

$$\boldsymbol{V_0 > \sqrt{4gR}}$$

Exercise-05

a. For horizontal plane

1) Representation of forces



2) Differential equation of movement

using 2nd law of Newton:

$$\sum \overrightarrow{F_{ext}} = m\vec{\gamma}$$

$$\vec{P} + \vec{R} + \vec{T} = m\vec{\gamma}$$

By projection:

$$\begin{cases} Ox: \quad -T = m\gamma \ldots\ldots\ldots (1) \\ Oy: \quad R - P = 0 \ldots\ldots\ldots (2) \end{cases}$$

From eq (1): $-T = m\gamma$

$$\Longrightarrow -kx = m\frac{d^2x}{dt^2}$$

$$\boldsymbol{\frac{d^2x}{dt^2} + \frac{k}{m}x} = 0$$

$$\boldsymbol{or:} \quad \boldsymbol{\ddot{x} + \frac{k}{m}x} = 0$$

it presents the differential equation of 2nd order, which takes the form: $\boldsymbol{\ddot{x} + \omega_0 x} = 0$

3) Natural oscillation frequency ω_0

From the form: $\ddot{x} + \omega_0 x = 0$

$$\boldsymbol{\omega_0} = \sqrt{\boldsymbol{\frac{k}{m}}}$$

4) Constants *A* and *B*

We have: $x(t) = A\ sin\ (\omega_0 t) + B\ cos\ (\omega_0 t)$

And: $\dot{x}(t) = A\omega_0\ cos\ (\omega_0 t) - B\omega_0\ sin\ (\omega_0 t)$

From the initial conditions: $\begin{cases} x(t=0) = x_m \\ \dot{x}(t=0) = 0 \end{cases}$

Therefore: $\begin{cases} x(0) = B = x_m \\ \dot{x}(0) = A\omega_0 \end{cases} \Rightarrow \begin{cases} \boldsymbol{B = x_m} \\ \boldsymbol{A = 0} \end{cases}$

Solution : $\boldsymbol{x(t) = x_m\ cos\ (\sqrt{\frac{k}{m}}t)}$

5) Natural oscillation period T_0

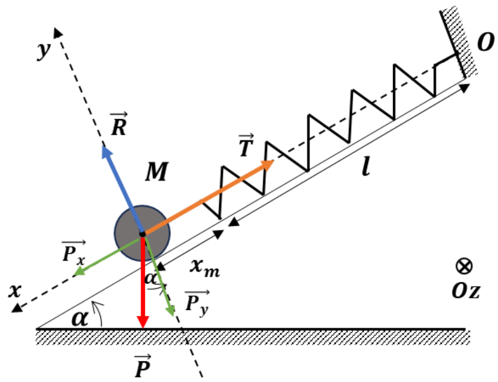
$$T_0 = \frac{2\pi}{\omega_0}$$

$$\boldsymbol{T_0 = 2\pi\sqrt{\frac{m}{k}}}$$

b. For inclined plane (α)

1) Representation of forces

Components: $\vec{P}\begin{pmatrix} mg\sin\alpha \\ mg\cos\alpha \\ 0 \end{pmatrix}; \vec{R}\begin{pmatrix} 0 \\ R \\ 0 \end{pmatrix}; \vec{T}\begin{pmatrix} -T \\ 0 \\ 0 \end{pmatrix}$



2) Equations of movement

using 2nd law of Newton:

$$\sum \overrightarrow{F_{ext}} = m\vec{\gamma}$$

$$\vec{P} + \vec{R} + \vec{T} = m\vec{\gamma}$$

By projection:

$$\begin{cases} Ox: \quad mg\sin\alpha - T = m\gamma \ldots\ldots\ldots (1) \\ Oy: \quad -mg\cos\alpha + R = 0 \ldots\ldots (2) \end{cases}$$

3) Static equilibrium position l

At equilibrium: $V = 0 \Longrightarrow \gamma = 0$

From eq (1): $mg\sin\alpha - T = 0$

$\Longrightarrow mg\sin\alpha = k(\Delta l)$

$\Longrightarrow mg\sin\alpha = k(l - l_0)$

$\Longrightarrow \boldsymbol{l = l_0 + \frac{mg}{k}\sin\alpha}$

3) Differential equation of movement

From eq (1): $mg\sin\alpha - T = m\gamma$

$$\Longrightarrow mg\sin\alpha - k(\Delta l + x) = m\frac{d^2x}{dt^2}$$

$$\Longrightarrow kx = m\frac{d^2x}{dt^2}$$

Because: $mg\sin\alpha = k(\Delta l)$

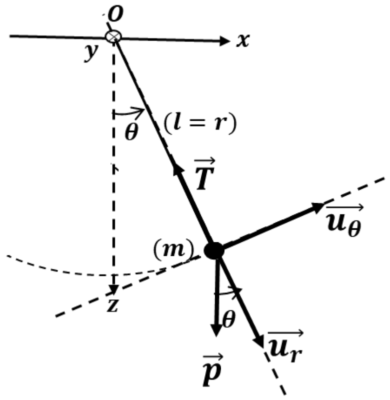
$$\Longrightarrow \boldsymbol{\frac{d^2x}{dt^2} + \frac{k}{m}x} = 0$$

$$\Longrightarrow \boldsymbol{\ddot{x} + \frac{k}{m}x} = 0$$

We obtain the same differential equation of 2nd order, which takes the form: $\ddot{x} + \omega_0 x = 0$ and having the same solution: $\boldsymbol{x(t) = x_m\ cos\ (\sqrt{\frac{k}{m}}t)}$

Exercise-06

1/ **The forces acting on the point M and their components in the polar R.**



The forces acting on point M are: $\vec{P}$ (the weight force) and $\vec{T}$ (the tension force). By projecting the forces onto the radial (r) and the angular (θ) axes we determine their component:

$$\vec{P}\begin{pmatrix} mg\cos\theta \\ -mg\sin\theta \end{pmatrix} \;;\; \vec{T}\begin{pmatrix} -T \\ 0 \end{pmatrix}$$

2/ **Differential Equation of movement**

a. **By applying the fundamental principle of dynamics (FPD)**

According to Newton's 2nd law (PDF):

$$\sum \vec{F}ext = \text{m}\vec{\gamma}$$

$$\Longrightarrow \vec{P} + \vec{T} = \text{m}\vec{\gamma}$$

By projection on the radial and angular axes:

$$\vec{P}\begin{cases} mg\cos\theta \\ -mg\sin\theta \end{cases} + \vec{T}\begin{cases} -T \\ 0 \end{cases} = \text{m}\vec{\gamma}\begin{cases} \boldsymbol{\gamma_r} \\ \boldsymbol{\gamma_\theta} \end{cases}$$

$$\begin{cases} \gamma_r = \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2 \\ \gamma_\theta = 2\dfrac{dr}{dt}\dfrac{d\theta}{dt} + r\dfrac{d^2\theta}{dt^2} \end{cases} \xRightarrow{r=l=cst} \begin{cases} \gamma_r = -l\left(\dfrac{d\theta}{dt}\right)^2 \\ \gamma_\theta = l\dfrac{d^2\theta}{dt^2} \end{cases}$$

The equations of movement will be as follows:

$$\Rightarrow \begin{cases} \boldsymbol{mg\cos\theta - T = -ml\left(\dfrac{d\theta}{dt}\right)^2} \;\dots\dots(\mathbf{1}) \\ \boldsymbol{-mg\sin\theta = ml\dfrac{d^2\theta}{dt^2}} \;\dots\dots\dots\dots\dots(\mathbf{2}) \end{cases}$$

According to eq (2), we write:

$$-mg\sin\theta = ml\frac{d^2\theta}{dt^2} \Rightarrow -g\sin\theta = l\frac{d^2\theta}{dt^2}$$

The differential equation of movement is therefore written:

$$\boldsymbol{\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0}$$

b. **By applying the kinetic moment theorem**

Kinetic moment theorem: the derivative of kinetic moment about a point O is equal to the sum of the moments of the external forces applied to material point M about a point O.

$$\boldsymbol{\frac{d\overrightarrow{L_O}}{dt} = \sum \overrightarrow{M_O}(\vec{F})}$$

$$\frac{d\overrightarrow{L_O}}{dt} = \overrightarrow{M_O}(\vec{P}) + \overrightarrow{M_O}(\vec{T}) \dots\dots\dots.\boldsymbol{eq}(*)$$

*Let's calculate the kinetic moment $\overrightarrow{L_O}$:

$$\overrightarrow{L_O} = \overrightarrow{OM} \wedge \vec{p}$$

$$\text{where :}\begin{cases} \overrightarrow{OM} = l\,\overrightarrow{u_r} \\ \vec{p} = m\,\vec{V} = m\,(\dfrac{dr}{dt}\overrightarrow{u_r} + r\,\dfrac{d\theta}{dt}\,\overrightarrow{u_\theta}) \end{cases}$$

knowing that: $r = l = cst$

$$\begin{cases} \overrightarrow{OM} = l\,\overrightarrow{u_r} \\ \vec{p} = ml\,\dfrac{d\theta}{dt}\,\overrightarrow{u_\theta} \end{cases} \Rightarrow \overrightarrow{L_O} = \overrightarrow{OM} \wedge \vec{p} = ml^2\,\frac{d\theta}{dt}\,\vec{k}$$

$$\boldsymbol{\overrightarrow{L_O} = ml^2\,\frac{d\theta}{dt}\,\vec{k}}$$

$$\boldsymbol{\frac{d\overrightarrow{L_O}}{dt} = ml^2\,\frac{d^2\theta}{dt^2}\,\vec{k}} \dots\dots\; (\mathbf{3})$$

Let's calculate the moment of forces: $\overrightarrow{M_O}(\vec{P})$, $\overrightarrow{M_O}(\vec{T})$

$$\overrightarrow{M_O}(\vec{P}) = \overrightarrow{OM} \wedge \vec{P}$$

$$\text{where :}\begin{cases} \overrightarrow{OM} = l\,\overrightarrow{u_r} \\ \vec{P} = mg\cos\theta\,\overrightarrow{u_r} - mg\sin\theta\,\overrightarrow{u_\theta}) \end{cases}$$

$$\Rightarrow \boldsymbol{\overrightarrow{M_O}(\vec{P}) = -mgl\sin\theta\;\vec{k}} \dots\dots(\mathbf{4})$$

$$\overrightarrow{M_O}(\vec{T}) = \overrightarrow{OM} \wedge \vec{T} \Rightarrow \begin{cases} \overrightarrow{OM} = l\,\overrightarrow{u_r} \\ \vec{T} = -T\,\overrightarrow{u_r} + 0\,\overrightarrow{u_\theta} \end{cases}$$

$$\Rightarrow \boldsymbol{\overrightarrow{M_O}(\vec{T}) = \vec{0}} \dots\dots..(\mathbf{5})$$

We replace eqs (3, 4, 5) in (*):

$$ml^2\,\frac{d^2\theta}{dt^2}\,\vec{k} = -mgl\sin\theta\,\vec{k}$$

The differential equation of movement is therefore written:

$$\boldsymbol{\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0}$$

c. **c. By applying the mechanical energy theorem**

$$E_m = E_c + E_P$$

$$\Rightarrow E_m = \frac{1}{2}m\,V^2 + mgh$$

$$\text{knowing that: }\begin{cases} V = r\,\dfrac{d\theta}{dt} = l\,\dfrac{d\theta}{dt} \\ h = l(1 - \cos\theta) \end{cases}$$

$$\Rightarrow E_m = \frac{1}{2} m\, l^2 \left(\frac{d\theta}{dt}\right)^2 + mgl - mgl\cos\theta$$

As long as there are no friction forces, the mechanical energy is therefore conserved, i.e.:

$$\frac{dE_m}{dt} = 0$$

$$\Rightarrow \frac{dE_m}{dt} = m\, l^2 \frac{d\theta}{dt}\frac{d^2\theta}{dt^2} + mgl\sin\theta\frac{d\theta}{dt} = 0$$

The differential equation of movement is therefore written:

$$\Rightarrow \boldsymbol{\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0}$$

3/ Equation of movement $\boldsymbol{\theta(t)}$

For very small amplitude oscillations $\sin\theta \approx \theta$, the differential equation is written in the form:

$$\Rightarrow \frac{d^2\theta}{dt^2} + \frac{g}{l}\ \theta\ = 0$$

This differential equation is of the 2nd degree, with the form: $y'' + \omega^2\, y\ = 0$, it has a solution of the form:

$$\theta(t) = A\sin(\omega t) + B\cos(\omega t), \quad \omega = \sqrt{\frac{g}{l}}$$

To determine A and B, we use the initial conditions:

$$at\ t = 0\ s: \begin{cases} \theta = \theta_0 \\ V_0 = 0 \end{cases} \qquad \Rightarrow \frac{d\theta}{dt} = \frac{V_0}{r} = 0$$

$$we\ have: \begin{cases} \theta(t) = A\sin(\omega t) + B\cos(\omega t) \\ \dot\theta = A\omega\,(\cos(\omega t) - Bsin\,(\omega t)) \end{cases}$$

So: $at\ t = 0\ s$:

$$\begin{cases} \theta(0) = A\sin(0) + B\cos(0)\ =\ \theta_0 \\ \dot\theta = A\omega\,(\cos(0) - Bsin\,(0)) = 0 \end{cases}$$

So: $\begin{cases} B\ =\ \theta_0 \\ A = 0 \end{cases}$ we replace them in the solution $\theta(t)$:

$$\boldsymbol{\theta(t) = \theta_0\cos\left(\sqrt{\frac{g}{l}}\,t\right)}$$