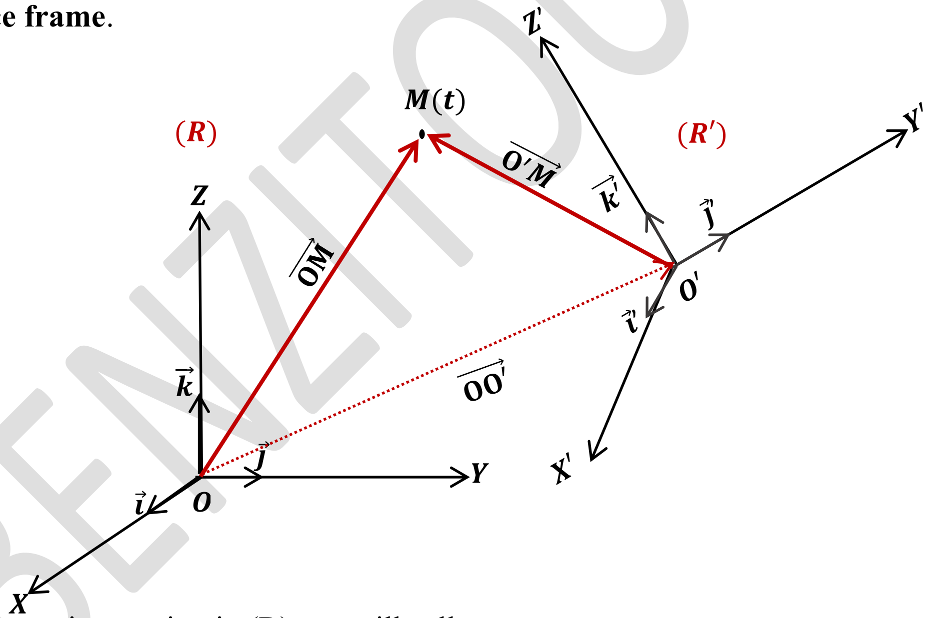
Chapter IV: Relative movement

Summary

1. Relative movement

1.1. Definition

We consider a reference frame (R) provided with a fixed orthonormal basis $R\ (O,\vec{\imath},\vec{\jmath},\vec{k})$. This fixed reference frame will be called an **absolute reference frame**. On the other hand, we consider another reference frame (R') provided with an orthonormal basis $R'(O',\vec{\imath'},\vec{\jmath'},\vec{k'})$ in motion with respect to (R), this moving reference frame will be called a **relative reference frame**.



Let M be a point moving in (R), we will call:

- **Absolute movement**, the movement of M with respect to (R)
- **Relative movement**, the movement of M with respect to (R')
- **Training movement**, the movement of (R') with respect to (R)

1.2. Calculation of velocity and acceleration

The calculation of the absolute velocity as well as the absolute acceleration of the mobile M with respect to the fixed reference frame (R) is carried out by two methods: the direct method (or direct derivative) and the compositios of velocities and accelerations method.

1.2.1. Direct method

The velocity and acceleration of the mobile M with respect to the absolute reference (R) and the relative reference (R') are given by:

a) Absolute quantities:

The movement of M with respect to the absolute reference frame (R) is characterized by:

Position vector: $\overrightarrow{OM} = x\,\vec{i} + y\,\vec{j} + z\,\vec{k}$

Absolute velocity : $\overrightarrow{V_a}(M) = \left.\dfrac{d\overrightarrow{OM}}{dt}\right|_R = \dfrac{dx}{dt}\,\vec{i} + \dfrac{dy}{dt}\,\vec{j} + \dfrac{dz}{dt}\,\vec{k}$

Absolute acceleration : $\overrightarrow{\gamma_a}(M) = \left.\dfrac{d\overrightarrow{V_a}}{dt}\right|_R = \left.\dfrac{d^2\overrightarrow{OM}}{dt^2}\right|_R = \dfrac{d^2x}{dt^2}\,\vec{i} + \dfrac{d^2y}{dt^2}\,\vec{j} + \dfrac{d^2z}{dt^2}\,\vec{k}$

b) Relative quantities:

The movement of M with respect to the relative reference frame (R') is characterized by:

Position vectory: $\overrightarrow{OM'} = x'\,\vec{i'} + y'\,\vec{j'} + z'\,\vec{k'}$

Relative velocity : $\overrightarrow{V_r}(M) = \left.\dfrac{d\overrightarrow{O'M}}{dt}\right|_{R'} = \dfrac{dx'}{dt}\,\vec{i'} + \dfrac{dy'}{dt}\,\vec{j'} + \dfrac{dz'}{dt}\,\vec{k'}$

Relative *acceleration* : $\overrightarrow{\gamma_r}(M) = \left.\dfrac{d\overrightarrow{V_r}}{dt}\right|_{R'} = \left.\dfrac{d^2\overrightarrow{O'M}}{dt^2}\right|_{R'} = \dfrac{d^2x'}{dt^2}\,\vec{i'} + \dfrac{d^2y'}{dt^2}\,\vec{j'} + \dfrac{d^2z'}{dt^2}\,\vec{k'}$

1.2.2. Composition of velocities and accelerations

1/ Composition of velocities

The composition of the absolute velocity vector $\overrightarrow{V_a}$ is expressed as a function of the relative velocity vector $\overrightarrow{V_r}$, as follows.

We have:

$$\vec{V_a} = \left.\frac{d\overrightarrow{OM}}{dt}\right|_R = \frac{d}{dt}\left(\overrightarrow{OO'} + \overrightarrow{O'M}\right) = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \left.\frac{d\overrightarrow{O'M}}{dt}\right|_R$$

$$\vec{V_a} = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \left[\frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'}\right] + \left[\frac{d\vec{\imath'}}{dt}x' + \frac{d\vec{\jmath'}}{dt}y' + \frac{d\vec{k'}}{dt}z'\right]$$

The reference frame (R') is mobile, the unit vectors $(\vec{\imath'}, \vec{\jmath'}, \vec{k'})$ are therefore not constant over time.

The composition of the absolute velocity is therefore written:

$$\vec{V_a} = \vec{V_r} + \left[\left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \left[\frac{d\vec{\imath'}}{dt}x' + \frac{d\vec{\jmath'}}{dt}y' + \frac{d\vec{k'}}{dt}z'\right]\right]$$

The term in the square brackets describes the movement of the reference $R'(O', \vec{\imath'}, \vec{\jmath'}, \vec{k'})$ with respect to the reference (R) and we call it the training velocity of (R') with respect to (R), denoted $\vec{V_e}$, and we write:

$$\vec{v_e} = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \left[\frac{d\vec{\imath'}}{dt}x' + \frac{d\vec{\jmath'}}{dt}y' + \frac{d\vec{k'}}{dt}z'\right]$$

Then, the absolute velocity is written by the law of composition of velocities, as follows:

$$\boldsymbol{\vec{V_a} = \vec{V_r} + \vec{V_e}}$$

Discussion on training velocity $\vec{V_e}$:

The term between the square brackets of the training velocity $\vec{V_e}$ contains the derivative of the unit vectors $(\vec{\imath'}, \vec{\jmath'}, \vec{k'})$ of the moving frame (R'). So, to identify this term we distinguish two following cases:

- **(R') in translation with respect to (R):**

  (R') in translation with respect to (R), i.e.: $\vec{\imath} = \vec{\imath'}, \vec{\jmath} = \vec{\jmath'}, \vec{k} = \vec{k'}$

$$\left[\frac{d\vec{\imath'}}{dt}x' + \frac{d\vec{\jmath'}}{dt}y' + \frac{d\vec{k'}}{dt}z'\right] = \vec{0}$$

  The training velocity will then write:

$$\vec{V_e} = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R$$

- **(R') in rotation with respect to (R):**

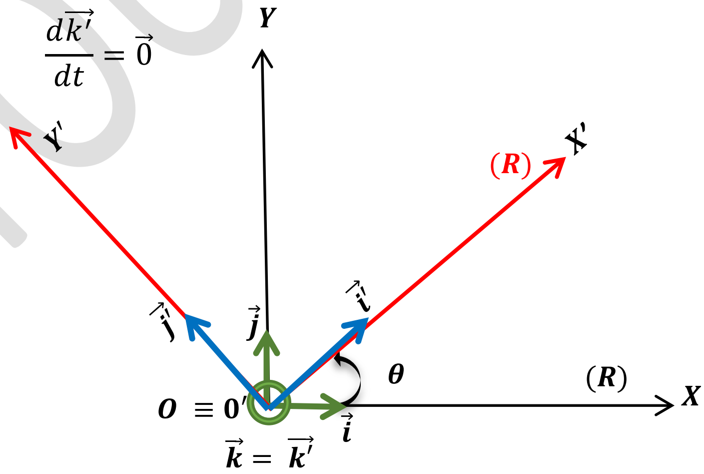
We consider that the axis of rotation of (R') with respect to (R) is Oz $(k = k')$. The angular velocity vector (rotation) is therefore: $\vec{\omega} = \frac{d\theta}{dt}\vec{k}$, or : $\vec{\omega} = \omega\,\vec{k}$. We know that:

$$\vec{V_e} = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \left[\frac{d\vec{i'}}{dt}x' + \frac{d\vec{j'}}{dt}y' + \frac{d\vec{k'}}{dt}z'\right]$$

The unit vectors of the moving reference frame (R') can be given as a function of that of the absolute reference frame according to the following relations:

$$\begin{cases} \vec{i'} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} \\ \vec{j'} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j} \end{cases}, \quad \text{where} \; : \begin{cases} \frac{d\vec{i'}}{dt} = \frac{d\vec{i'}}{d\theta}\frac{d\theta}{dt} = \omega\vec{j'} \\ \frac{d\vec{j'}}{dt} = \frac{d\vec{j'}}{d\theta}\frac{d\theta}{dt} = -\,\omega\vec{i'} \\ \frac{d\vec{k'}}{dt} = \vec{0} \end{cases}$$

We replace them in the expression of the training velocity:



$$\vec{V_e} = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \left[\omega\vec{j'}\,x' - \omega\vec{i'}\,y' + \vec{0}.z'\right]$$

$$\Rightarrow \quad \vec{V_e} = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_R + \omega\left[x'\vec{j'} - y'\vec{i'}\right]$$

The expression in parentheses can be determined by the vector product between the angular velocity vector $\vec{\omega}$ and the position vector $\overrightarrow{O'M}$:

$$\Rightarrow \boldsymbol{\vec{V_e}} = \left.\frac{\boldsymbol{d\overrightarrow{OO'}}}{\boldsymbol{dt}}\right|_{\boldsymbol{R}} + \boldsymbol{\vec{\omega}} \wedge \boldsymbol{\overrightarrow{O'M}}$$

We summarize:

$$\boldsymbol{\vec{V_a} = \vec{V_r} + \vec{V_e}}$$

$$\vec{V_r} = \left.\frac{d\overrightarrow{O'M}}{dt}\right|_{R'} = \frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'}$$

$$\vec{V_e} = \left.\frac{d\overrightarrow{OO'}}{dt}\right|_{R} + \vec{\omega} \wedge \overrightarrow{O'M}$$

To determine the derivative of a vector which belongs to the moving reference frame (R') with respect to the fixed reference frame (R), we use the following procedure:

$$\left.\frac{d\overrightarrow{O'M}}{dt}\right|_{R} = \left[\frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'}\right] + \left[\frac{d\vec{\imath'}}{dt}x' + \frac{d\vec{\jmath'}}{dt}y' + \frac{d\vec{k'}}{dt}z'\right]$$

We finally write:

$$\boldsymbol{\left.\frac{d\overrightarrow{O'M}}{dt}\right|_{R} = \left.\frac{d\overrightarrow{O'M}}{dt}\right|_{R'} + \vec{\omega} \wedge \overrightarrow{O'M}}$$

2/ Composition of accelerations

The composition of the absolute acceleration vector $\vec{\gamma_a}$ is expressed as a function of the relative acceleration vector $\vec{\gamma_r}$:

we have : $$\vec{\gamma_a} = \left.\frac{d\vec{V_a}}{dt}\right|_{R} = \left.\frac{d(\vec{V_r} + \vec{V_e})}{dt}\right|_{R} = \left.\frac{d\vec{V_r}}{dt}\right|_{R} + \left.\frac{d\vec{V_e}}{dt}\right|_{R}$$

We will now search to $\left.\frac{d\vec{V_r}}{dt}\right|_{R}$ and $\left.\frac{d\vec{V_e}}{dt}\right|_{R}$:

$\vec{V_r}$: is a vector defined in the frame (R'), therefore its derivative with respect to (R) takes the form:

$$\left.\frac{d\vec{V_r}}{dt}\right|_{R} = \left.\frac{d\vec{V_r}}{dt}\right|_{R'} + \vec{\omega} \wedge \vec{V_r};$$

$$\left.\frac{d\vec{V_e}}{dt}\right|_{R} = \frac{d}{dt}\left[\left.\frac{d\overrightarrow{OO'}}{dt}\right|_{R} + \omega \wedge \overrightarrow{O'M}\right]_{R} = \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_{R} + \left.\frac{d}{dt}(\vec{\omega} \wedge \overrightarrow{O'M})\right|_{R}$$

$$\left.\frac{d\vec{V_e}}{dt}\right|_{R} = \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_{R} + \left.\frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O'M}\right|_{R} + \vec{\omega} \wedge \left.\frac{d\overrightarrow{O'M}}{dt}\right|_{R}$$

$$\left.\frac{d\overrightarrow{V_e}}{dt}\right|_R = \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R + \left.\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M}\right|_R + \vec{\omega}\wedge\left[\left.\frac{d\overrightarrow{O'M}}{dt}\right|_{R'} + \vec{\omega}\wedge\overrightarrow{O'M}\right]$$

$$\left.\frac{d\overrightarrow{V_e}}{dt}\right|_R = \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R + \left.\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M}\right|_R + \vec{\omega}\wedge\left[\overrightarrow{V_r} + \vec{\omega}\wedge\overrightarrow{O'M}\right]$$

$$\left.\frac{d\overrightarrow{V_e}}{dt}\right|_R = \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R + \left.\frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M}\right|_R + \vec{\omega}\wedge\overrightarrow{v_r} + \vec{\omega}\wedge\left(\vec{\omega}\wedge\overrightarrow{O'M}\right)$$

We do now the sum between $\left.\frac{d\overrightarrow{V_r}}{dt}\right|_R$ and $\left.\frac{d\overrightarrow{V_e}}{dt}\right|_R$:

$$\overrightarrow{\gamma_a} = \left.\frac{d\overrightarrow{V_r}}{dt}\right|_R + \left.\frac{d\overrightarrow{V_e}}{dt}\right|_R$$

$$\overrightarrow{\gamma_a} = \left.\frac{d\overrightarrow{V_r}}{dt}\right|_{R'} + \vec{\omega}\wedge\overrightarrow{V_r} + \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R + \frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M} + \vec{\omega}\wedge\overrightarrow{V_r} + \vec{\omega}\wedge\left(\vec{\omega}\wedge\overrightarrow{O'M}\right)$$

$$\boldsymbol{\overrightarrow{\gamma_a} = \left.\frac{d\overrightarrow{V_r}}{dt}\right|_{R'} + 2\vec{\omega}\wedge\overrightarrow{V_r} + \left[\left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R + \frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M} + \vec{\omega}\wedge\left(\vec{\omega}\wedge\overrightarrow{O'M}\right)\right]}$$

We define:

- Relative acceleration: $\overrightarrow{\gamma_r} = \left.\frac{d\overrightarrow{V_r}}{dt}\right|_{R'}$
- Coriolis acceleration: $\overrightarrow{\gamma_c} = 2\vec{\omega}\wedge\overrightarrow{V_r}$
- Training acceleration: $\overrightarrow{\gamma_e} = \left.\frac{d^2\overrightarrow{OO'}}{dt^2}\right|_R + \frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M} + \vec{\omega}\wedge\left(\vec{\omega}\wedge\overrightarrow{O'M}\right)$.

Then, the absolute acceleration is written by the law of composition of accelerations, as follows:

$$\boldsymbol{\overrightarrow{\gamma_a} = \overrightarrow{\gamma_r} + \overrightarrow{\gamma_c} + \overrightarrow{\gamma_e}}$$