

Series of exercises 3 (questions marked * left to the students)

Exercise 01

Using the definition, prove that:

$$1) \lim_{x \rightarrow 3} \frac{x^2-1}{x^2+1} = \frac{4}{5}, \quad 2) \lim_{x \rightarrow \infty} \frac{x^2-1}{x^2+1} = 1, \quad 3) \lim_{x \rightarrow 1} \frac{x+2}{(x-1)^2} = +\infty,$$
$$4)^* \lim_{x \rightarrow -1} \frac{x+3}{x+2} = 2, \quad 5)^* \lim_{x \rightarrow \infty} \frac{2x^2+x+1}{x^2-3x} = 2, \quad 6)^* \lim_{\substack{x \rightarrow 1 \\ x > 1}} \frac{2x^2-x-2}{x^2-x} = -\infty.$$

Exercise 02 Calculate the following limits:

$$1) \lim_{x \rightarrow 1} \left(\frac{1}{1-x} - \frac{3}{1-x^3} \right), \quad 2^*) \lim_{x \rightarrow 1} \left(\frac{\sqrt{x} + \sqrt{x-1} - 1}{\sqrt{x^2-1}} \right) \quad 4) \lim_{x \rightarrow +\infty} (\sqrt{x^2-2x} - \sqrt{x^2-1}),$$
$$5) \lim_{x \rightarrow +\infty} x \ln \frac{x}{x+2}, \quad 6)^* \lim_{x \rightarrow +\infty} x \ln \frac{x^2+x}{x^2+2x+3}, \quad 7^*) \lim_{x \rightarrow +\infty} \left(\frac{x^2+1}{x^2-2} \right)^{x^2},$$
$$8) \lim_{x \rightarrow +\infty} \left(\frac{x+3}{x-2} \right)^{2x+1}, \quad 9) \lim_{x \rightarrow \pm\infty} \frac{x}{E(x)+1}, \quad 10)^* \lim_{x \rightarrow \pm\infty} x E(x),$$
$$11)^* \lim_{x \rightarrow +\infty} \frac{x + \cos(x^2+2x)}{x^2-2}.$$

Remark: For questions 5,6,7 and 8, we use the limit: $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$.

Exercise 04

Prove that the function f does not have a limit at x_0 in each of the following cases:

$$1) x_0 = 0 \text{ and } f(x) = \sin \frac{1}{x}, \quad 2)^* x_0 = \infty \text{ and } f(x) = \cos x,$$
$$3)^* x_0 = 0, \quad f = v \circ u, \quad u(x) = x \cos \frac{1}{x} \text{ and } v(x) = \begin{cases} 1, & x = 0 \\ 0, & x \neq 0 \end{cases}.$$

Exercise 05

Can the function f be continuously extended at x_0 in each of the following cases:

$$1) f(x) = \frac{x^3+5x+6}{x^5+1} \text{ and } x_0 = -1, \quad 2)^* f(x) = \frac{x^2-x}{\sin(x-1)} \text{ and } x_0 = 1,$$
$$3)^* f(x) = 1 - x \sin \frac{1}{x} \text{ and } x_0 = 0.$$

Exercise 06

Determine a and b such that the function f is continuous in its domain.

$$1) f(x) = \begin{cases} x & \text{si } |x| \leq 1 \\ x^2 + ax + b & \text{si } |x| > 1 \end{cases}, \quad 2)^* f(x) = \begin{cases} (x-1)^3 & \text{si } x \leq 0 \\ ax + b & \text{si } 0 < x < 1 \\ \sqrt{x} & \text{si } x > 1 \end{cases}.$$

Exercise 07

Apply the Mean value theorem to prove the following:

$$1) \forall x \in]0; 1[: 1 + x < e^x < \frac{1}{1-x}, \quad 2) \forall x \geq 0 : 2 \leq \sqrt{4+x^2} \leq 2 + \frac{x^2}{2}$$

3)* $na^{n-1}(b-a) < b^n - a^n < nb^{n-1}(b-a)$ where $0 < a < b$ and $n \geq 2$ be an integer.

Exercise 08 Calculate the following limits using L'Hôpital's rule:

$$1) \lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}, \quad 2) \lim_{x \rightarrow \frac{\pi}{3}} \frac{\sin(x - \frac{\pi}{3})}{1 - 2 \cos x}, \quad 3) \lim_{x \rightarrow +\infty} x \left(e^{\frac{2x+1}{x^2}} - 1 \right), \quad 4) \lim_{x \rightarrow 0} \frac{\tan x - x}{x - \sin x}$$

$$5) \lim_{x \rightarrow +\infty} \frac{\ln(2x^3 - 3x^2 + x - 1)}{2x - 1}, \quad 6) \lim_{x \rightarrow +\infty} x \ln \left[\tan \left(\frac{\pi}{4} + \frac{\pi}{x} \right) \right], \quad 7) \lim_{x \rightarrow 1} \left(\frac{1}{\ln x} - \pi \cotan \pi x \right)$$

$$8) \lim_{x \rightarrow 0} \frac{3 \tan 4x - 12 \tan x}{3 \sin 4x - 12 \sin x}, \quad 9) \lim_{x \rightarrow \frac{\pi}{4}} \frac{2\sqrt{2} - (\cos x + \sin x)^3}{1 - \sin 2x}, \quad 10) \lim_{x \rightarrow 0} \frac{1}{x} \ln \left(\frac{e^x - 1}{x} \right).$$

Exercise 09 Let the function f be defined on $\mathbb{R}$ as follows:
$$f(x) = \begin{cases} \frac{-x^2}{x+2} & \text{si } x \geq 0 \\ \ln \frac{2x^2+1}{x^2+1} & \text{si } x < 0 \end{cases}.$$

1) Examine the continuity of f over $\mathbb{R}$.

2) Is the function f differentiable at 0?

3) Express $f'(x)$ in terms of x .

4) Prove that f has an reciprocal function f^{-1} . This involves defining its definition domain and expressing $f^{-1}(x)$ in terms of x .

Exercise 10 * Let the function f be defined on $\mathbb{R}$ as follows
$$f(x) = \begin{cases} \frac{\sin ax}{x} & \text{si } x < 0 \\ e^{bx} - x & \text{si } x \geq 0 \end{cases}$$

where a and b are real numbers.

1) Determine a such that f is continuous at 0.

2) Determine the value of b so that f is differentiable at 0.

Exercise 11

I) Prove the following:

$$1) \forall x \in]-1, 1[: \tan \text{Arc sin } x = \frac{x}{\sqrt{1-x^2}}.$$

$$2) \forall x \geq 0 : \text{Arc tan}(x+1) - \text{Arc tan } x = \text{Arc tan} \frac{1}{1+x+x^2}.$$

$$3) \forall x \in [-1, 1] : \text{Arc cos } x + \text{Arc cos}(-x) = \pi.$$

II) Calculate the following limits using L'Hôpital's rule:

$$1) \lim_{x \rightarrow 0} \frac{x \text{Arc sin } x^2}{x \cos x - \sin x}, \quad 2) \lim_{x \rightarrow 0} \left(\frac{1}{x \text{Arc tan } x} - \frac{1}{x^2} \right), \quad 3) \lim_{x \rightarrow 0} (e^x + x)^{\frac{1}{x}}, \quad 4) \lim_{x \rightarrow 0} \left(\tan \frac{\pi x}{2x+1} \right)^{\frac{1}{x}},$$

$$5) \lim_{x \rightarrow \frac{\pi}{2}} (\tan x)^{2 \cos x}, \quad 6) \lim_{x \rightarrow 0} \frac{\cosh x - \cos x - x^2}{x^6}.$$