

Sheet of exercises N°0

For a set X , we denote $\mathcal{P}(X)$ the set of its subsets, and for $A \in \mathcal{P}(X)$ we denote A^c its complement in X .

Exercise 1 Describe $\mathcal{P}(X)$ and $\mathcal{P}(\mathcal{P}(X))$ for

- (a) $X = \emptyset$; (b) $X = \{1\}$; (c) $X = \{1, 2\}$.

Exercise 2 For all subsets A and B of a set X , show the following equivalences:

1. $A \subset B \iff A \cap B = A$
2. $A \subset B \iff A \cup B = B$
3. $A \subset B \iff A \setminus B = \emptyset$
4. $A \cap B = \emptyset \iff A \subset B^c$
5. $A \setminus B = A \iff A \cap B = \emptyset$
6. $A \subset B \iff A^c \supset B^c$.

Exercise 3 Let X be a set and let $(A_i)_{i \in I}$ be any family in $\mathcal{P}(X)$. For $I = \emptyset$, we agree to set $\bigcup_{i \in I} A_i = \emptyset$ and $\bigcap_{i \in I} A_i = X$. Establish the relations:

1. $\left(\bigcup_{i \in I} A_i\right)^c = \bigcap_{i \in I} A_i^c$
2. $\left(\bigcap_{i \in I} A_i\right)^c = \bigcup_{i \in I} A_i^c$
3. $A \cap \left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} (A \cap A_i)$
4. $A \cup \left(\bigcap_{i \in I} A_i\right) = \bigcap_{i \in I} (A \cup A_i)$

Exercise 4

1. Consider the map $f : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto x + 1$. Determine the (direct) image of each of the following sets:

$$A = \{1, 2, 4\}, \quad B = \{x \in \mathbb{Z} ; x \geq -1\}, \quad C = \mathbb{N}.$$

2. Consider the map $f : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto x^2 - 1$. Determine the preimage of each of the following sets:

$$A = \{2, 3, 7\}, \quad B = \mathbb{N}, \quad C = [-1, +\infty[.$$

3. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the map defined by:

$$f(x) = \begin{cases} x + 1, & x \geq 0, \\ 1, & x < 0. \end{cases}$$

Determine the direct image of $\mathbb{R}$ and then the preimage of the interval $[1, 2]$.

Exercise 5 Let X and Y be two sets and f a map from X to Y . Show that for any subset A of X and any subset B of Y , we have

1. $f(f^{-1}(B)) \subset B$ and $A \subset f^{-1}(f(A))$,
2. $f^{-1}(B^c) = (f^{-1}(B))^c$.

Exercise 6 Let X and Y be two sets and f a map from X to Y .

1. For any families $(A_i)_{i \in I}$ of $\mathcal{P}(X)$ and $(B_i)_{i \in I}$ of $\mathcal{P}(Y)$, show that we have:
 - i) $f(\bigcup_{i \in I} A_i) = \bigcup_{i \in I} f(A_i)$
 - ii) $f(\bigcap_{i \in I} A_i) \subset \bigcap_{i \in I} f(A_i)$
 - iii) $f^{-1}(\bigcup_{i \in I} B_i) = \bigcup_{i \in I} f^{-1}(B_i)$
 - iv) $f^{-1}(\bigcap_{i \in I} B_i) = \bigcap_{i \in I} f^{-1}(B_i)$
2. Give an example of strict inclusion for the second relation.
3. Show that equality holds in the second relation if and only if f is injective.

Exercise 7 Let the map $f : \mathbb{R} \rightarrow \mathbb{R}$, $x \mapsto |x| + 2x$.

1. Show that f is bijective.
2. Determine the inverse map f^{-1} .