University of Oum El Bouaghi | Academic year: 2025/2026
Introduction to topology | License 2 - Mathematics

Sheet of exercises N°1

Exercise 1 *Let* $X =]0, +\infty[$. *For* $x, y \in X$, *we set*

$$d(x,y) = \left|\frac{1}{x} - \frac{1}{y}\right|.$$

1. *Show that* d *is a distance on* X.
2. *Show that, in the metric space* (X, d), $\mathbb{N}^*$ *is a bounded subset and* $\left\{\frac{1}{n}\ ;\ n \in \mathbb{N}^*\right\}$ *is an unbounded subset.*
3. *Is the metric space* (X, d) *complete? (Indication. Consider the real sequence* $x_n = n$*).*

Exercise 2 (Construction of distances) *We call a gauge a map* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *which is non-decreasing, only vanishing at* 0 *and sub-additive (i.e.* $\varphi(s+t) \leq \varphi(s) + \varphi(t)$, $\forall s, t \in \mathbb{R}_+$*).*

1. *Let* (X, d) *be a metric space and* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *a gauge. Show that* $d' = \varphi \circ d$ *is a distance on* X.
2. *Application: Deduce that*

   $$d_1 = \frac{d}{1+d}, \qquad d_2 = \min\{1, d\}, \qquad d_3 = \ln(1+d), \qquad \text{and} \qquad d_4 = d^\alpha \text{ with } 0 < \alpha < 1$$

   are distances on X.
3. *Show that* d *and* $d_1 = \frac{d}{1+d}$ *are topologically equivalent.*
4. *Show that* d *and* $d' = \varphi \circ d$ *are metrically equivalent if there exists a real constant* $C > 0$ *such that*

   $$C^{-1} t \leq \varphi(t) \leq Ct, \qquad \forall t \in \mathbb{R}^+.$$

Exercise 3 *Show that in a metric space every finite subset is closed.*

Exercise 4 *We equip* $\mathbb{R}$ *with the usual distance.*

1. *Describe the interior and the closure of the set* $A = \left\{\frac{1}{n}\ ;\ n \in \mathbb{N}^*\right\}$.
2. *Specify the accumulation points and the isolated points of* A.

Exercise 5 *In* $\mathbb{R}$ *equipped with the usual distance, consider the part*

$$A =]-\infty, -1[\cup \left\{0, \frac{\pi}{4}, \sqrt{3}\right\} \cup \left\{3 - \frac{1}{n}\ ;\ n \in \mathbb{N}^*\right\}.$$

1. *Determine* $\mathring{A}$ *and* $\overline{A}$.

2. *Determine the accumulation points and the isolated points of* A.

Exercise 6 *Let* (X, d) *be a metric space and let* A *and* B *be two non-empty subsets of* X. *Show that:*

1. $\forall x, y \in X : \quad |d(x, A) - d(y, A)| \leq d(x, y)$,

2. $\forall x \in X : \quad x \in \overline{A} \iff d(x, A) = 0$,

3. $\operatorname{diam}(\overline{A}) = \operatorname{diam}(A)$.

Exercise 7 *Show that in a metric space,*

1. *Every convergent sequence is a Cauchy sequence. The converse is false.*

2. *Every Cauchy sequence is bounded. The converse is false.*

3. *Every subsequence of a convergent sequence is convergent.*

4. *Every subsequence of a Cauchy sequence is a Cauchy sequence.*

5. *Every Cauchy sequence with a convergent subsequence is convergent.*

Exercise 8 *Let* d *and* d' *be two metrically equivalent distances on a set* X. *Show that:*

1. (X, d) *and* (X, d') *have the same bounded parts.*

2. (X, d) *and* (X, d') *have the same convergent sequences and the same Cauchy sequences.*

3. (X, d) *is complete if and only if* (X, d') *is complete.*

Exercise 9 *Let* d *and* d' *be two distances on* X. *Show that*

1. d *is finer than* d' *if and only if the identity map* $\mathrm{id} : (X, d) \to (X, d')$ *is continuous.*

2. d *and* d' *are topologically equivalent if and only if the identity map* $\mathrm{id} : X \to X$ *is a homeomorphism from* (X, d) *onto* (X, d').

Exercise 10 *Let* (X, d) *be a metric space.*

1. *Given a non-empty subset* A *of* X, *show that the map* $d(\cdot, A) : x \mapsto d(x, A)$ *from* X *to* $\mathbb{R}$ *is Lipschitzian.*

2. *We equip the product set* $X \times X$ *with the distance*

$$D_\infty((x_1, x_2), (y_1, y_2)) = \max\{d(x_1, y_1), d(x_2, y_2)\}$$

Show that the map $d : X \times X \to \mathbb{R}$ *is Lipschitzian.*

Exercise 11 *Show that the map* $f : \mathbb{R} \to \mathbb{R},\ x \mapsto \sqrt{x^2+1}$ *is Lipschitzian with Lipschitz constant* $k = 1$.

Exercise 12 *Let* $f : (X, d) \to (Y, d')$ *and* $g : (Y, d') \to (Z, d'')$ *be two uniformly continuous maps,* f *on* X *and* g *on* Y. *Show that the composition* $h = g \circ f : (X, d) \to (Z, d'')$ *is a uniformly continuous map on* X.

Exercise 13 *Let* (X, d) *be a metric space. Show that:*

1. *Any intersection of complete parts of* (X, d) *is complete.*
2. *A finite union of complete subsets of* (X, d) *is complete.*
3. *If all closed and bounded subsets are complete, then* (X, d) *is complete.*

Exercise 14 (Distance of the uniform convergence) *Let* X *be any non-empty set and let* (Y, d) *be a metric space. We denote by* $\mathcal{B}(X, Y)$ *the set of all maps* $f : X \to Y$ *which are bounded, that is to say which verify:* $\operatorname{diam}[f(X)] < +\infty$.

For $f, g \in \mathcal{B}(X, Y)$, *we denote*

$$d_\infty(f, g) = \sup_{x \in X} d\left(f(x), g(x)\right).$$

1. *Show that* d_∞ *is a distance on* $\mathcal{B}(X, Y)$.
2. *Show that* $(\mathcal{B}(X, Y), d_\infty)$ *is a complete space if* (Y, d) *is complete.*

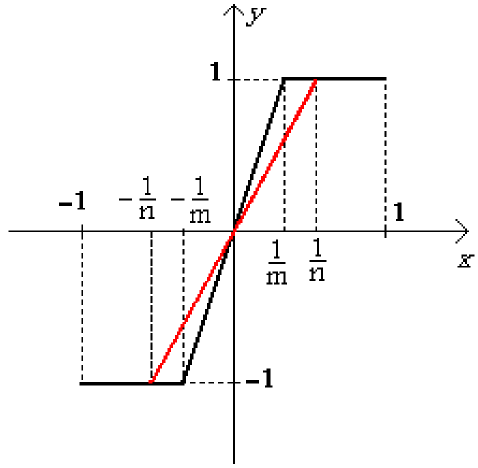
Exercise 15 *Show that for all bounded interval* $[a, b]$ *of* $\mathbb{R}$, *the space* $C\left([a, b], \mathbb{R}\right)$, *of the continuous functions on* $[a, b]$ *with values in* $\mathbb{R}$, *is a complete space for the distance of the uniform convergence* d_∞.

Exercise 16 *Let* $C([-1, 1], \mathbb{R})$ *be the set of continuous functions* $f : [-1, 1] \to \mathbb{R}$. *We equip* $C([-1, 1], \mathbb{R})$ *with the distance* d_1 *defined by*

$$d_1(f, g) = \int_{-1}^{1} |f(x) - g(x)|\, dx.$$

We will show that $C([-1, 1], \mathbb{R})$ *equipped with this distance is not complete. For this, we consider the sequence* $(f_n)_{n \geq 1}$ *of functions defined by*

$$f_n(x) = \begin{cases} -1 & \text{for } x \in [-1, -1/n], \\ nx & \text{for } x \in [-1/n, 1/n], \\ 1 & \text{for } x \in [1/n, 1]. \end{cases}$$



1. *Verify that* $f_n \in C([-1,1], \mathbb{R})$ *for all* $n \geq 1$.

2. *Show that for all* $n, m \geq 1$ *:*

$$|f_m(x) - f_n(x)| \leq 2, \ \forall x \in [-1,1].$$

   Deduce that $(f_n)_{n\geq 1}$ *is a Cauchy sequence.*

3. *Suppose there exists a function* $f \in C([-1,1], \mathbb{R})$ *such that* $(f_n)_{n\geq 1}$ *converges to* f *in* $(C([-1,1], \mathbb{R}), d_1)$. *Show that we then have*

$$\lim_{n\to+\infty} \int_{-1}^{-\alpha} |f_n(x) - f(x)|\, dx = 0 \qquad \text{and} \qquad \lim_{n\to+\infty} \int_{\alpha}^{1} |f_n(x) - f(x)|\, dx = 0$$

   for all $0 < \alpha < 1$.

4. *Show that*

$$\lim_{n\to+\infty} \int_{-1}^{-\alpha} |f_n(x) + 1|\, dx = 0 \qquad \text{and} \qquad \lim_{n\to+\infty} \int_{\alpha}^{1} |f_n(x) - 1|\, dx = 0$$

   for all $0 < \alpha < 1$.

5. *Deduce that*

$$f(x) = \begin{cases} -1, & x \in [-1, 0[, \\ 1, & x \in]0, 1]. \end{cases}$$

   Conclude.

Exercise 17 (Fixed point theorem and system resolution) *We equip* $\mathbb{R}^2$ *with the distance* d_1 :

$$d_1((x,y),(x',y')) = |x - x'| + |y - y'|.$$

and we define the map $f : \mathbb{R}^2 \to \mathbb{R}^2$ *by:*

$$f(x,y) = \left(\frac{1}{4}\sin(x+y), 1 + \frac{2}{3}\arctan(x-y)\right).$$

1. *Show that there exists a constant* $k \in]0,1[$ *such that, whatever* (x,y)*,* $(x',y') \in \mathbb{R}^2$*, we have*

$$d_1\left(f(x,y), f(x',y')\right) \leq k \ d_1\left((x,y),(x',y')\right).$$

2. *Deduce that the system*

$$\begin{aligned} \tfrac{1}{4}\sin(x+y) &= x \\ 1 + \tfrac{2}{3}\arctan(x-y) &= y \end{aligned} \qquad (S)$$

admits a unique solution in $\mathbb{R}^2$*.*

We recall the Banach-Picard contraction fixed point theorem:

Theorem 1 *Let* (X,d) *be a complete metric space. If the map* $f : X \to X$ *is a contraction with ratio* k*, then it admits a unique fixed point* $x^* \in X$*,* $f(x^*) = x^*$*.*
Moreover, any recurrent sequence given by

$$\begin{cases} x_{n+1} = f(x_n), \\ x_0 \in X, \end{cases}$$

converges to x^**, and we have*

$$\forall n \in \mathbb{N} : \quad d(x_n, x^*) \leq k^n \ d(x_0, x^*) \qquad \text{and} \qquad d(x_n, x^*) \leq \frac{k^n}{1-k} \ d(x_0, x_1).$$