Tutorial n°4: Relative movement

(With change of reference)

Exercise 1

The coordinates of a moving particle M in a fixed reference frame $R_1(O,\ \vec{i},\ \vec{j},\ \vec{k})$ are given by $(x_1,\ y_1\ and\ \ z_1)$ and in the moving reference frame $R_2(O',\ \vec{i},\ \vec{j},\ \vec{k})$ are given by $(x_2,\ y_2\ and\ \ z_2)$. The reference frame (R_2) is in translation with respect to (R_1).

$$\begin{cases} x_1 = 2t^3 - 3 \\ y_1 = 5t^3 - 2t \\ z_1 = t^5 \end{cases} \quad ; \quad \begin{cases} x_2 = 2t^3 - 5t \\ y_2 = 5t^3 + 7 \\ z_2 = t^5 - t + 1 \end{cases}$$

1- Write the expression of the position vectors with respect to R_1 and R_2.

2- Give the expression of the velocity vectors with respect to R_1 and R_2.

3- Deduce the training velocity of R_2 with respect to R_1.

4- Give the expression of acceleration vectors with respect to R_1 and R_2.

5- Find the Coriolis acceleration and training acceleration of R_2 with respect to R_1.

Exercise 2

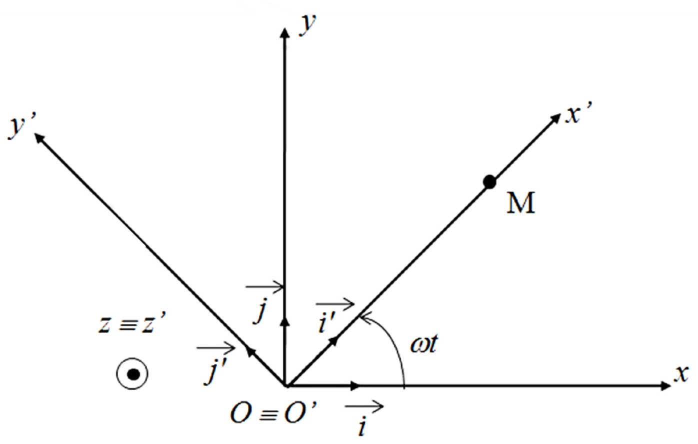
Let $R'(O'\ x'\ y'\ z')$ be a moving reference frame provided with a base $(\vec{i'},\vec{j'},\vec{k'})$ that turns around the (Oz) axis of a fixed reference frame $R(O\ x\ y\ z)$ provided with a base $(\vec{i},\vec{j},\vec{k})$ with an angular velocity $\vec{\omega} = \omega\vec{k}$. A mobile M $(OM\ = r)$ moves on the straight line Ox' according to the following hourly equation: $\boldsymbol{r\ =\ a\sin\omega t}$.

a and ω are real and positive constants.

1- Write the expression of the position vectors of M with respect to R and R'.

2- Calculate the absolute velocity $\vec{V_a}$ and the absolute acceleration $\vec{\gamma_a}$ of mobile M by the direct method.

3- Calculate the absolute velocity $\vec{V_a}$ and the absolute acceleration $\vec{\gamma_a}$ of mobile M by the composition of velocities and accelerations method.



Exercise 3

In the fixed reference frame $R(O, \vec{\imath}, \vec{\jmath}, \vec{k})$, a circle of radius $O'A$ rotates around the center O with a constant angular velocity $\vec{\omega} = \omega\vec{k}$. At the center O' of the circle, a moving reference frame $R'(O', \vec{\imath'}, \vec{\jmath'}, \vec{k'})$ is attached.

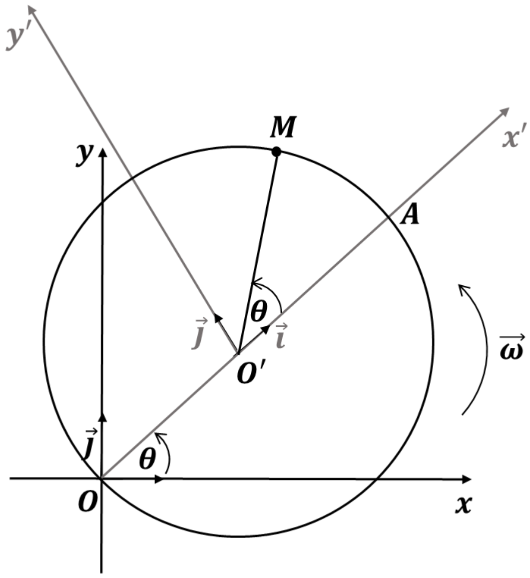
Consider a mobile M moves along the perimeter of the circle with the same angular velocity $\vec{\omega}$. At $t = 0$ s, M coincides with point A, on the Ox axis.

1- Write the expression of the position vectors of M with respect to R and R'.

2- Calculate the absolute velocity $\vec{V_a}$ and the absolute acceleration $\vec{\gamma_a}$ of mobile M by:

a. The direct method.

b. The composition of velocities and accelerations method.



Exercise-01

1) **The position vectors with respect to R_1 and R_2**

$\overrightarrow{OM}\Big|_{R_1} = (2t^3 - 3)\vec{\imath} + (5t^3 - 2t)\vec{\jmath} + t^5\vec{k}$

$\overrightarrow{O'M}\Big|_{R_2} = (2t^3 - 5t)\vec{\imath} + (5t^3 + 7)\vec{\jmath} + (t^5 - t + 1)\vec{k}$

2) **The velocity vectors with respect to R_1 and R_2**

$$\overrightarrow{V_a} = \frac{d\overrightarrow{OM}}{dt}\Big|_{R_1} = (6t^2)\vec{\imath} + (15t^2 - 2)\vec{\jmath} + 5t^4\vec{k}$$

$$\overrightarrow{V_r} = \frac{d\overrightarrow{O'M}}{dt}\Big|_{R_2} = (6t^2 - 5)\vec{\imath} + (15t^2)\vec{\jmath} + (5t^4 - 1)\vec{k}$$

3) **The training velocity of R_2 with respect to R_1**

Using the law of composition of velocities:

$\overrightarrow{V_e} = \overrightarrow{V_a} - \overrightarrow{V_r}$

$\overrightarrow{V_e} = 5\,\vec{\imath} - 2\,\vec{\jmath} + \vec{k}$

4) **The acceleration vectors with respect to R_1 and R_2**

$$\overrightarrow{\gamma_a} = \frac{d\overrightarrow{V_a}}{dt}\Big|_{R_1} = (12t^2)\vec{\imath} + (30t)\vec{\jmath} + (20t^4)\vec{k}$$

$$\overrightarrow{\gamma_r} = \frac{d\overrightarrow{V_r}}{dt}\Big|_{R_2} = (12t^2)\vec{\imath} + (30t)\vec{\jmath} + (20t^4)\vec{k}$$

5) **Coriolis acceleration and training acceleration**

Using the law of composition of accelerations:

$\overrightarrow{\gamma_a} = \overrightarrow{\gamma_r} + \overrightarrow{\gamma_c} + \overrightarrow{\gamma_e}$

Coriolis acceleration $\overrightarrow{\gamma_c}$

$\overrightarrow{\gamma_c} = 2\,\vec{\omega} \wedge \overrightarrow{V_r} = \vec{0}$,

The mvt of R_2 with respect to R_1 is a translational mvt, i.e. the angular velocity of rotation is zero ($\vec{\omega} = \vec{0}$)

$\boldsymbol{\overrightarrow{\gamma_c} = \vec{0}}$

Training acceleration $\overrightarrow{\gamma_e}$

$\overrightarrow{\gamma_e} = \overrightarrow{\gamma_a} - \overrightarrow{\gamma_r} - \overrightarrow{\gamma_c}$, $\Rightarrow \boldsymbol{\overrightarrow{\gamma_e} = \vec{0}}$

Exercise-02

1) **The position vectors with respect to R and R'**

We have: $\overrightarrow{O'M}\Big|_{R'} = \overrightarrow{OM}\Big|_{R'} = a\sin\omega t\,\vec{\imath'}$

Knowing that: $\begin{cases} \vec{\imath'} = \cos\omega t\,\vec{\imath} + \sin\omega t\,\vec{\jmath} \\ \vec{\jmath'} = -\sin\omega t\,\vec{\imath} + \cos\omega t\,\vec{\jmath} \end{cases}$

By replacing $\vec{\imath'}$ in the expression $\overrightarrow{OM}$, we obtain:

$$\begin{cases} \overrightarrow{OM}\Big|_R = a\sin\omega t\cos\omega t\,\vec{\imath} + a\sin^2\omega t\,\vec{\jmath} \\ \overrightarrow{O'M}\Big|_{R'} = a\sin\omega t\,\vec{\imath'} \end{cases}$$

2) **Absolute velocity $\overrightarrow{V_a}$ and absolute acceleration $\overrightarrow{\gamma_a}$ by using a direct method:**

$$\overrightarrow{V_a} = \frac{d\overrightarrow{OM}}{dt}\Big|_R = a\omega\,(\cos 2\omega t\ \vec{\imath} + \sin 2\omega t\,\vec{\jmath})$$

$$\overrightarrow{\gamma_a} = \frac{d\overrightarrow{V_a}}{dt}\Big|_R = 2a\omega^2\,(-\sin 2\omega t\ \vec{\imath} + \cos 2\omega t\,\vec{\jmath})$$

3) **Absolute velocity $\overrightarrow{V_a}$ by using the composition of velocities**

By using the law: $\boldsymbol{\overrightarrow{V_a} = \overrightarrow{V_r} + \overrightarrow{V_e}}$

- **Relative velocity $\overrightarrow{V_r}$**

$$\overrightarrow{V_r} = \frac{d\overrightarrow{O'M}}{dt}\Big|_{R'} = a\omega\cos\omega t\,\vec{\imath'}$$

$$\Rightarrow \boldsymbol{\overrightarrow{V_r} = a\,\omega\,(\cos^2\omega t\,\vec{\imath} + \sin\omega t\cos\omega t\ \vec{\jmath})}$$

- **Training velocity $\overrightarrow{V_e}$**

$\overrightarrow{V_e} = \frac{d\overrightarrow{O'O}}{dt} + \vec{\omega} \wedge \overrightarrow{O'M}$ Où : $\overrightarrow{O'O} = \vec{0}$ and $\vec{\omega} = \omega\vec{k'}$

$\Rightarrow \overrightarrow{V_e} = a\,\omega\sin\omega t\ \vec{\jmath'}$

$\Rightarrow \boldsymbol{\overrightarrow{V_e} = a\,\omega\,(-\sin^2\omega t\,\vec{\imath} + \sin\omega t\cos\omega t\ \vec{\jmath})}$

- **Absolute velocity $\overrightarrow{V_a}$**

$\boldsymbol{\overrightarrow{V_a} = a\,\omega\,(\cos 2\omega t\,\vec{\imath} + \sin 2\omega t\,\vec{\jmath})}$

4) **Absolute acceleration $\overrightarrow{\gamma_a}$ by using the composition of accelerations**

By using the law: $\boldsymbol{\overrightarrow{\gamma_a} = \overrightarrow{\gamma_r} + \overrightarrow{\gamma_c} + \overrightarrow{\gamma_e}}$

- **Relative acceleration $\overrightarrow{\gamma_r}$**

$$\overrightarrow{\gamma_r} = \frac{d\overrightarrow{V_r}}{dt}\Big|_{R'} = -\,a\omega^2\sin\omega t\,\vec{\imath'}$$

$$\Rightarrow \boldsymbol{\overrightarrow{\gamma_r} = -\,a\omega^2(\sin\omega t\cos\omega t\,\vec{\imath} + \sin^2\omega t\,\vec{\jmath})}$$

- **Training acceleration $\overrightarrow{\gamma_e}$**

$$\overrightarrow{\gamma_e} = \frac{d^2\overrightarrow{O'O}}{dt^2} + \frac{d\vec{\omega}}{dt} \wedge \overrightarrow{O'M} + \vec{\omega} \wedge (\vec{\omega} \wedge \overrightarrow{O'M})$$

where : $\overrightarrow{O'O} = \vec{0}$ *and* $\omega = c^t$

$\Rightarrow \overrightarrow{\gamma_e} = \vec{\omega} \wedge (\vec{\omega} \wedge \overrightarrow{O'M})$

$\Rightarrow \overrightarrow{\gamma_e} = \omega\vec{k'} \wedge (\omega\vec{k'} \wedge a\sin\omega t\,\vec{\imath'})$

$\Rightarrow \overrightarrow{\gamma_e} = -a\,\omega^2\sin\omega t\,\vec{\imath'}$

$\Rightarrow \boldsymbol{\overrightarrow{\gamma_e} = -a\,\omega^2(\sin\omega t\cos\omega t\,\vec{\imath} + \sin^2\omega t\,\vec{\jmath}\ )}$

- **Coriolis acceleration $\overrightarrow{\gamma_c}$**

$\overrightarrow{\gamma_c} = 2\ \vec{\omega} \wedge \overrightarrow{V_r} = 2\,\omega\,\vec{k'} \wedge a\omega\cos\omega t\,\vec{\imath'}$

$\Rightarrow \overrightarrow{\gamma_c} = 2\,a\,\omega^2\ \cos\omega t\,\vec{\jmath'}$

$\Rightarrow \boldsymbol{\overrightarrow{\gamma_c} = 2\,a\,\omega^2\ (-\cos\omega t\sin\omega t\,\vec{\imath} + \cos^2\omega t\,\vec{\jmath})}$

- **Absolute acceleration $\overrightarrow{\gamma_a}$**

$\Rightarrow \overrightarrow{\gamma_a} = -2\,a\omega^2(2\sin\omega t\cos\omega t\,\vec{\imath} + (\cos^2\omega t\,\vec{\jmath})$
$\quad - \sin^2\omega t\,\vec{\jmath})$

$\boldsymbol{\overrightarrow{\gamma_a} = -\,2\,a\omega^2(\sin 2\omega t\,\vec{\imath} + \cos 2\omega t\,\vec{\jmath}\,)}$

Exercise-03

$$\left\|\overrightarrow{O'M}\right\| = \left\|\overrightarrow{OO'}\right\| = R$$
$\vec{\omega} = \omega\vec{k};\ \theta = \omega t;\ Oz \equiv O'z$

1) Position vectors $\overrightarrow{OM}, \overrightarrow{O'M}, \overrightarrow{OO'}$

$$\begin{cases} \overrightarrow{OO'}\Big|_R = R\cos\theta\,\vec{\imath} + R\sin\theta\,\vec{\jmath} \\ \overrightarrow{O'M}\Big|_{R'} = R\cos\theta\,\vec{\imath'} + R\sin\theta\,\vec{\jmath'} \\ \overrightarrow{OM}\Big|_R = \overrightarrow{OO'} + \overrightarrow{O'M} \end{cases}$$

$\overrightarrow{OM}, \overrightarrow{O'M}, \overrightarrow{OO'}$ should be defined in the fixed R, so, we need to convert the vectors i', j' into i, j

$$\begin{cases} \vec{\imath'} = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath} \\ \vec{\jmath'} = -\sin\theta\,\vec{\imath} + \cos\theta\,\vec{\jmath} \end{cases}$$

$\overrightarrow{O'M} = R\,(\cos^2\theta - \sin^2\theta)\vec{\imath} + 2R\cos\theta\sin\theta\,\vec{\jmath}$

Therefore:

$$\begin{cases} \overrightarrow{\boldsymbol{OO'}}\Big|_R = \boldsymbol{R\cos\theta\,\vec{\imath} + R\sin\theta\,\vec{\jmath}} \\ \overrightarrow{\boldsymbol{O'M}}\Big|_{R'} = \boldsymbol{R\cos2\theta\,\vec{\imath} + R\sin2\theta\,\vec{\jmath}} \\ \overrightarrow{\boldsymbol{OM}}\Big|_R = \boldsymbol{R(\cos\theta + \cos2\theta)\,\vec{\imath} + R(\sin\theta + \sin2\theta)\,\vec{\jmath}} \end{cases}$$

2) Absolute velocity and acceleration by using a direct method:

$$\vec{V_a} = \frac{d\overrightarrow{OM}}{dt}\Big|_R = \frac{d\theta}{dt}\frac{d\overrightarrow{OM}}{d\theta} = \omega\frac{d\overrightarrow{OM}}{d\theta}$$

$$\boldsymbol{\vec{V_a} = -R\omega(\sin\theta + 2\sin2\theta)\,\vec{\imath} + R\omega(\cos\theta + 2\cos2\theta)\,\vec{\jmath}}$$

$$\vec{\gamma_a} = \frac{d\vec{V_a}}{dt}\Big|_R = \frac{d\theta}{dt}\frac{d\vec{V_a}}{d\theta} = \omega\frac{d\vec{V_a}}{d\theta}$$

$$\boldsymbol{\vec{\gamma_a} = -R\omega^2(\cos\theta + 4\cos2\theta)\,\vec{\imath} - R\omega^2(\sin\theta + 4\sin2\theta)\,\vec{\jmath}}$$

3) Absolute velocity and acceleration by using the composition of velocities:

Absolute velocity $\vec{V_a}$

By using the law: $\boldsymbol{\vec{V_a} = \vec{V_r} + \vec{V_e}}$

- **Relative velocity $\vec{V_r}$**

$$\vec{V_r} = \frac{d\overrightarrow{O'M}}{dt}\Big|_{R'}$$

$$\Rightarrow \boldsymbol{\vec{V_r} = -R\omega\sin2\theta\,\vec{\imath} + R\omega\cos2\theta\,\vec{\jmath}}$$

- **Training velocity $\vec{V_e}$**

$$\vec{V_e} = \frac{d\overrightarrow{O'O}}{dt} + \vec{\omega}\wedge\overrightarrow{O'M}$$

$$\boldsymbol{\vec{V_e} = -R\omega(\sin\theta + \sin2\theta)\,\vec{\imath} + R\omega(\cos\theta + \cos2\theta)\,\vec{\jmath}}$$

- **Absolute velocity $\vec{V_a}$**

$$\boldsymbol{\vec{V_a} = -R\omega(\sin\theta + 2\sin2\theta)\,\vec{\imath} + R\omega(\cos\theta + 2\cos2\theta)\,\vec{\jmath}}$$

Absolute acceleration $\vec{\gamma_a}$

By using the law: $\boldsymbol{\vec{\gamma_a} = \vec{\gamma_r} + \vec{\gamma_c} + \vec{\gamma_e}}$

- **Relative acceleration $\vec{\gamma_r}$**

$$\vec{\gamma_r} = \frac{d\vec{V_r}}{dt}\Big|_{R'}$$

$$\Rightarrow \boldsymbol{\vec{\gamma_r} = -R\omega^2\cos2\theta\,\vec{\imath} - R\omega^2\sin2\theta\,\vec{\jmath}}$$

- **Training acceleration $\vec{\gamma_e}$**

$$\vec{\gamma_e} = \frac{d^2\overrightarrow{O'O}}{dt^2} + \frac{d\vec{\omega}}{dt}\wedge\overrightarrow{O'M} + \vec{\omega}\wedge(\vec{\omega}\wedge\overrightarrow{O'M})$$

$\omega = c^t$

$$\Rightarrow \vec{\gamma_e} = \frac{d^2\overrightarrow{O'O}}{dt^2} + \vec{\omega}\wedge(\vec{\omega}\wedge\overrightarrow{O'M})$$

$$\boldsymbol{\vec{\gamma_e} = -R\omega^2(\cos\theta + \cos2\theta)\,\vec{\imath} - R\omega^2(\sin\theta + \sin2\theta)\,\vec{\jmath}}$$

- **Coriolis acceleration $\vec{\gamma_c}$**

$\vec{\gamma_c} = 2\,\vec{\omega}\wedge\vec{V_r}$

$$\Rightarrow \boldsymbol{\vec{\gamma_c} = -2R\omega^2\cos2\theta\,\vec{\imath} - 2R\omega^2\sin2\theta\,\vec{\jmath}}$$

- **Absolute acceleration $\vec{\gamma_a}$**

$$\vec{\gamma_a} = \boldsymbol{-R\omega^2(\cos\theta + 4\cos2\theta)\,\vec{\imath} - R\omega^2(\sin\theta + 4\sin2\theta)\,\vec{\jmath}}$$

Trigonometric relationships used:

$\boldsymbol{\cos(a+b) = \cos a\,\cos b - \sin a\sin b}$

$\boldsymbol{\sin(a+b) = \sin a\cos b + \cos a\sin b}$

$\boldsymbol{\cos(2a) = \cos^2 a - \sin^2 b}$

$\boldsymbol{\sin(2a) = 2\cos a\sin a}$